Physics 696
Topics in Advanced Accelerator Design I
Monday, September 10 2012

E&M: Plane Waves, Waveguides, Boundary Absorption

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Happy Birthday to Arthur Compton (1927 Nobel), Joey Votto, Colin Firth, and the LHC!
Happy Swap Ideas Day, International Make-Up Day, and Cheap Advice Day!

Maxwell's Equations

Electric charge density

$$\vec{\nabla} \cdot \vec{D} = \rho$$ Electric charge generates electric displacement

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law

$$\vec{\nabla} \cdot \vec{B} = 0$$ No magnetic charges

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ Magnetic field generated by real or displacement current density

Electric current density

Maxwell's Equations are linear in the source terms ρ and $\vec{J}$. In general will generate linear (partial) differential equations to solve. Superposition valid!

Constitutive Relations

$$\boxed{\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}}{\mathrm{N \cdot m^2}}$$

$\vec{D}$: Electric displacement field

$\vec{E}$: Electric field

ε : Permittivity

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\boxed{\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}}$$

$$\mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{N \cdot s^2}}{\mathrm{C^2}}$$

$\vec{B}$: Magnetic field

$\vec{H}$: Magnetizing field

ε : Permeability

Source-Free Maxwell Equations

- When there are no sources ($\vec{J} = 0, \rho = 0$) then Maxwell's equations become much more symmetric:

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Harmonic Dependence of Fields

- The linearity of the first order time derivatives suggests that we can assume a harmonic time dependence of our magnetic and electric fields

$$\vec{B}(\vec{x},t) = \vec{B}_0(\vec{x})\,\mathrm{e}^{-i\omega t}$$
$$\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x})\,\mathrm{e}^{-i\omega t}$$

This gives the spatial Maxwell Equations:

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$$
$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} + i\omega\mu\epsilon\vec{E} = 0$$

Plane Waves

- Taking the curl of each curl equation and using the identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

then gives us two **identical** spatial wave equations with straightforward solutions:

$$[\nabla^2 + \mu\epsilon\omega^2]\vec{E} = 0 \quad \Rightarrow \quad \boxed{\vec{E}(\vec{x},t) = \vec{E}_0 \, \mathrm{e}^{i\vec{k}\cdot\vec{x} - i\omega t}}$$

$$[\nabla^2 + \mu\epsilon\omega^2]\vec{B} = 0 \quad \Rightarrow \quad \boxed{\vec{B}(\vec{x},t) = \vec{B}_0 \, \mathrm{e}^{i\vec{k}\cdot\vec{x} - i\omega t}}$$

These are plane waves traveling in direction $\vec{k}$

Plane Wave Properties

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

The source-free divergence equations imply

$$\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$$

so the fields are both transverse to the direction $\vec{k}$

Faraday's law also implies that both fields are spatially transverse to each other

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{n\hat{n}\times\vec{E}_0}{c} \qquad \hat{n}\equiv\frac{\vec{k}}{k}$$

Here n is the index of refraction.

Standing Waves

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} \\ \vec{B}(\vec{x},t) &= \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}\end{aligned}}$$

Note that Maxwell's equations are linear, so any linear combination of magnetic/electric fields is also a solution.

Thus a **standing wave** solution is also acceptable, where there are two plane waves moving in opposite directions:

$$\vec{E}(\vec{x},t) = \vec{E}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{-i\vec{k}\cdot\vec{x}-i\omega t}\right)$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{-i\vec{k}\cdot\vec{x}-i\omega t}\right)$$

Polarization

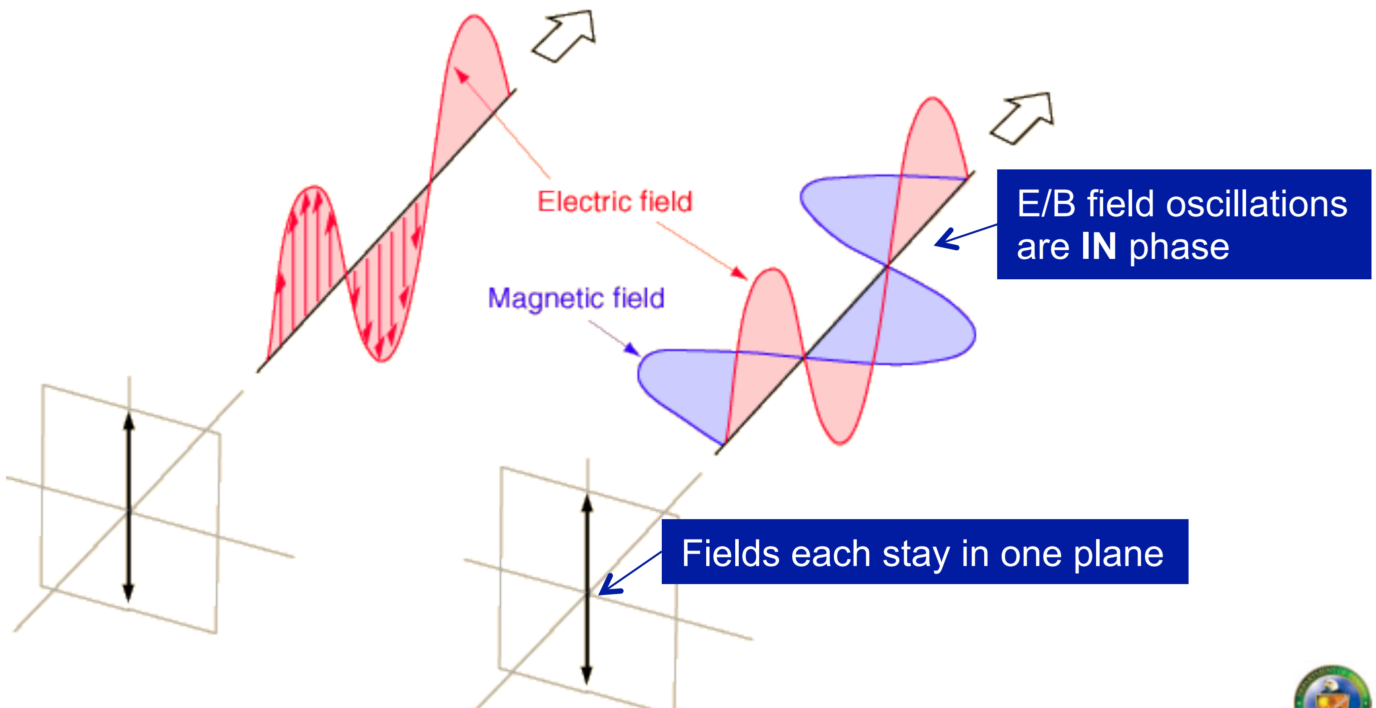
$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

- As long as the $\vec{E}$ and $\vec{B}$ fields are transverse, they can still have different transverse components.
  - So our description of these fields is also incomplete until we specify the transverse coordinates at all locations in space
  - This is equivalent to an uncertainty in phase of rotation of $\vec{E}$ and $\vec{B}$ around the wave vector $\vec{k}$.
  - The identification of this transverse field coordinate basis defines the **polarization** of the field.

Linear Polarization

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

- So, for example, if $\vec{E}$ and $\vec{B}$ transverse directions are constant and do not change through the plane wave, the wave is said to be **linearly polarized**.

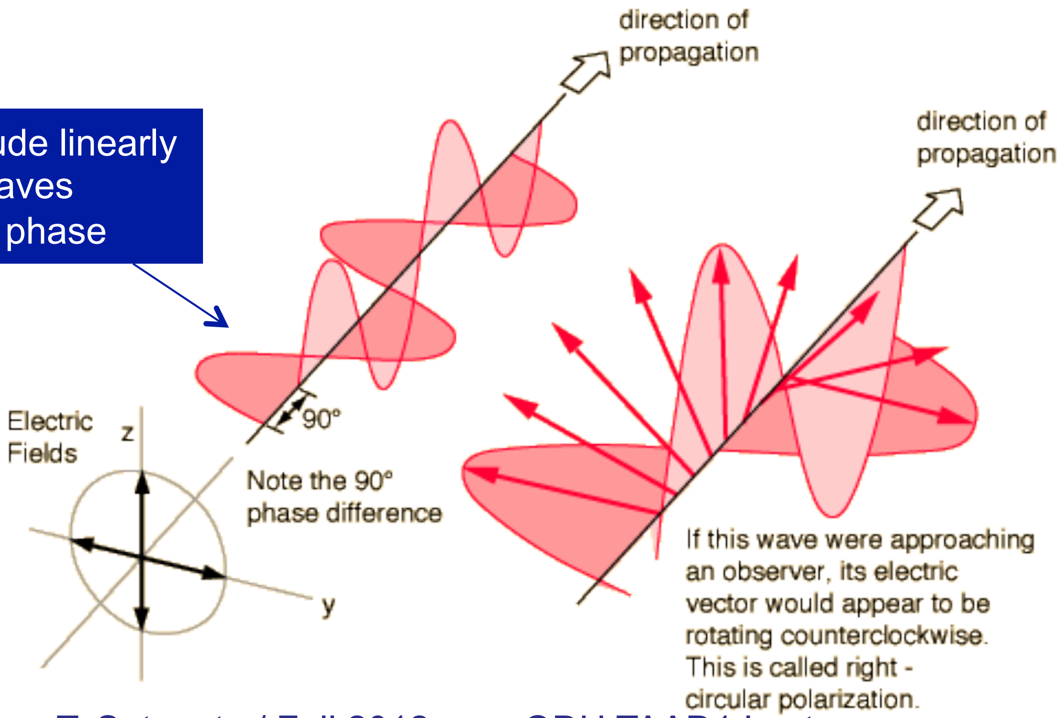


Circular Polarization

$$\vec{E}(\vec{x},t) = \vec{E}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t+\pi/2} \right)$$
$$\vec{B}(\vec{x},t) = \vec{B}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t+\pi/2} \right)$$

- If $\vec{E}$ and $\vec{B}$ transverse directions vary with time, they can appear as two plane waves traveling out of phase. This phase difference is 90 degrees for **circular polarization**.

Two equal-amplitude linearly polarized plane waves 90 degrees out of phase



Back to Maxwell, Now in Accelerators

- Not all space is source-free
  - We'll have to deal with boundary conditions
  - We'll even have to deal with complicated (linear) combinations of our plane waves in source-free space
- Accelerators have privileged directions
  - E.g. the design momentum direction of the beam
  - This is not always the same as our k direction!
  - So let's break down Maxwell's equations again, now into transverse and longitudinal (z) components

Transverse Separation of Coordinates

- Here $\hat{z}$ labels the direction along the beam, and t labels transverse coordinates perpendicular to $\hat{z}$

$$\vec{E} = E_z \hat{z} + \vec{E}_t \qquad \vec{E}_t = \vec{E} - \left(\vec{E}\cdot\hat{z}\right)\hat{z}$$

$$\vec{B} = B_z \hat{z} + \vec{B}_t \qquad \vec{B}_t = \vec{B} - \left(\vec{B}\cdot\hat{z}\right)\hat{z}$$

$$\left[\nabla_t^2 + \left(\mu\varepsilon\omega^2\right) - k^2\right]\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

$$\frac{\partial \vec{E}_t}{\partial z} + i\omega\,\hat{z}\times\vec{B}_t = \vec{\nabla}_t E_z \qquad \hat{z}\cdot\left(\vec{\nabla}_t\times\vec{E}_t\right) = i\omega B_z$$

$$\frac{\partial \vec{B}_t}{\partial z} - i\mu\varepsilon\omega\,\hat{z}\times\vec{E}_t = \vec{\nabla}_t B_z \qquad \hat{z}\cdot\left(\vec{\nabla}_t\times\vec{B}_t\right) = -i\mu\varepsilon\omega E_z$$

$$\vec{\nabla}_t\cdot\vec{E}_t = -\frac{\partial E_z}{\partial z} \qquad \vec{\nabla}_t\cdot\vec{B}_t = -\frac{\partial B_z}{\partial z}$$

Transverse Separation of Coordinates II

- We can even write down the transverse fields **completely** in terms of the longitudinal fields

$$\vec{E}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z\right]$$

$$\vec{B}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t B_z - \mu\varepsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right]$$

Notice how beautifully symmetric these equations are.

This is a helpful decomposition of plane wave solutions to Maxwell's equations into one longitudinal direction and two transverse directions.

Boundary Conditions: TM and TE Modes

$$\vec{E}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z\right]$$

$$\vec{B}_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\vec{\nabla}_t B_z - \mu\varepsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right]$$

- We can come up with some boundary conditions for a couple of simple cases where either electric (TE) or magnetic (TM) fields are completely transverse to $\hat{z}$

Transverse Magnetic (TM)

$$B_z = 0; \qquad \text{Boundary Condition } \left. E_z\right|_S = 0$$

Transverse Electric (TE)

$$E_z = 0; \qquad \text{Boundary Condition } \left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

Waveguides

- Consider the relationship between the transverse field components

$$\vec{H}_t = \frac{\pm 1}{Z}\hat{z}\times\vec{E}_t$$

Impedance Z

Wave Impedance

$$Z = \begin{cases} \dfrac{k}{\varepsilon\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\varepsilon}} & \text{(TM)} \quad B_z = 0 \\ \dfrac{\mu\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\varepsilon}} & \text{(TE)} \quad E_z = 0 \end{cases}$$

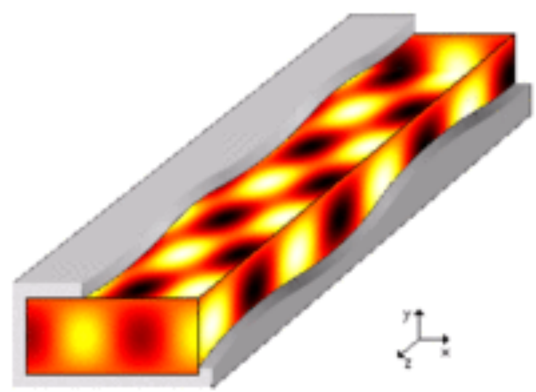
Waveguides

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

- Plane waves carry electromagnetic energy in the $\vec{k}$ direction and are periodic in all three spatial dimensions
  - We can use that spatial periodicity in transverse boundary conditions, such as at a conductor wall.
  - (Near-)perfect conductors allow us to match transverse boundary conditions

This gives a plane wave propogating longitudinally in a finite transverse volume



Transverse Fields as an Eigenvalue Problem

TM Waves

$$\vec{E}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi \qquad \gamma \equiv (\mu\varepsilon\omega^2 - k^2) \quad \psi \equiv E_z$$

TE Waves

$$\vec{H}_t = \pm \frac{ik}{\gamma^2} \vec{\nabla}_t \psi \qquad \gamma \equiv (\mu\varepsilon\omega^2 - k^2) \quad \psi \equiv B_z$$

Transverse Helmholz Equation

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi = 0$$

Differential equation with boundary conditions on E_z or B_z

Boundary Conditions

- The boundary conditions are different for TM and TE

$$(\psi \equiv E_z)|_S = 0 \quad (\mathrm{TM}) \qquad \left(\frac{\partial(\psi \equiv B_z)}{\partial n}\right)|_S = 0 \quad (\mathrm{TE})$$

- For many plane waves of various k values, there are a spectrum of eigenvalues γ_λ and eigenfunctions $\psi_\lambda(x,y)$ that solve the Helmholz equation

$$\left(\nabla_t^2 + \gamma_\lambda^2\right)\psi_\lambda(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma_\lambda\right)\psi_\lambda(x,y) = 0$$

Here the wave number $\vec{k}_\lambda$ of a particular mode λ is

$$k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$$

Cutoff Frequency

$$k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$$

- What happens when a mode has a frequency such that the right hand side above is negative?
  - Then k_λ is imaginary and the mode does not propagate
  - This property is called **cutoff**
  - This happens for frequencies **below the cutoff frequency**

$$\omega_\lambda^2 \equiv \frac{\gamma_\lambda^2}{\mu\epsilon} \qquad k_\lambda^2 = \mu\epsilon(\omega^2 - \omega_\lambda^2)$$

- If you want to propogate a single frequency mode (best for narrow-band response)
  - It's best to build your waveguide so its frequency is below cutoff for the fundamental mode and below cutoff for all other modes