Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 3/7: Quadrupoles, Dipole Edge Focusing, Periodic Motion, Lattice Functions

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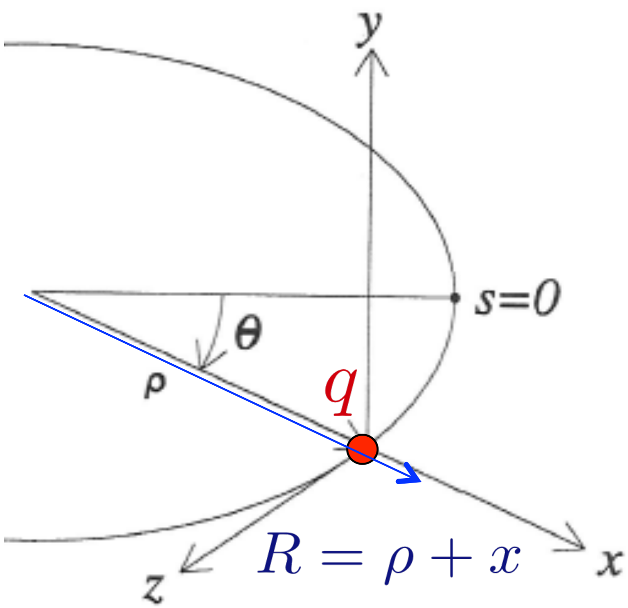
Thursday, September 29, 2011

MePAS Accelerator Physics Syllabus

- 1-2: Wednesday
  - Relativity/EM review, coordinates, cyclotrons
  - Weak focusing, transport matrices, dipole magnets, dispersion
- **3: Thursday**
  - Edge focusing, quadrupoles, accelerator lattices, start FODO
- 4: Friday
  - Periodic lattices, FODO optics, emittance, phase space
- 5: Saturday
  - Insertions, beta functions, tunes, dispersion, chromaticity
- 6: Monday
  - Dispersion suppression, light source optics (DBA, TBA, TME)
- 7: Tuesday
  - (Nonlinear dynamics), Putting it all together

Parameterizing Particle Motion: Approximations

coordinate system



- We have specified a coordinate system and made a few reasonable approximations:
  0) No local currents (beam in a near-vacuum)
  1) Paraxial approximation:
  $$x', y' \ll 1 \quad \text{or} \quad p_x, p_y \ll p_s$$
  2) Perturbative coordinates:
  $$x, y \ll \rho$$
  3) Transverse linear B field:
  $$\vec{B} = B_0 \hat{y} + (x\hat{y} + y\hat{x}) \left(\frac{\partial B_y}{\partial x} \right)$$
  4) Negligible E field:
  $$\gamma \approx \text{constant}$$

Review

- Drift transport matrix: $B = 0$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- Dipole transport matrix without focusing: $B = B_0 \hat{y}$

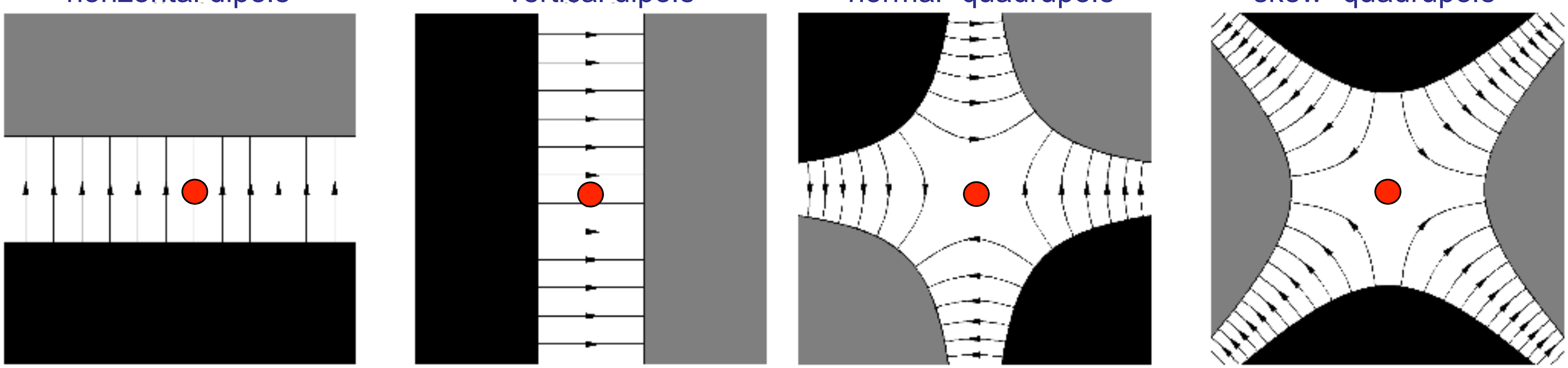
$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

- Dipole horizontal transport matrix including focusing and dispersion:

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

Focusing Without Bending

- Quadrupole magnets have $B_0 = 0$ but $\left(\frac{\partial B_y}{\partial x}\right) \neq 0$
- No dipole field: design trajectory is straight
  - Like taking $\rho \to \infty$ in our previous analysis
  - This is one reason why we changed our parameterizations from $\frac{dx}{d\theta} \to \frac{dx}{ds} \equiv x'$



Quadrupole Equations of Motion

$$\frac{d^2x}{d\theta^2} - nx = 0 \qquad \frac{d^2y}{d\theta^2} + ny = 0 \qquad n \equiv -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)$$

(A2)

$$\frac{d}{d\theta} = \frac{1}{R}\frac{d}{ds} \qquad \Rightarrow \qquad x'' + \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)x = 0 \qquad y'' - \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)y = 0$$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right) \qquad [K] = [\text{length}]^{-2}$$

- This is truly a simple harmonic oscillator when K is constant: for a quadrupole of length L

$$\begin{pmatrix} x(L) \\ x'(L) \\ y(L) \\ y'(L) \end{pmatrix} = \begin{pmatrix} \cos(L\sqrt{K}) & \frac{1}{\sqrt{K}}\sin(L\sqrt{K}) & 0 & 0 \\ -\sqrt{K}\sin(L\sqrt{K}) & \cos(L\sqrt{K}) & 0 & 0 \\ 0 & 0 & \cosh(L\sqrt{K}) & \frac{1}{\sqrt{K}}\sinh(L\sqrt{K}) \\ 0 & 0 & \sqrt{K}\sinh(L\sqrt{K}) & \cosh(L\sqrt{K}) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

focusing   defocusing

Thick quadrupole transport matrix

Swap places when K goes to -K

Thin Quadrupoles

- In most accelerator uses, we can take L->0 with KL constant

Use small-angle approximation to rewrite as a "thin" quadrupole

$$\begin{pmatrix} x(L) \\ x'(L) \\ y(L) \\ y'(L) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -KL & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & KL & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

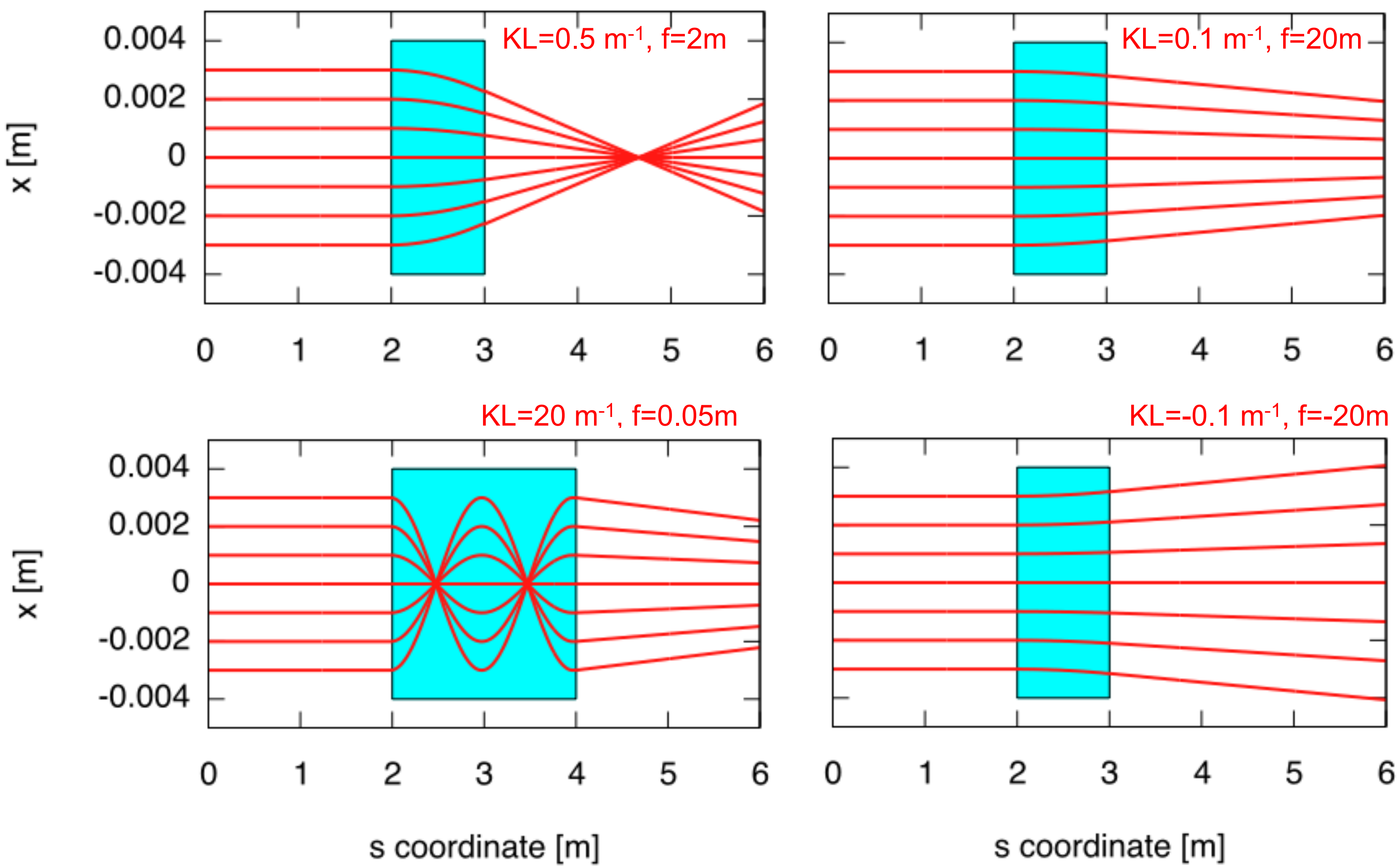
This is just like a lens in classical optics with a focal length $f \equiv \frac{1}{KL}$

$$\begin{pmatrix} x(L) \\ x'(L) \\ y(L) \\ y'(L) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

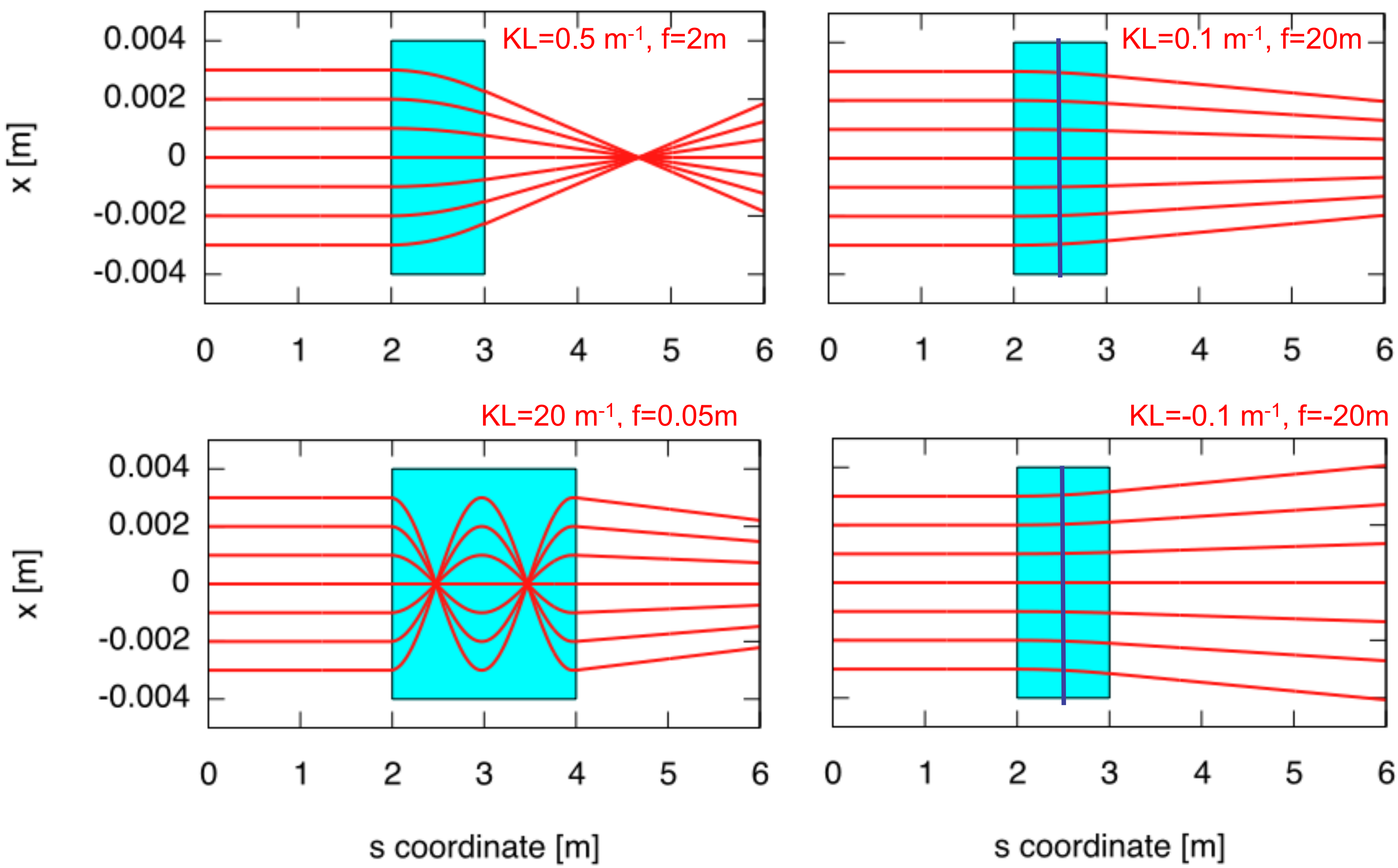
Thin quadrupole transport matrix

Swap places when K goes to -K

Picturing Drift and Quadrupole Motion

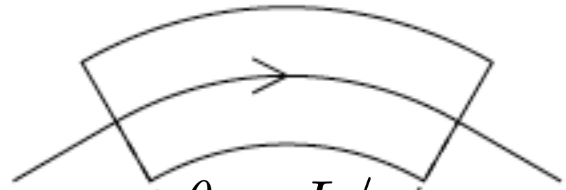


Picturing Drift and Quadrupole Motion



Thin Quadrupole Approximations

Dipole Edge Focusing



$$\theta = L/\rho$$

- Quadrupoles are not the only place we get focusing!
- Recall our 3x3 sector dipole matrix

  Vertical motion is just a drift of length $L = \rho\theta$

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

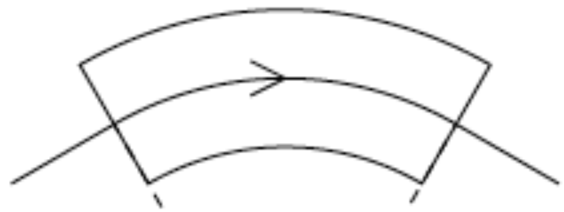
$$n = 0 \quad \Rightarrow \quad M_H(\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

- But this magnet is curved and therefore not easy to build

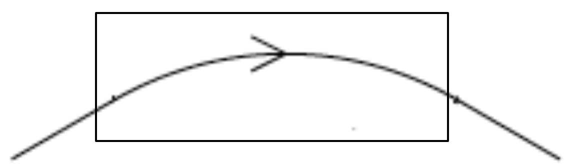
  In particular, the ends are “tilted” to be $\perp$ to design trajectory

Sector and Rectangular Bends

- Sector bend (sbend)
  - Beam design entry/exit angles are ⊥ to end faces

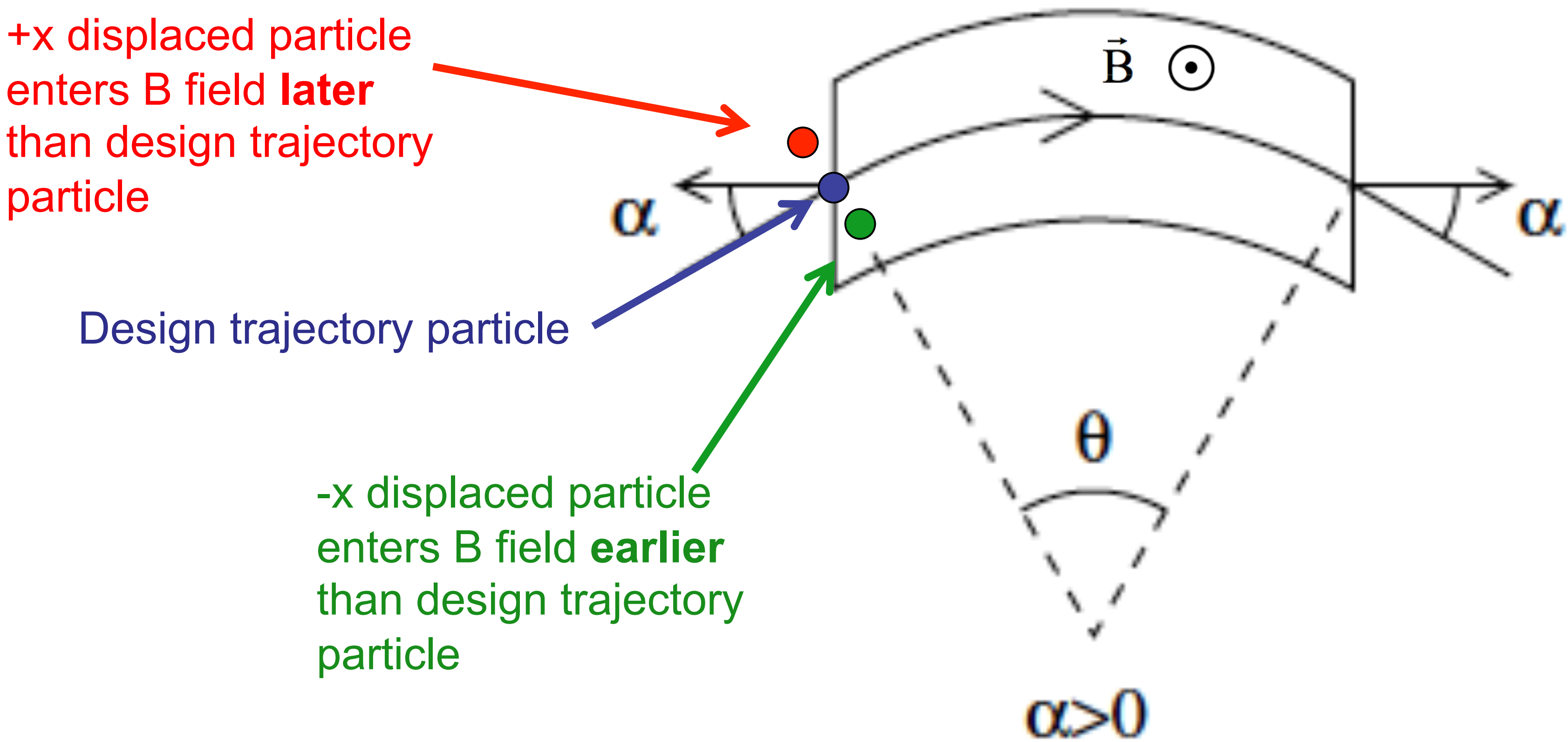


  - Simpler to conceptualize, but harder to build
- Rectangular bend (rbend)
  - Beam design entry/exit angles are half of bend angle



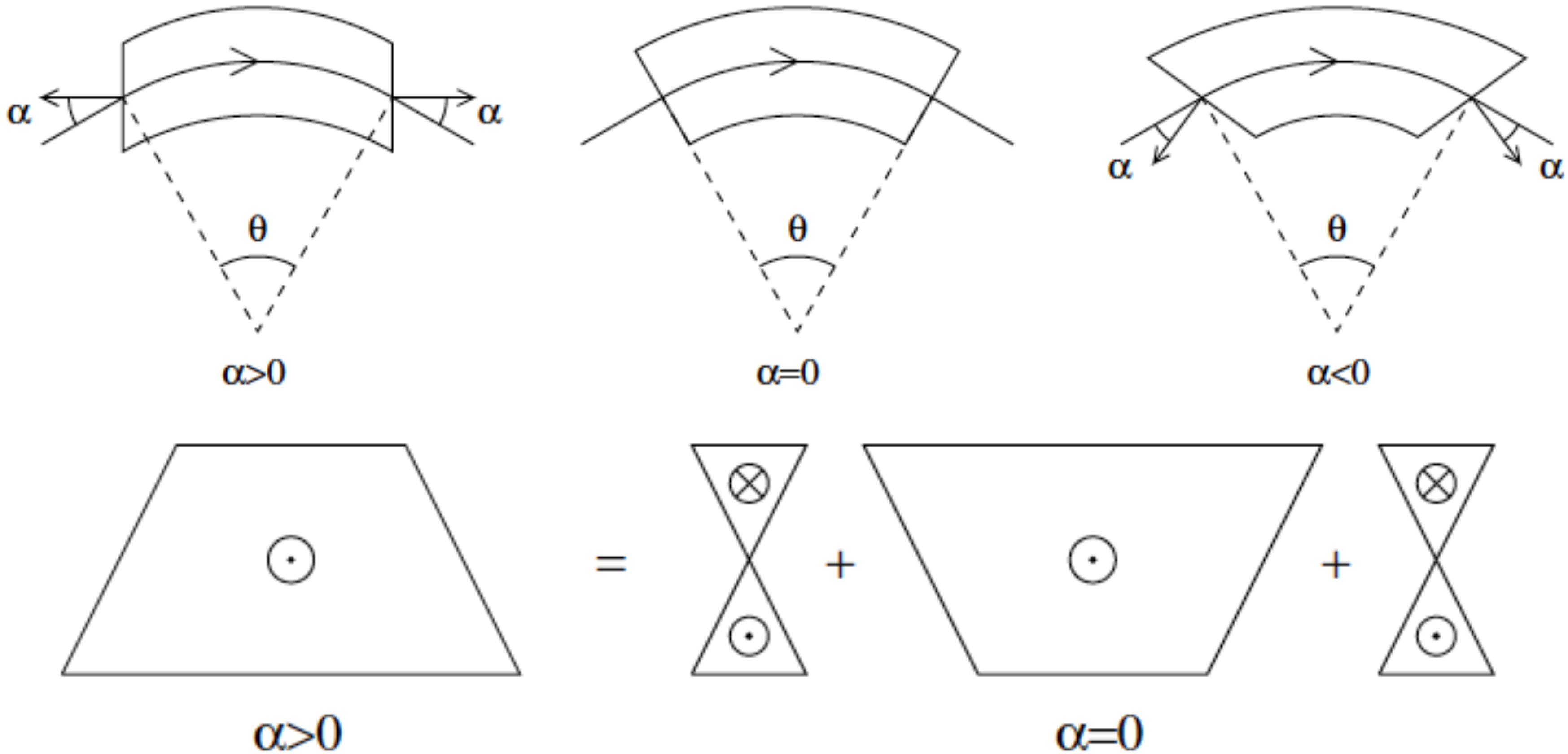
  - Easier to build, but must include effects of edge focusing

Dipole End Angles



- Different transverse positions see different B field!
  - Particles displaced by +x see B field later than design
  - Particles displaced by –x see B field earlier than design

Dipole End Angles



- We treat general case of symmetric dipole end angles
  - Superposition: looks like wedges on end of sector dipole
  - Rectangular bends are a special case

Kick from a Thin Wedge

- The edge focusing calculation requires the kick from a thin wedge

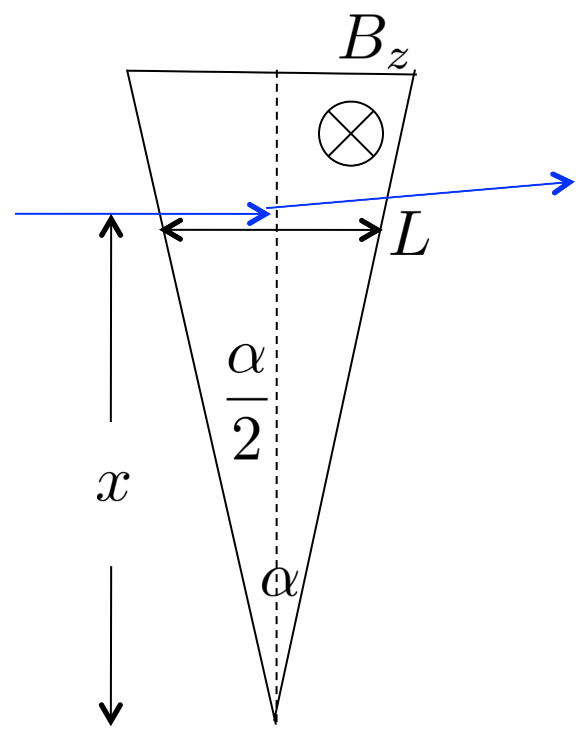
$$\Delta x' = \frac{B_z L}{(B\rho)}$$

What is L? (distance in wedge)

$$\tan\left(\frac{\alpha}{2}\right) = \frac{L/2}{x}$$

$$L = 2x \tan\left(\frac{\alpha}{2}\right) \approx x \tan\alpha$$

So $\boxed{\Delta x' = \frac{B_z \tan\alpha}{(B\rho)} x = \frac{\tan\alpha}{\rho} x}$



Quadrupole-like defocusing term, linear in position

Dipole Matrix with Ends

- The matrix of a dipole with thick ends is then

$$M_{\text{sector dipole}}(x, x', \delta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\text{end lens}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tan\alpha}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

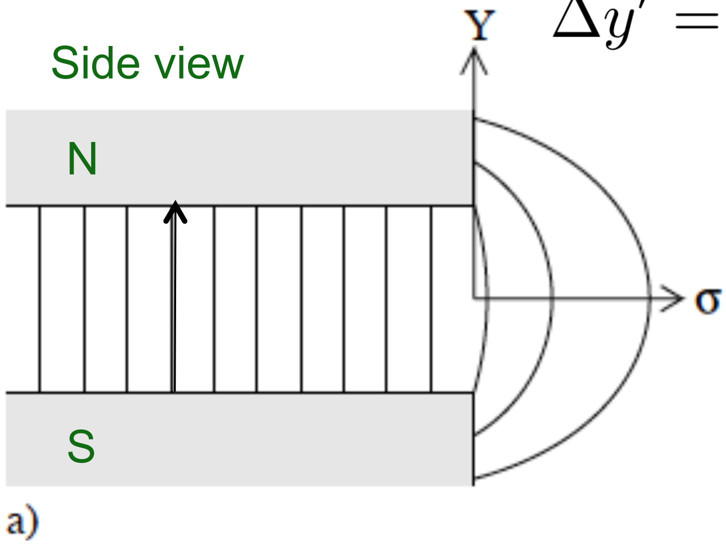
$$M_{\text{edge}-\text{focused dipole}} = M_{\text{end lens}} M_{\text{sector dipole}} M_{\text{end lens}}$$

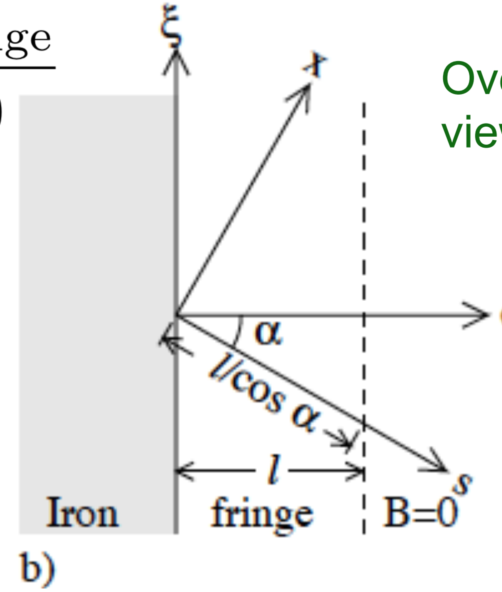
$$M_{\text{edge}-\text{focused dipole}} = \begin{pmatrix} \frac{\cos(\theta-\alpha)}{\cos\alpha} & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin(\theta-2\alpha)}{\rho\cos^2\alpha} & \frac{\cos(\theta-\alpha)}{\cos\alpha} & \frac{\sin(\theta-\alpha)+\sin\alpha}{\cos\alpha} \\ 0 & 0 & 1 \end{pmatrix}$$

- Rectangular bend is special case where $\alpha=\theta/2$

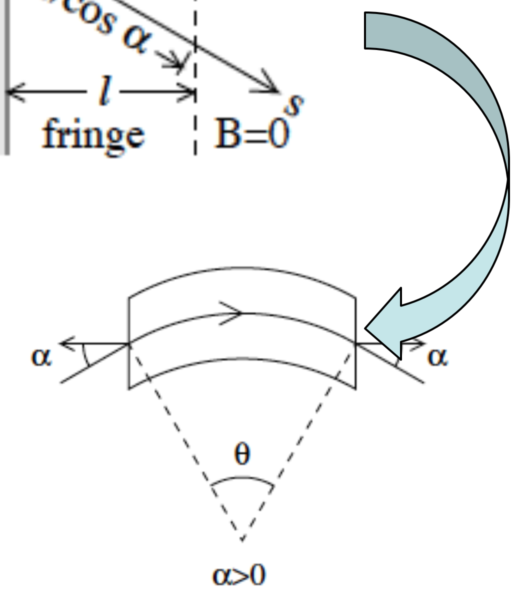
What About Vertical Edge Focusing?

$$\Delta y' = \frac{B_x l_{\rm fringe}}{(B\rho)}$$





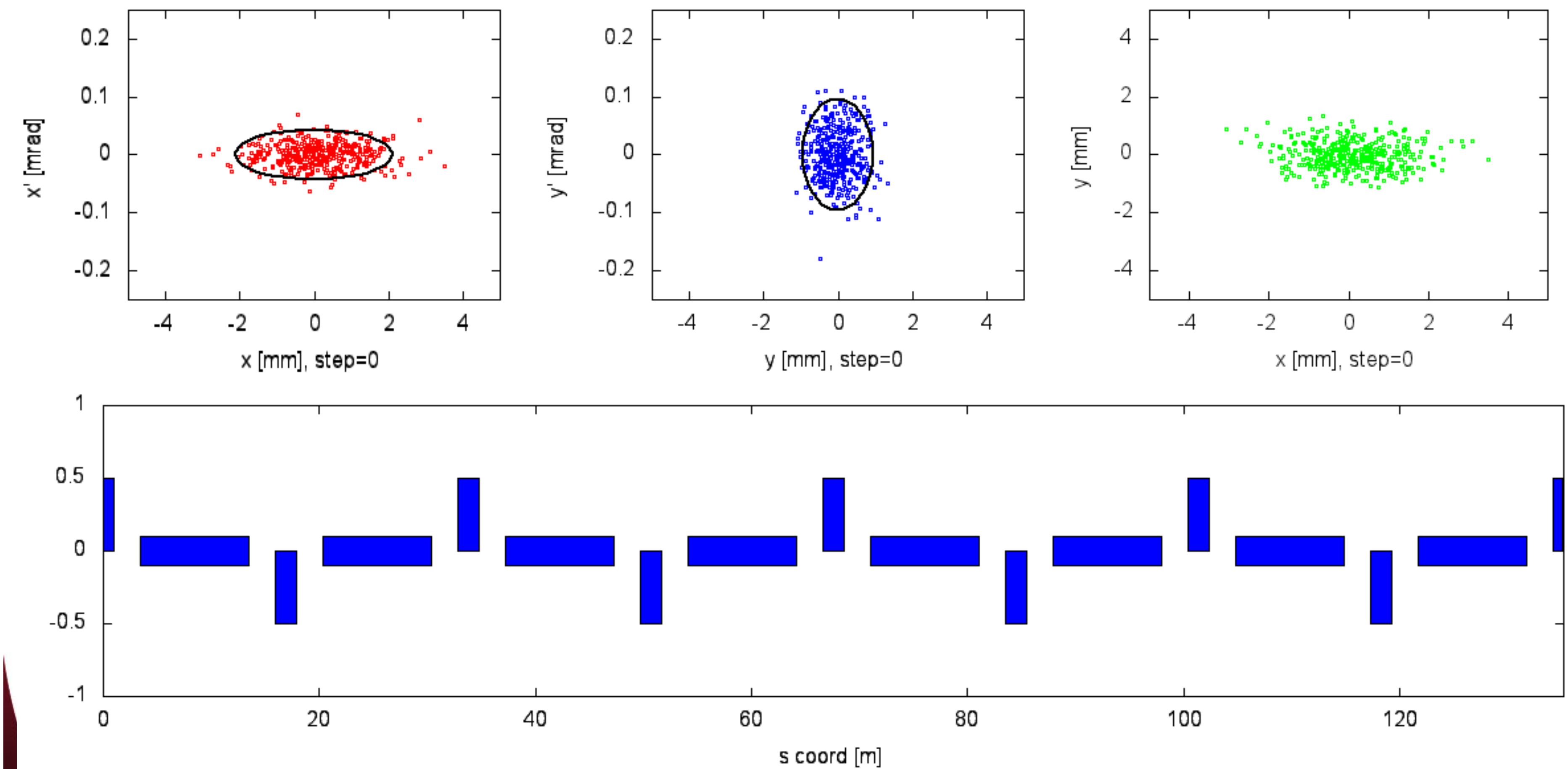
Overhead view, $\alpha>0$



- Field lines go from –y to +y for a positively charged particle
  - $B_x<0$ for $y>0$; $B_x>0$ for $y<0$
    - Net focusing!
  - Field goes like $\sin(\alpha)$
    - get $\cos(\alpha)$ from integral length
  - Quadrupole-like focusing

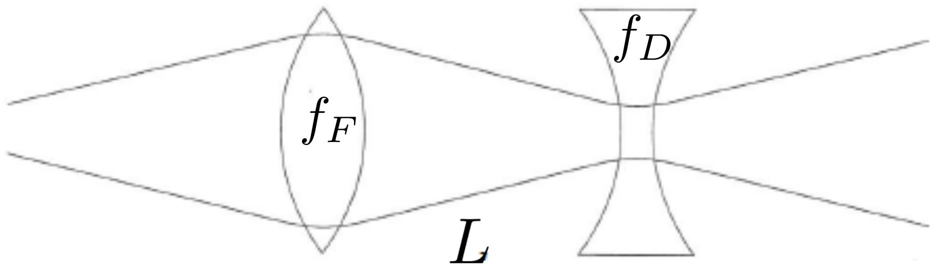
$$\Delta y' = \frac{(-B_x y \sin\alpha / l)(l/\cos\alpha)}{(B\rho)} = -\frac{\tan\alpha}{\rho} y$$

=========== Almost There ===========



Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift L



$$M_{\rm doublet} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

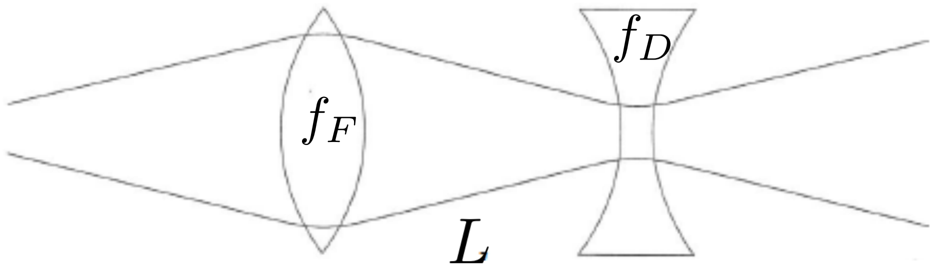
$$\frac{1}{f_{\rm doublet}} = \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D}$$

$$f_D = f_F = f \quad \Rightarrow \quad \frac{1}{f_{\rm doublet}} = -\frac{L}{f^2}$$

There is **net focusing** given by this **alternating gradient** system

A fundamental point of optics, and of accelerator **strong focusing**

Strong Focusing: Another View



$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

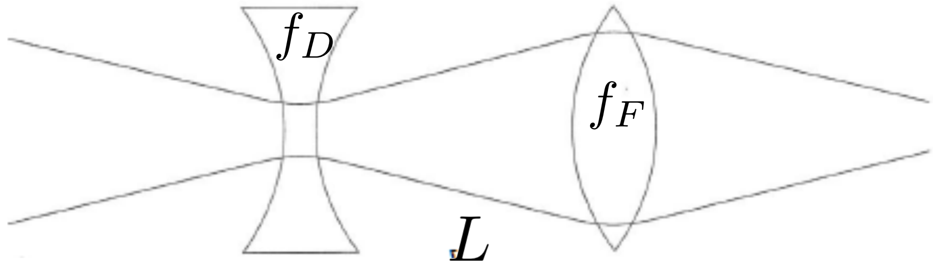
incoming paraxial ray $$\begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \end{pmatrix} x_0$$

For this to be focusing, x’ must have opposite sign of x

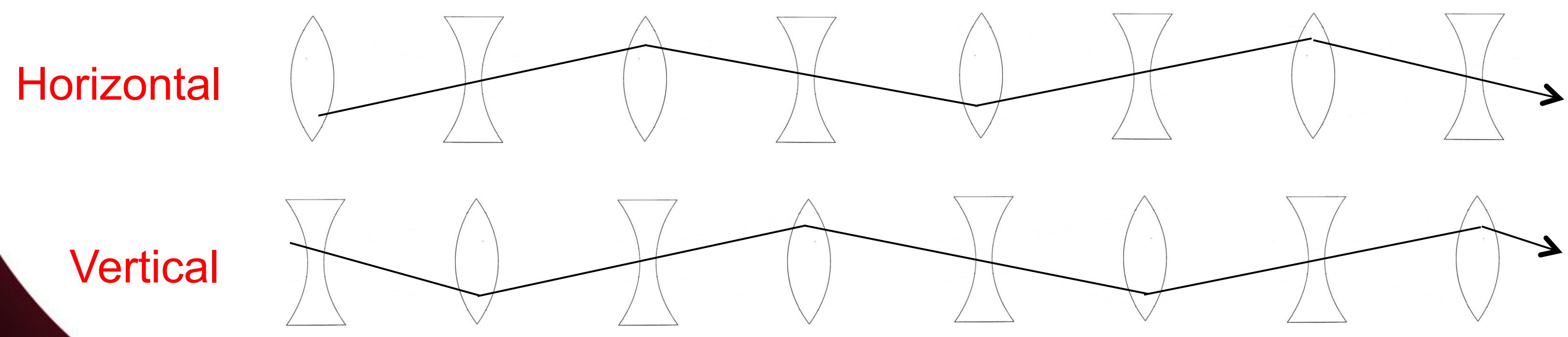
$$f_F = f_D \quad x' < 0 \quad \textbf{BUT} \quad x > 0 \text{ iff } f_F > L$$

Equal strength doublet is net focusing under condition that each lens’s focal length is greater than distance between them

Strong Focusing Homework



- The previous argument also works when the defocusing quadrupole comes before the focusing quadrupole
  - **Homework:** Calculate the net focusing condition for this system
  - Since quadrupoles focus in one plane and defocus in the other, alternating quadrupoles continuously produces a system that is **overall net focusing and stable**



FODO lattice: Periodic!

More Math: Hill's Equation

- Let's go back to our equations of motion for $R \to \infty$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)$$

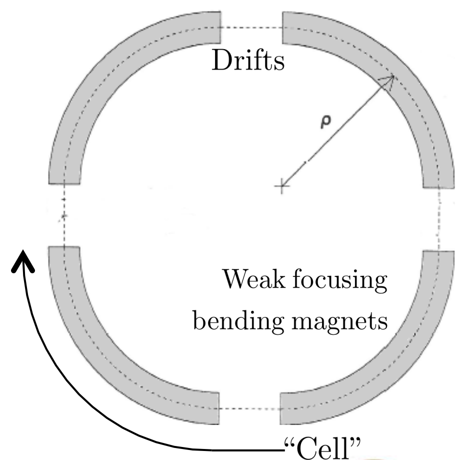
What happens when we let the focusing K vary with s?

Also assume K is **periodic** in s with some periodicity C

$$x'' + K(s)x = 0 \qquad K(s) \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)(s) \qquad K(s+C) = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated "cell" of the layout

(Such as a FODO cell in the previous slide)



The simple harmonic oscillator equation with a periodically varying spring constant K(s) is known as **Hill's Equation**

Hill's Equation Solution Ansatz

$$x'' + K(s)x = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)(s)$$

- Solution is a quasi-periodic harmonic oscillator

$$x(s) = A\, w(s)\ \cos[\phi(s) + \phi_0]$$

where w(s) is periodic in C but the phase ϕ is not!!

Substitute this educated guess ("ansatz") to find

$$x' = Aw' \cos[\phi + \phi_0] - Aw\phi' \sin[\phi + \phi_0]$$

$$x'' = A(w'' - w\phi'^2) \cos[\phi + \phi_0] - A(2w'\phi' + w\phi'') \sin[\phi + \phi_0]$$

$$x'' + K(s)x = -A(2w'\phi' + w\phi'') \sin(\phi + \phi_0) + A(w'' - w\phi'^2 + Kw) \cos(\phi + \phi_0) = 0$$

For $w(s)$ and $\phi(s)$ to be independent of ϕ_0, coefficients of sin and cos terms must vanish identically

Courant-Snyder Parameters

$$2ww'\phi' + w^2\phi'' = (w^2\phi')' = 0 \quad \Rightarrow \quad \phi' = \frac{k}{w(s)^2}$$

$$w'' - (k^2/w^3) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2$$

- Notice that in both equations $w^2 \propto k$ so we can scale this out and define a new set of functions, **Courant-Snyder Parameters** or **Twiss Parameters**

$$\beta(s) \equiv \frac{w^2(s)}{k}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s)$$

$$\gamma(s) \equiv \frac{1 + \alpha(s)^2}{\beta(s)}$$

$$\phi' = \frac{1}{\beta(s)} \qquad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\Rightarrow \quad K\beta = \gamma + \alpha'$$

$\beta(s), \alpha(s), \gamma(s)$ are all periodic in C

$\phi(s)$ is **not** periodic in C

Towards The Matrix Solution

- What is the matrix for this Hill's Equation solution?

$$x(s) = A\sqrt{\beta(s)}\cos\phi(s) + B\sqrt{\beta(s)}\sin\phi(s)$$

$$\phi' = \frac{1}{\beta(s)} \qquad x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\phi(s) - [A + \alpha(s)B]\sin\phi(s)\}$$

$$A = \frac{x_0}{\sqrt{\beta(s)}} \qquad B = \frac{1}{\sqrt{\beta(s)}}[\beta(s)x_0' + \alpha(s)x_0]$$

This all looks pretty familiar and pretty tedious…

We have done this many times so we skip to the solution

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Delta\phi_C = \int_{s_0}^{s_0+C}\frac{ds}{\beta(s)}$$

Interesting Observations

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

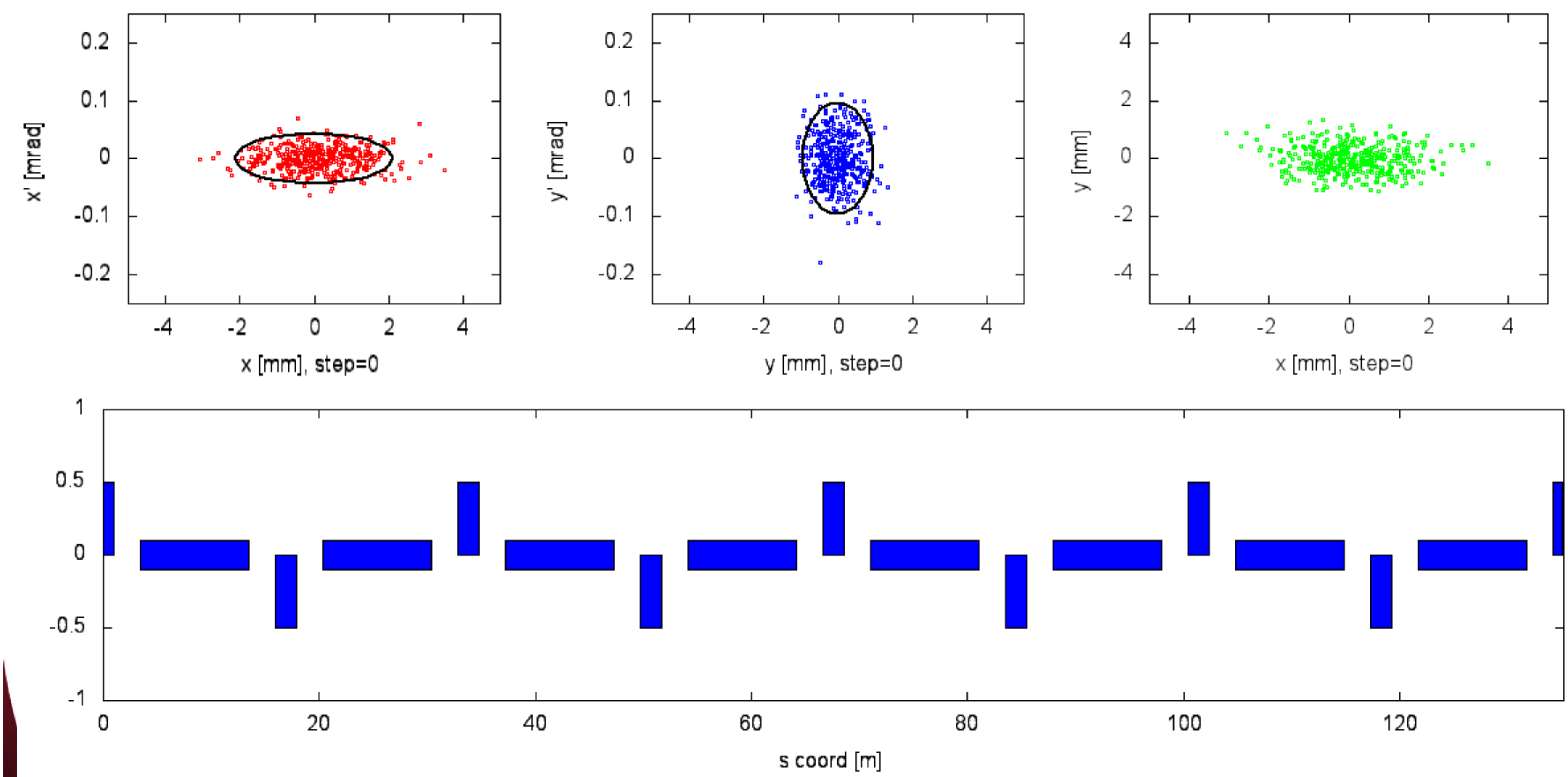
- $\Delta\phi_C$ is independent of s: **betatron phase advance** again
- Determinant of matrix M is still 1! (Check!)
- Still looks like a rotation and some scaling
- M can be written down in a beautiful and deep way

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\Delta\phi_C}$$

and remember $x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$

========== Once Again ☺ ==========



0.2
0.1
0
-0.1
-0.2
x' [mrad]
-4 -2 0 2 4
x [mm], step=0
y' [mrad]
y [mm], step=0
4
2
-2
-4
y [mm]
x [mm], step=0
1
0.5
-0.5
-1
0 20 40 60 80 100 120
s coord [m]