Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 7/7: Some "Fun": Corrections, Nonlinear Dynamics, Putting It All Together

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Review

$$\mathrm{Hill's\ equation}\quad x'' + K(s)x = 0$$

$$\mathrm{quasi-periodic\ ansatz\ solution}\quad x(s) = \sqrt{\epsilon\beta(s)}\cos[\phi(s)+\phi_0]$$

$$\beta(s) = \beta(s+C) \qquad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\quad \Delta\phi_C = \displaystyle\int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \qquad \mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

Courant – Snyder invariant $\quad \mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$

General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using the lattice parameters

$$M(s_2) = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\phi + \alpha(s_1)\sin\Delta\phi] & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\phi + [1+\alpha(s_1)\alpha(s_2)]\sin\Delta\phi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi] \end{pmatrix}$$

- This does not have a pretty form like the periodic matrix

However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

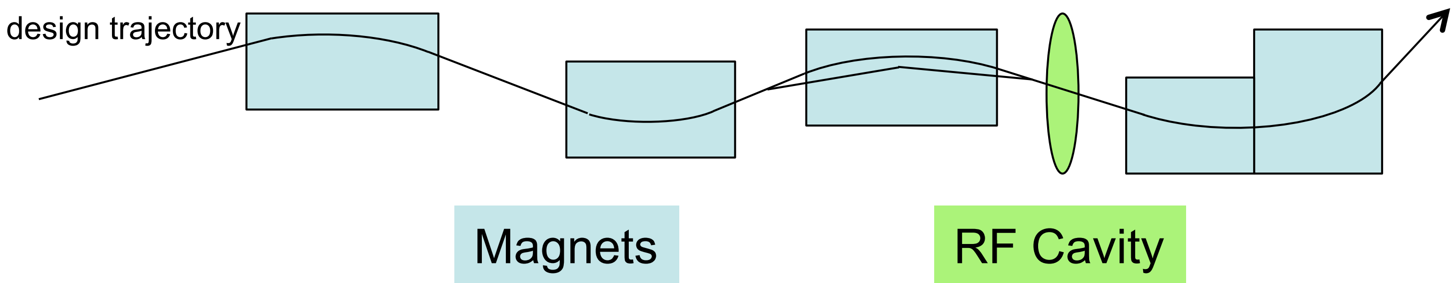
where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

The most common use of this matrix is the m_{12} term:

$$\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\,\sin(\Delta\phi)\,x'(s_1)$$

Effect of angle kick on downstream position

Design Orbit Perturbations



- Sometimes need a local change $\Delta x(s)$ to the design orbit
  - But we really only get changes in angle $\Delta x'$ from magnets
  - e.g. small dipole "corrector": $\Delta x' = B_{\text{corrector}} L_{\text{corrector}} / (B\rho)$
  - Changes to/corrections of design orbit from dipole correctors
  - Linear errors add up via linear superposition

$$\begin{pmatrix} \Delta x(s_2) \\ \Delta x(s_1) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x'(s_1) \end{pmatrix}$$

$$\Delta x(s_2) = \Delta x'(s_1) \sqrt{\beta(s_1)\beta(s_2)} \sin \Delta\phi_{12}$$

$$\Delta x'(s_2) = \Delta x'(s_1) \sqrt{\frac{\beta(s_1)}{\beta(s_2)}} [\cos \Delta\phi_{12} - \alpha(s_2) \sin \Delta\phi_{12}]$$

Two-Bump

$$\Delta x(s_2) = \Delta x'(s_1)\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi_{12}$$

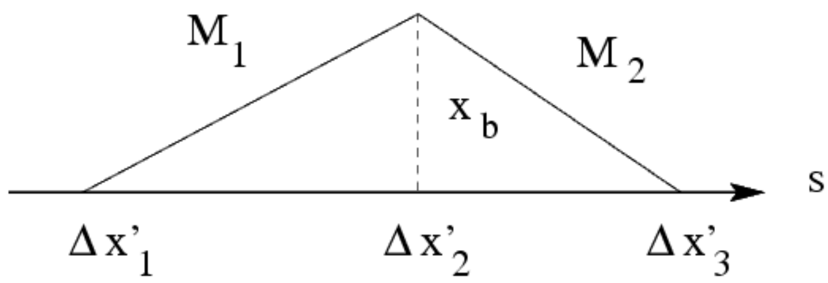
$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi_{12} - \alpha(s_2)\sin\Delta\phi_{12}]$$

- But this orbit error now changes all later positions and angles
  - Add another dipole corrector at a location where $\Delta\phi_{12} = k\pi$ At this point the distortion from the original dipole corrector is all x’ that we can cancel with the second dipole corrector.

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}} + \text{angle from s}_2\text{ dipole}$$

  - Called a **two-bump**: localized orbit distortion from two correctors
  - But requires $\Delta\phi_{12} = k\pi$ between correctors

Three-Bump



- A general local orbit distortion from three dipole correctors
  - Constraint is that net orbit change from sum of all three kicks must be zero

$$\begin{pmatrix} C_2 & S_2 \\ C_2' & S_2' \end{pmatrix} \left[\begin{pmatrix} C_1 & S_1 \\ C_1' & S_1' \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x_1' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x_2' \end{pmatrix} \right] + \begin{pmatrix} 0 \\ \Delta x_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Delta x_1' = \frac{x_b}{S_1} \quad \Delta x_2' = -\frac{C_2 S_1 + S_2 S_1'}{S_1 S_2} x_b \quad \Delta x_3' = \frac{S_2}{S_1^2} x_b$$

  - Bump amplitude $x_b = S_1 \Delta x_1'$
  - Only **three-bump** requirement is that S_1, $S_2 \neq 0$

Steering Error in Synchrotron Ring

- Short steering error $\Delta x'$ in a ring with periodic matrix M
  - Solve for new periodic solution or design orbit (x_0,x'_0)

$$M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

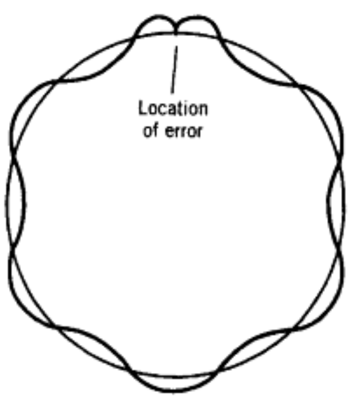
  - Note that (x_0=0,x'_0=0) is not the periodic solution any more!

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = (I-M)^{-1} \begin{pmatrix} 0 \\ \Delta x'_0 \end{pmatrix}$$

$$(I-M)^{-1} = (I - e^{(2\pi Q)J})^{-1} = \left([e^{\pi QJ}\left(e^{-\pi QJ} - e^{\pi QJ}\right)]\right)^{-1}$$

$$= -\left(2J\sin(\pi Q)\right)^{-1}\left(e^{\pi QJ}\right)^{-1}$$

$$= \frac{1}{2\sin(\pi Q)}(J\cos(\pi Q) + I\sin(\pi Q))$$



$$x_0 = \frac{\Delta x'_0}{2}\tan(\pi Q) \qquad \rightarrow \infty \text{ if } Q = k\pi$$

integer resonances

Focusing Error in Synchrotron Ring

- Short focusing error in a ring with periodic matrix M
  - Now solve for Tr M to find effects on tune Q

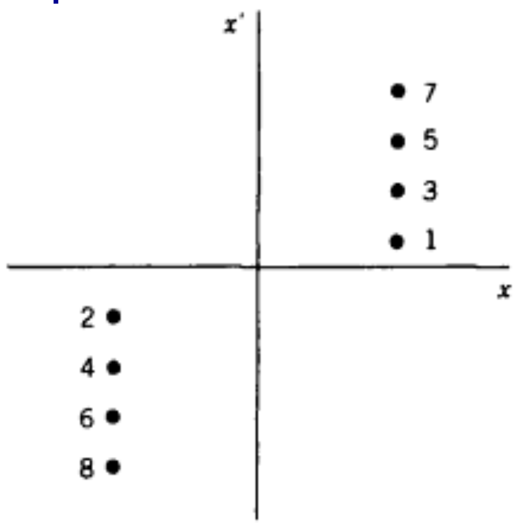
$$M_{\text{new}} = M \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\frac{1}{2}\text{Tr } M = \cos(2\pi Q_{\text{new}}) = \cos(2\pi Q_0) - \frac{1}{2}\frac{\beta_0}{f}\sin(2\pi Q_0)$$

  - For small errors $Q_{\text{new}} = Q_0 + \Delta Q$ we can expand to find

$$\Delta Q \approx \frac{1}{4\pi}\frac{\beta_0}{f}$$

  - Quadrupole errors also cause resonances when $Q = k/2$: **half-integer resonances**



Chromaticity Correction

Natural chromaticity $\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q} \oint K(s)\beta(s)\, ds$

- How can we control chromaticity in our synchrotron ring?
  - We need a way to connect momentum offset δ to focusing
  - Dispersion (momentum-dependent position) and sextupoles (nonlinear focusing depending on position) come to rescue

$$x(s) = x_{\text{betatron}}(s) + \eta_x(s)\delta$$

$$\text{Sextupole B field} \quad B_y = b_2 x^2$$

$$B_y(\text{sext}) = b_2[x_{\text{betatron}}(s) + \eta_x(s)\delta]^2 \approx b_2 x_{\text{betatron}}^2 + 2b_2 x_{\text{betatron}}(s)\eta_x(s)\delta$$

Nonlinear! like a quadrupole K(s)!

  - Total chromaticity from all sources is then

$$\xi = -\frac{1}{4\pi Q} \oint [K(s) - b_2(s)\eta_x(s)]ds$$

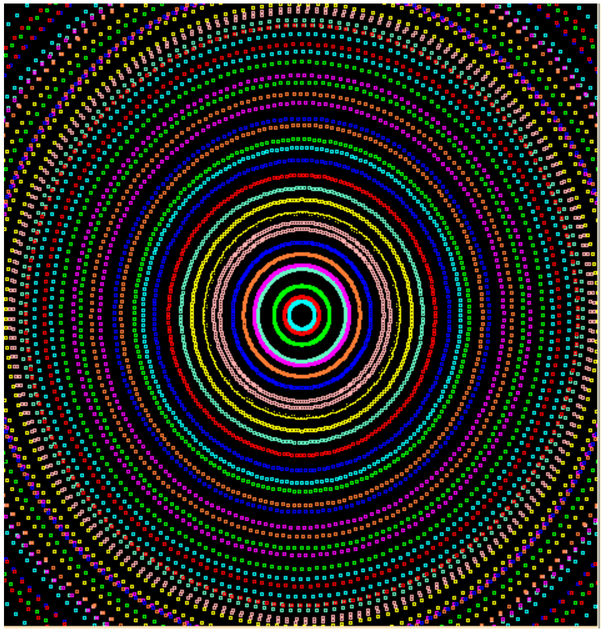
  - Strong focusing (large K) requires large sextupoles, **nonlinearity**!

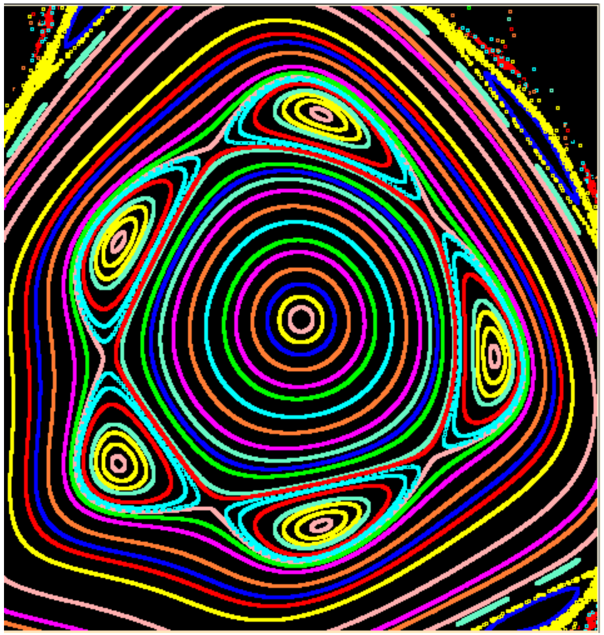
How Bad Could It Be?

- We expanded so far assuming nonlinearities are small
- How bad could it really be?
  - Even **one** large nonlinear (sextupole, octupole, …) error in a ring can drive **all** nonlinear resonances $kQ_x + lQ_y = m$
  - Large-amplitude particle motion will almost invariably be **unstable**, even **chaotic** in the true dynamical sense

Q=10.205
Linear
No sextupole

High amplitude motion stable





Q=10.205
Nonlinear
One sextupole
b_2=0.564

High amplitude motion unstable

$5Q_x$=m resonance islands

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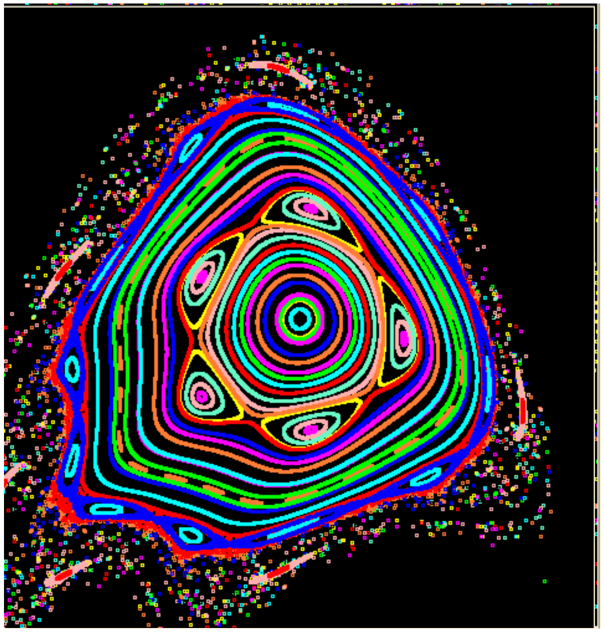
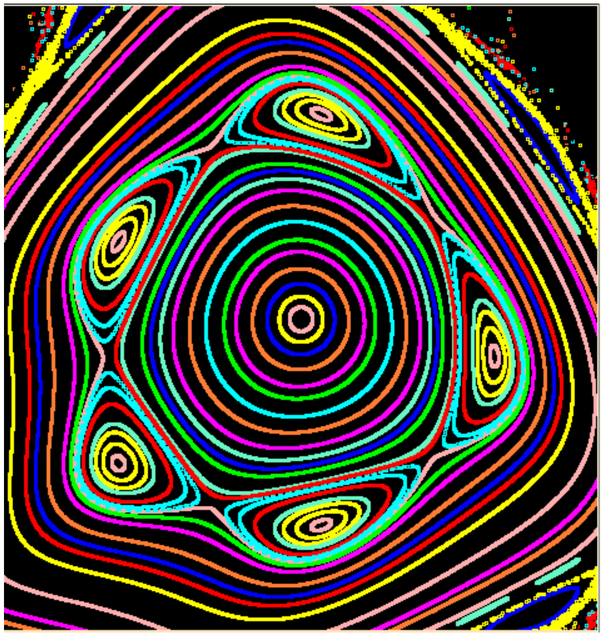
Q=10.205
Nonlinear
One sextupole
b_2=1041





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Tune Diagram

- We often plot (Q_x,Q_y) on a tune diagram

  Also plot resonance lines $kQ_x + lQ_y = m$

  We can still analyze nonlinear resonances with perturbation theory

  Various resonances are driven to various "orders" in perturbative expansions

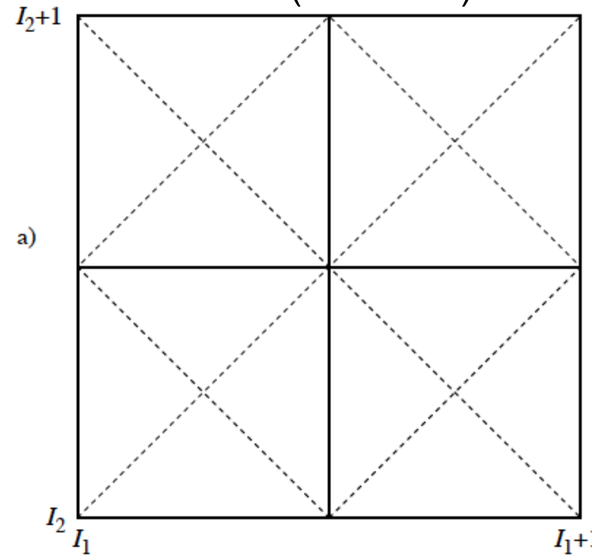
  If nonlinearities are kept small, then higher order nonlinearities can be neglected

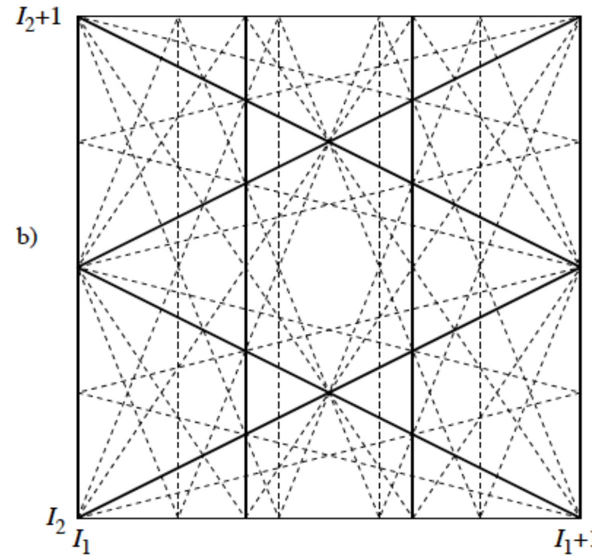
  Thanks for the great magnets, Cherryl and Liz!

  Remember tune here is a blob, not a point!

  Beam tunes (Q_x,Q_y) have distributions

Normal quad, octupole errors (first order)





Normal sextupole, decapole errors (first order)

Hamiltonians

- The dynamical treatment of nonlinearities particle accelerators is a broad subfield of its own
  - Predictions of nonlinear behavior directly affect specifications, cost, and even feasibility
  - Cost of facility scales with magnet aperture, dynamic aperture
  - Interactions with other fields (e.g. astrophysics, Laskar)
- Perturbative approach is usually done using Hamiltonians
  - Matrices are not very convenient for analysis of nonlinearities

$$H(x, x'; s) = \frac{x'^2}{2} + K(s)\frac{x^2}{2}$$

Quadratic quadrupole focusing!

Hamilton's Equations $$x' = \frac{\partial H}{\partial x'} \qquad (x')' = -\frac{\partial H}{\partial x}$$

$\Rightarrow$ Hill's Equation again $$x''(s) + K(s)x(s) = 0$$

Action-Angle Coordinates

- Certain types of coordinate transforms keep Hamilton's Equations for the dynamics
  - These are called **canonical transformations** in dynamics
  - Can be generated from **generating functions**
- One very useful coordinate transformation is to **action-angle** coordinates
  - We want to treat the betatron phase ϕ as a coordinate
  - Its canonical conjugate variable is an action J

$$J = \frac{x^2}{2\beta(s)} \sec^2 \phi(s) = \frac{1}{2\beta(s)} [x(s)^2 + (\beta(s)x'(s) + \alpha(s)x(s))^2]$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta(s)} \cos \phi(s) \\ -\sqrt{1/\beta(s)}[\sin \phi(s) + \alpha(s) \cos \phi(s)] \end{pmatrix}$$

  - This looks horrible, but now the Hamiltonian is simpler!

Action-Angle Coordinates

- How simple is the action-angle Hamiltonian?

$$H(J, \phi; s) = \frac{J}{\beta(s)}$$

  - Periodic in s (like Hill's Equation)
  - No dependence on ϕ so J is a constant of the motion
  - Integrating this over one turn gives a "one-turn" Hamiltonian

$$H_{\text{one turn}} = J \oint \frac{ds}{\beta(s)} = 2\pi Q J$$

  - Hamilton's equations give $\Delta\phi_{\text{one turn}} = 2\pi Q \quad \Delta J_{\text{one turn}} = 0$
- The real power is that we can add nonlinear perturbation potentials to the Hamiltonian for nonlinear fields and solve in exactly the same ways

$$H(x, x', y, y'; s) = \frac{x'^2}{2} + K_x(s)\frac{x^2}{2} + \frac{y'^2}{2} + K_y(s)\frac{y^2}{2} + V(x, y; s)$$

$$H(J_x, \phi_x, J_y, \phi_y; s) = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + V(J_x, \phi_x, J_y, \phi_y; s)$$

“Simple” Nonlinearity: A Single Sextupole

$$H(x,x',y,y';s) = \frac{x'^2}{2} + K_x(s)\frac{x^2}{2} + \frac{y'^2}{2} + K_y(s)\frac{y^2}{2} + \frac{b_2}{6}(x^3 - 3xy^2)$$

$$\text{one dimensional (y} = 0) \quad H(x,x';s) = \frac{x'^2}{2} + K_x(s)\frac{x^2}{2} + \frac{b_2}{6}x^3$$

- The extra term will add dependencies on ϕ so J is no longer a constant of the motion: **phase space distortion**
- Using $x = \sqrt{2J_x\beta_x}\cos\phi_x$ we can expand the nonlinear term

$$\text{nonlinear term} \quad V = \frac{b_2}{6}x^3 = \frac{b_2}{6}(2J_x\beta_x)^{3/2}\cos^3\phi_x$$

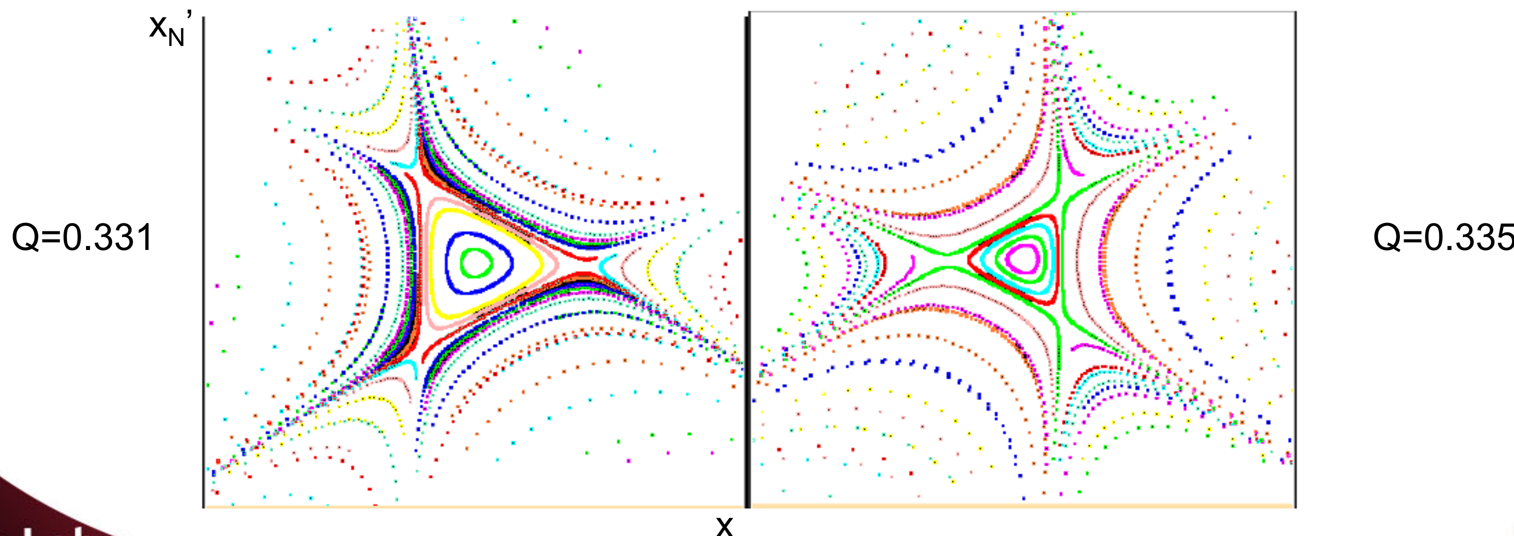
$$\cos^3\phi = \frac{1}{4}(3\cos\phi + \cos 3\phi)$$

$$\text{nonlinear term} \quad V = \frac{b_2}{6}x^3 = \frac{b_2\sqrt{2}}{12}(J_x\beta_x)^{3/2}(3\cos\phi_x + \cos 3\phi_x)$$

- Betatron phase dependences drive Q_x=k, $3Q_x$=k resonances

Sextupole $3Q_x$=k Resonance

- For tunes close to $3Q_x$=k, we can solve for the motion
  - Often just iterate Hamilton's equations to plot them
  - There end up being three fixed points of this "map" at three betatron phases separated by $2\pi/3$ (120 degrees)
  - These three fixed points also end up being locally unstable
    - Nearby motion is hyperbolic rather than elliptical
  - Area of stable particle motion is distorted and reduced
    - So don't operate near $3Q_x$=k with strong sextupoles!



Putting It All Together

- Some thoughts from 20 years in accelerators and seeing several new accelerator projects start up

- Organize
  - Users, physicists, engineers, funding groups/agencies
- Discuss
  - What sort of facility is best for needs? Initial specifications
- Write
  - Iterate more and more details of facility design, engineering
  - Eventually becomes a “conceptual design” report
  - Balance risk, feasibility, cost, benefits
- Sell
  - Pitch the project to funding agencies

Organize

- You have an excellent start towards organizing a team
- Users are your customers!
  - Their needs must be met to politically support your project
  - Some must be “talked back down to Earth” – or talked up!
  - Identifying science benefits for facility organizes priorities
  - They primarily provide the **goals** of the facility
- Scientists and engineers
  - You have an excellent start here, particularly many students!
  - Have them travel and learn from other experts, facilities
  - Interact with users to build a strong community, communication
  - Primarily provide the **technical constraints** of the facility
- Funding agencies
  - **Sometimes** good to have them involved from early stages
  - They can provide important early **resource constraints**

Discuss

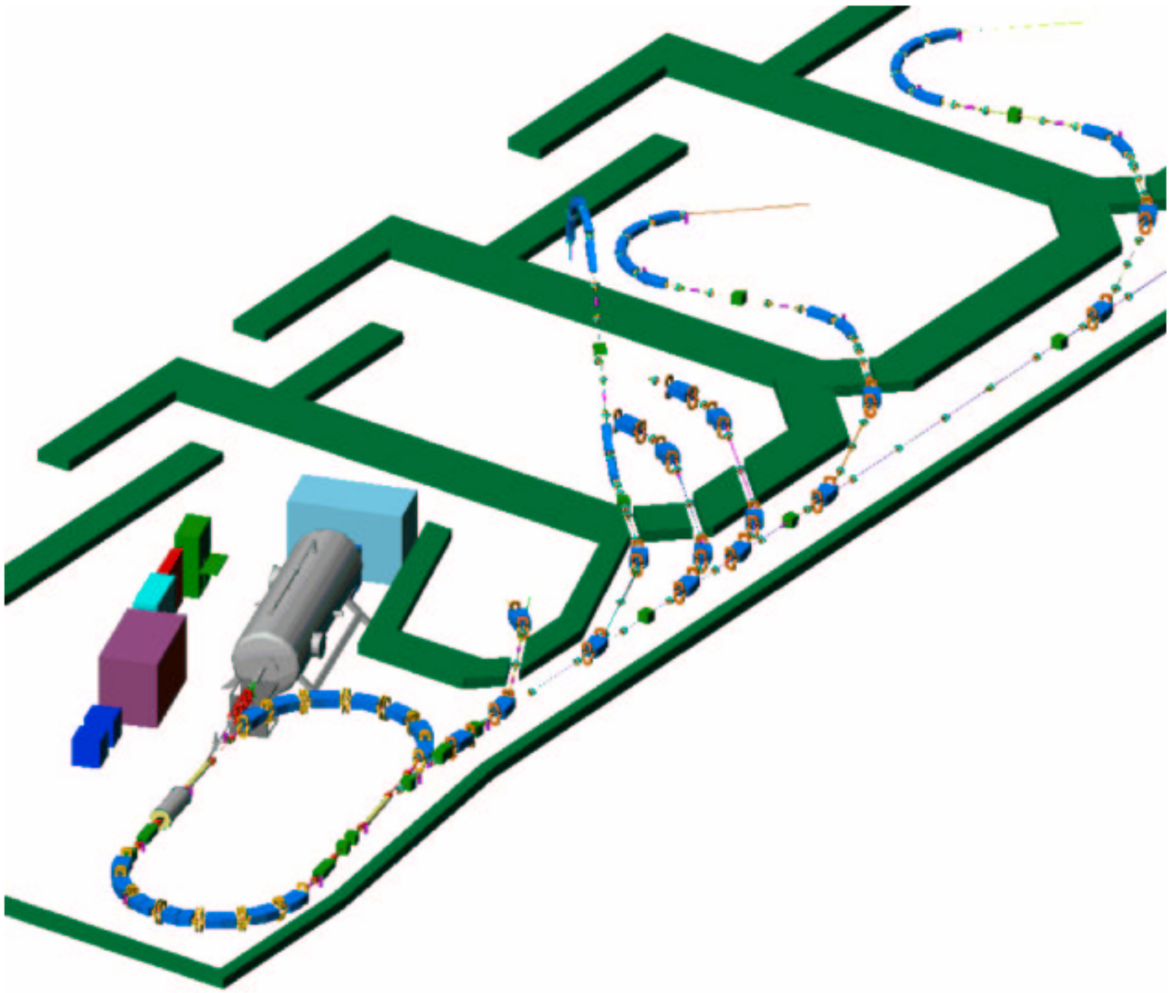
- Iterate ideas among all groups
  - Negotiations provide a framework for communication, trust
  - Developing teamwork
  - Writing justification document for facility need
- Small group teams naturally develop
  - Focus on one subset of problem or particular area of challenge, technical risk
  - Common areas of expertise and interest
  - Teams remain in place to contribute to conceptual design
- Strong central management
  - Integrates input from teams, guides overall vision
  - Stays focused on end goal, but flexible in face of change
  - Makes critical decisions to establish baseline parameters

Write

- Conceptual Design Report (CDR)
  - An assembly of physics and engineering specifications of the facility
  - Not necessarily “build to print” or “build to spec” yet
  - No detailed costing, site specifics
- The CDR should clearly identify
  - Physics goals and user community of the facility
  - All parameters necessary to meet those goals
  - How those parameters are technically produced
    - Identify areas of risk and/or required R&D
    - Identify areas of negligible risk and off-shelf availability
  - Breadth of subsystem physics/engineering design
    - Magnets, lattice, RF, injector, vacuum, cryogenics, …
    - Desired beam properties, brilliance, number of beamlines…

Medical Accelerator CDR Cover

Conceptual Design of the RCMS



Pitch

- The CDR should convince other people "we know how to build this"
  - Including identifying risks and benefits of R&D
  - CDR serves as a central focus of team efforts to remain consistent across the project: a good management tool

- CDR can also convince others "we know why we want to build this"
  - Sometimes a physics/benefit justification section is included
  - But this is often a separate document used to justify and mobilize the resources necessary to produce the CDR