University Physics 226N/231N
Old Dominion University

Vectors and Motion in Two Dimensions

First "Midterm" is Wednesday, September 19!

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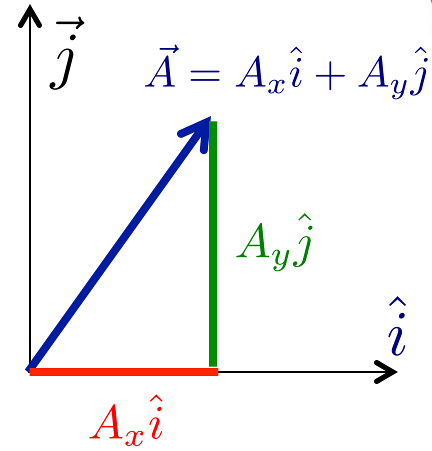
Happy Birthday to Arthur Compton (1927 Nobel), Joey Votto, Colin Firth, and the LHC!

Happy Swap Ideas Day, International Make-Up Day, and Cheap Advice Day!

Vector Arithmetic with Components

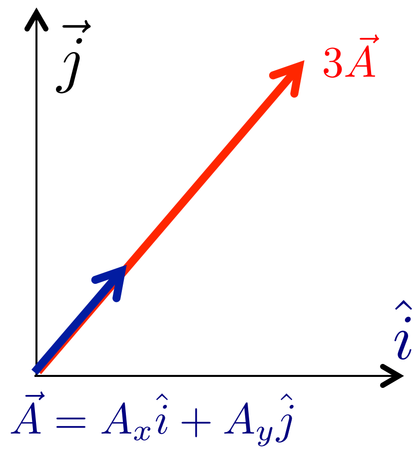
- To add vectors, add the individual components:
  - If $\vec{A} = A_x\hat{i} + A_y\hat{j}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j}$

$$\boxed{\begin{aligned}\vec{A} + \vec{B} &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \\ \vec{A} - \vec{B} &= (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}\end{aligned}}$$



- To multiply a vector $\vec{A}$ by a scaler (constant) c:

$$\boxed{c\vec{A} = (cA_x)\vec{i} + (cA_y)\vec{j}}$$



  - We can also multiply vectors in a variety of ways, but that's beyond the scope of this course…
    - And thankfully we don't need that for the physics we're learning!

Adding Vectors Graphically

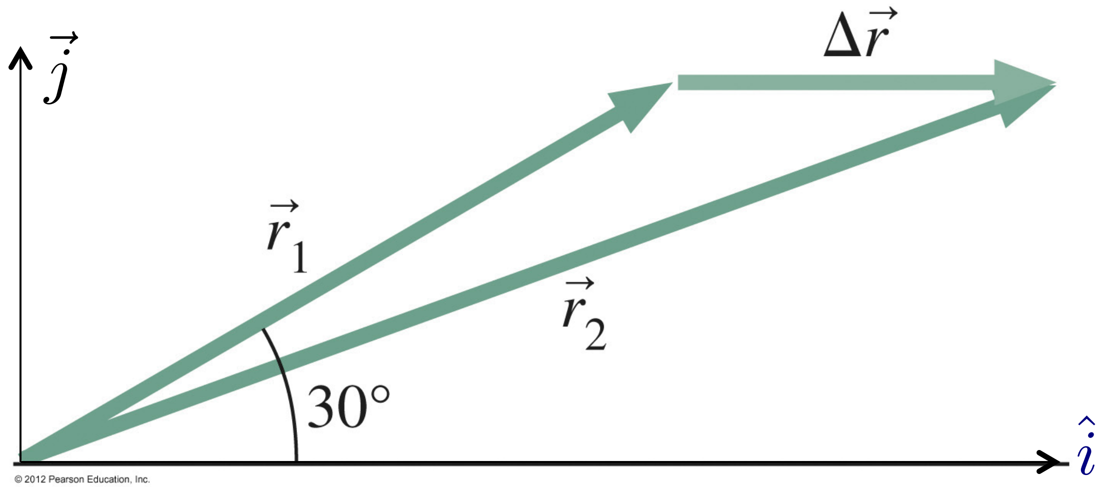
- To add vectors graphically, place the tail of the first vector at the head of the second.
  - Their sum is then the vector from the tail of the first vector to the head of the second.

– Here $\vec{r}_2$ is the sum of $\vec{r}_1$ and $\Delta\vec{r}$.

$$\boxed{\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}}$$

$$\vec{r}_2 = r_{2x}\hat{i} + r_{2y}\hat{j}$$

$$\boxed{\begin{aligned} r_{2x} &= r_{1x} + \Delta r_x \\ r_{2y} &= r_{1y} + \Delta r_y \end{aligned}}$$

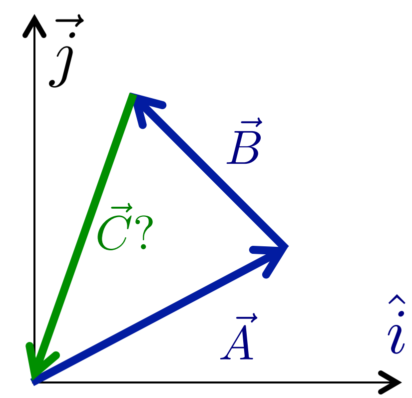


Example: Tricity Flight Pattern

- A plane flies 30° north of east for 300 miles, then 200 miles 45° north of west. How far and in what direction does it need to travel to return to its original destination?

Here let's **define** east as $\hat{\imath}$ and north as $\vec{\jmath}$

If we call our vectors $\vec{A}, \vec{B}$ and $\vec{C}$ then we have



$$\vec{A} + \vec{B} + \vec{C} = 0 \quad \Rightarrow \quad \vec{C} = -\vec{A} - \vec{B}$$

$$A_x = (300 \text{ miles}) \cos(30^\circ) = 259.8 \text{ miles}$$

$$A_y = (300 \text{ miles}) \sin(30^\circ) = 150.0 \text{ miles}$$

$$B_x = (200 \text{ miles}) \cos(135^\circ) = -141.4 \text{ miles}$$

$$B_y = (200 \text{ miles}) \sin(135^\circ) = 141.4 \text{ miles}$$

$$\vec{C} = (-A_x - B_x)\hat{\imath} + (-A_y - B_y)\hat{\jmath} = (-118.4 \text{ miles})\hat{\imath} + (-291.4 \text{ miles})\hat{\jmath}$$

$$|C| = \sqrt{(-118.4 \text{ miles})^2 + (-291.4 \text{ miles})^2} = \boxed{314.5 \text{ miles} = |C|}$$

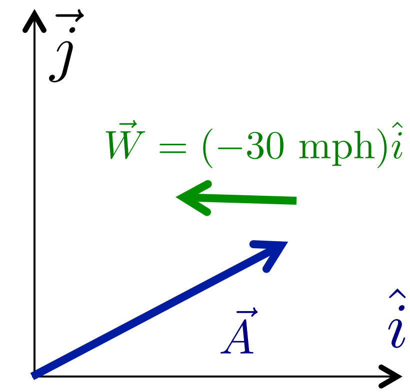
$$\text{angle} = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{291.4}{118.4}\right) = \boxed{67.9 \text{ degrees south of west} = \text{angle}}$$

Position, Velocity, and Acceleration Vectors

- You're now familiar with several **vector** quantities
  - Position, velocity, acceleration
    - Force and momentum, introduced soon, are also vectors (with new units)
    - All vectors have magnitude and direction
- You're also familiar with several **scalar** quantities
  - Distance, length, speed, angles, "just plain individual numbers"
    - Energy, work, and power, introduced soon, are also scalars (also with new units)
  - The components of vectors are scalars
- It is meaningless and nonsensical to
  - Add any scalar to any vector (but they can be multiplied!)
  - Add any scalars or vectors with different types of units
    - What is 1 meter + 1 second? (It's meaningless)
  - But we can add velocities together just like we added positions in the previous example

Example: Windy Tricity Flight Pattern

- Our Tricity pilot wants to fly 30° north of east for 300 miles but there is a constant wind blowing east to west at 25 mph.
  - If the plane's speed on a calm day is 300 mph, what direction must our pilot point his plane to fly to her destination?
  - How much longer does it take our pilot to get to her destination?



Here we want our **final** velocity to point in the $\vec{A}$ direction.

Call our windy day flight velocity $\vec{V}$ and wind velocity $\vec{W}$

$$\vec{A} = \vec{V} + \vec{W} \quad \Rightarrow \quad \vec{V} = \vec{A} - \vec{W}$$

$$\vec{V} = [(300\ \text{mph})\cos(30^\circ) - (-30\ \text{mph})]\hat{i} + [(300\ \text{mph}\sin(30^\circ)]\hat{j}$$

$$\vec{V} = (290\ \text{mph})\hat{i} + (150\ \text{mph})\hat{j}$$

$$\tan(\theta_V) = \frac{(150\ \text{mph})}{(290\ \text{mph})} = 0.51 \quad \Rightarrow \quad \boxed{\theta_V = 27.35^\circ \text{ north of east}}$$

$$|V| = \sqrt{V_x^2 + V_y^2} = 326\ \text{mph} \qquad t = (1\ \text{hour})\frac{(326\ \text{mph})}{(300\ \text{mph})} = 1.09\ \text{hours}$$

$$\boxed{\text{The trip takes 0.09 hours or about 5 minutes longer}}$$

Ponderable: Crossing the James River (10 min)

- The James River near my house is 4 miles wide, and its current is 3 mph. I can row a kayak at 5 mph in still water.
  - What direction do I need to row in to end up at a point straight across the river from where I started?
  - How much longer does it take me to get across the river?
  - If I'm tired and can only row 3 mph, can I still get across the river straight to the opposite point? Can I still get across the river at all?

river current 3 mph

direction I want to go

Ponderable: Crossing the James River (Solution)

- The James River near my house is 4 miles wide, and its current is 3 mph. I can row a kayak at 5 mph in still water.
  - What direction do I need to row in to end up at a point straight across the river from where I started?
  - How much longer does it take me to get across the river?
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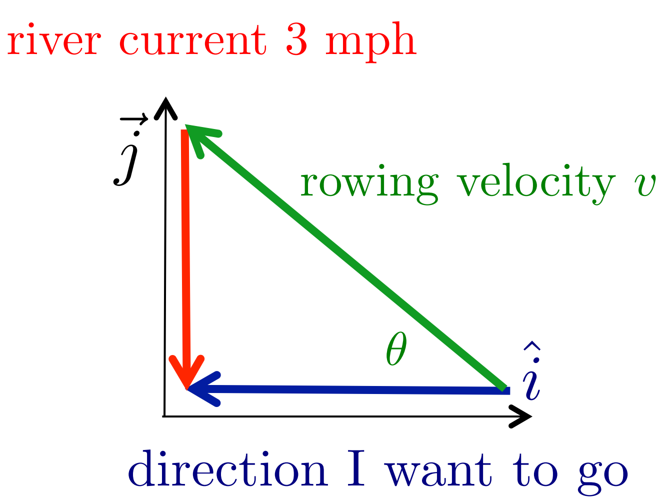


The $\vec{j}$ component of my rowing velocity must cancel out the push of the river current for me to have a net velocity of straight across

$$0 \text{ mph (net } \hat{\text{j}} \text{ velocity)} = v_y - (3 \text{ mph})$$

$$v_y = v \sin\theta = (3 \text{ mph})$$

$$\sin\theta = \frac{(3 \text{ mph})}{(5 \text{ mph})} \quad \Rightarrow \quad \boxed{\theta = 37^\circ}$$

$$t(\text{no current}) = \frac{4 \text{ miles}}{5 \text{ mph}} = 0.8 \text{ hours}$$

$$t(\text{current}) = \frac{4 \text{ miles}}{(5 \text{ mph})\cos\theta} = 1 \text{ hour}$$

Two Dimensional Motion With Gravity

- Now we add acceleration in one direction
  - Gravity influencing projectile motion is one classic example
  - We can decompose the motion into individual directions

$$\left.\begin{aligned} v_x &= v_{x0} \\ x &= x_0 + v_{x0}t \end{aligned}\right\}$$

Horizontal motion

(no acceleration)

$$\left.\begin{aligned} v_y &= v_{y0} + at \\ y &= y_0 + v_{y0}t + \frac{1}{2}at^2 \end{aligned}\right\}$$

Vertical motion

(gravitational acceleration,

$a = g = -9.8$ m/s^2)

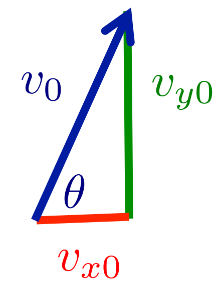
  - We've already described motion in both of these cases
  - So you already know the "hard parts" of projectile motion
    - And with that, all motion with constant acceleration in each direction
  - We have a lot more unknowns now:

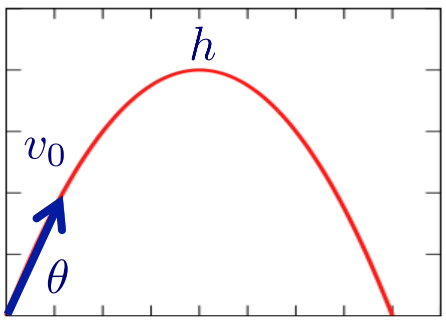
$$v_x, v_{x0}, (x - x_0) \qquad v_y, v_{y0}, (y - y_0) \qquad t$$

Example: Projectile Motion (Football Season)

- A quarterback throws a football at 45 mph. What angle does he need to throw it to hit a receiver 30 yards (90 feet) downfield?
  (Neglect air resistance and assume a level field. 45 mph is 66 feet/sec.)

- Call the angle θ - then we have $v_{x0} = v_0 \cos\theta$ and $v_{y0} = v_0 \sin\theta$
- We can employ a trick: from **symmetry**, $v_y = -v_{y0}$





$$v_y = v_{y0} + at \quad \Rightarrow \quad t = -\frac{2v_{y0}}{a}$$

$$x - x_0 = v_{0x} t = \left(-\frac{2v_{y0}}{a}\right) v_{x0} = -\frac{2v_0^2}{a} \cos\theta \sin\theta$$

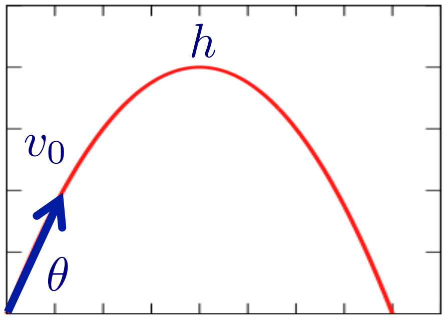
$$\text{Trigonometric identity}: \quad \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$$

$$\sin(2\theta) = -\frac{a(x-x_0)}{v_0^2} = -\frac{(-32 \text{ feet/s}^2)(90 \text{ feet})}{(66 \text{ feet/s})^2} = 0.66$$

$$\boxed{\theta = 20.7^\circ}$$

Projectile Motion: Observations and Animation

- For a projectile launched from ground level to ground level, and using $v_y = -v_{y0}$, we can **derive** some interesting results



$$x - x_0 \text{ (range)} = -\frac{v_0^2}{a}\sin(2\theta)$$

Max at 45 degrees

$$h \text{ (max height)} = -\frac{v_0^2 \sin^2\theta}{2a}$$

Max at 90 degrees

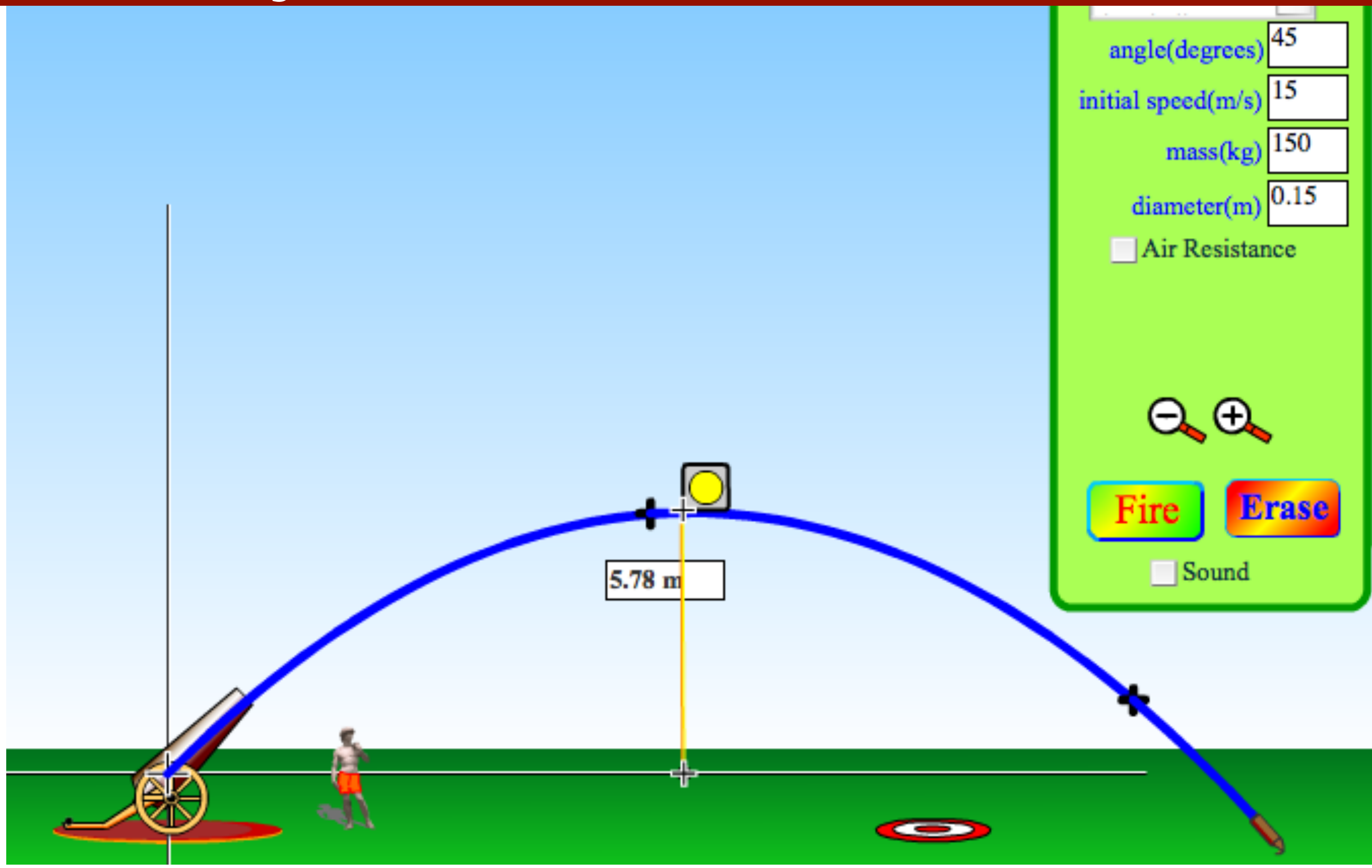
$$\frac{h}{x - x_0} = \frac{\sin^2\theta}{\sin(2\theta)} = \frac{1}{4}\tan\theta \quad \text{depends only on } \theta!$$

At a launch angle of 45 degrees to maximize the range of our projectile, the range $x - x_0 = -v_0^2/a$ is four times the achieved height.

We can experiment with this (and other projectile questions) with an applet located at

http://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html

Projectile Motion: Confirmation

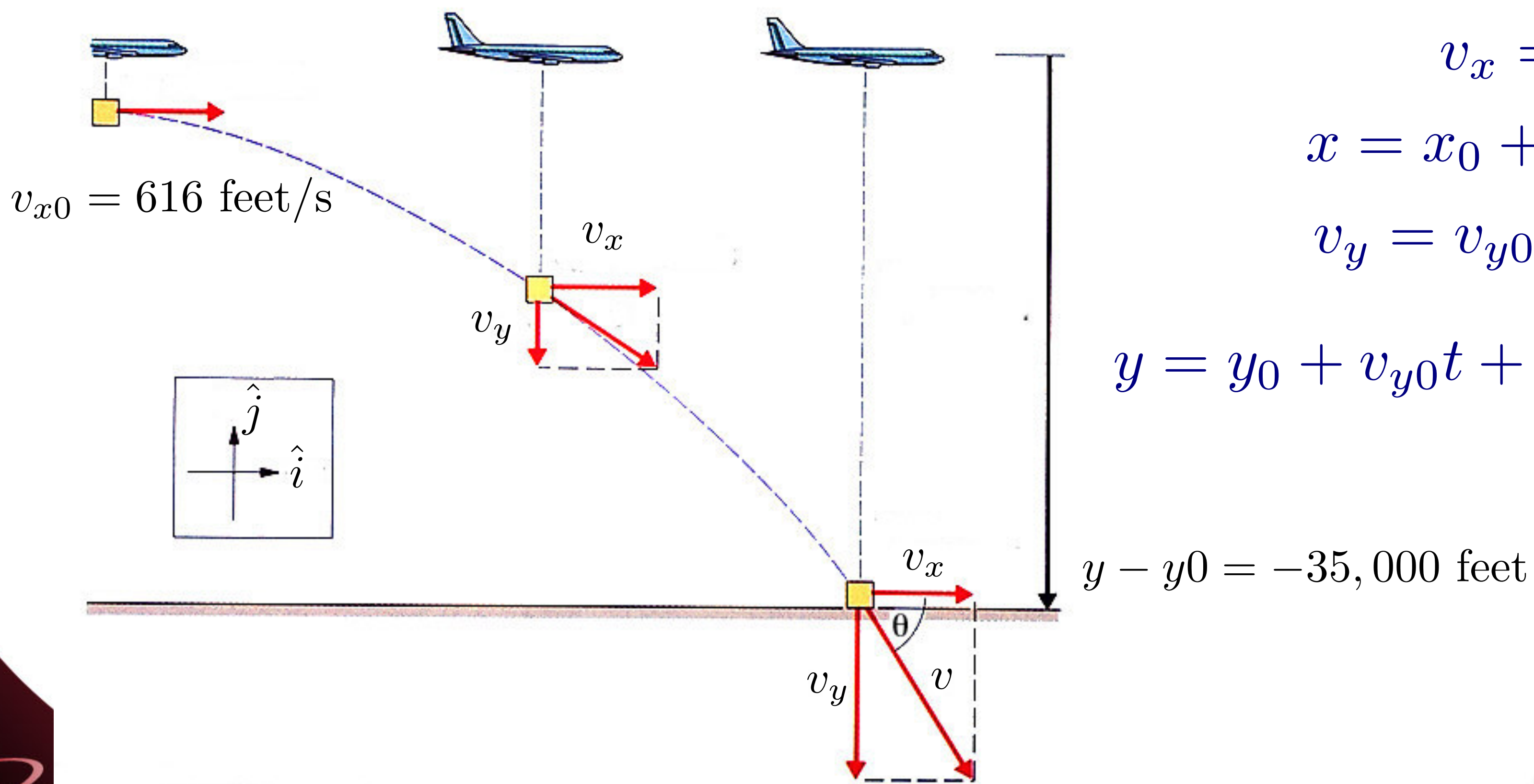


- Shoot projectile at 15 m/s at 45 degree angle
  - Height achieved is h=5.78 m, range achieved is $x-x_0$=23 m
  - Very close to $x - x_0 = 4h$ and

$$x - x_0 = -v_0^2/a = -(15\ \mathrm{m/s})^2/(9.8\ \mathrm{m/s^2}) = 22.96\ \mathrm{m}$$

Ponderable: Bomb Drop (10 minutes)

- A bomber is flying 420 mph (616 feet/s) at 35,000 feet altitude towards a ground-level target. a=g=-32 ft/s^2
  - How far before the target should the bomber drop the bomb?
  - Where is the bomber relative to the explosion when the bomb hits?



$$v_x = v_{x0}$$

$$x = x_0 + v_{x0}t$$

$$v_y = v_{y0} + at$$

$$y = y_0 + v_{y0}t + \frac{1}{2}at^2$$

Ponderable: Bomb Drop (Solution)

- A bomber is flying 420 mph (616 feet/s) at 35,000 feet altitude towards a ground-level target.
  - How far before the target should the bomber drop the bomb?
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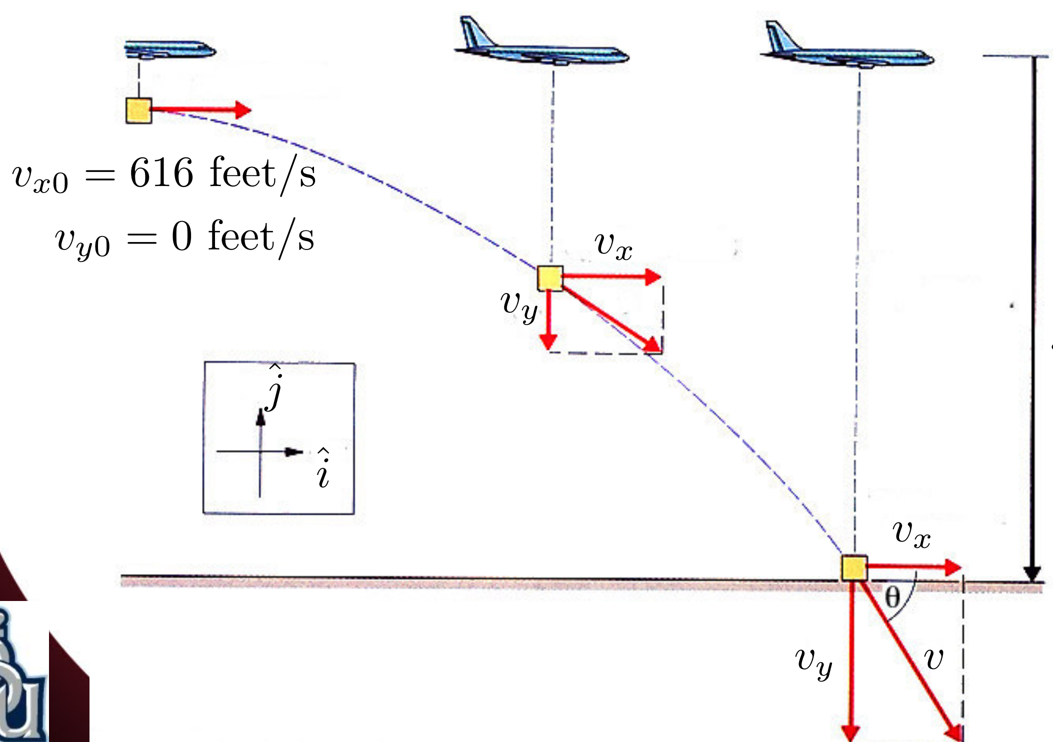


$$y - y_0 = \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2(y - y_0)}{a}} = 46.8 \text{ s}$$

$$x - x_0 = v_{x0}t = (616 \text{ feet/s})(46.8 \text{ s})$$
$$= \boxed{5.45 \text{ miles}}$$

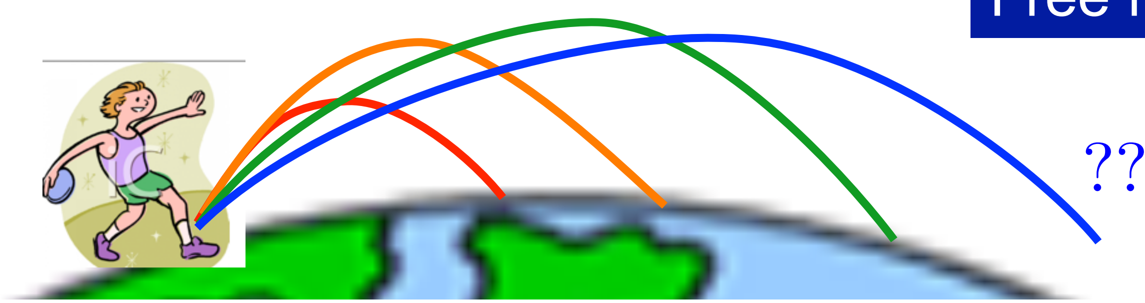
The bomber is directly above the explosion unless it turns or takes evasive maneuvers!

Ponderable: Falling Around The Earth

- We're talking a lot about projectile motion over level ground
  - The Earth is not flat: it is not "level ground" forever
  - Experience tells us that gravity always points towards the center of the Earth, wherever I am on the Earth

  - So it might be possible to shoot something really small and aerodynamic fast enough to "miss" the Earth even while the acceleration of gravity continuously makes it "fall"

Free fall



Ponderable: Falling Around The Earth (10 minutes)

- Estimate (or guesstimate) how fast I have to shoot a projectile **horizontally** for it to circle around a perfectly spherical Earth and come back to me.

All you need are:

$$g = -9.8 \text{ m/s}^2$$
$$r_e \approx 6.37 \times 10^6 \text{ m}$$

Hint: The "drop" in a horizontal distance d is

$$\Delta y \approx -\frac{d^2}{r_e}$$

Ponderable: Falling Around The Earth (Solution)

- Estimate (or guesstimate) how fast I have to shoot a projectile **horizontally** for it to circle around a perfectly spherical Earth and come back to me.

$$\Delta y \approx -\frac{d^2}{r_e}$$

Over a small segment of time Δt,vertical velocity is

$$v_y = \frac{\Delta y}{\Delta t} = -\frac{d^2}{r_e \Delta t}$$

and vertical acceleration is

$$a_y = g = \frac{\Delta v}{\Delta t} = -\frac{d^2}{r_e \Delta t^2}$$

Ponderable: Falling Around The Earth (Solution)

- Estimate (or guesstimate) how fast I have to shoot a projectile **horizontally** for it to circle around a perfectly spherical Earth and come back to me.

"Horizontally" we're not accelerating, so horizontal velocity is distance/time:

$$v_x = \frac{d}{\Delta t} \quad \Rightarrow \quad d = v_x \Delta t$$

So

$$g = -\frac{v_x^2}{r_e}$$

$$v_x = \sqrt{-g r_e} = \boxed{7.9\ \mathrm{km/s} = v_x}$$

Over 17000 miles/hour!

Circular Motion and Centripetal Acceleration

- Notice that nothing here depended on the height!
- In fact, nothing depended on "gravity" except the constant acceleration towards the center of the circle
- It turns out that is a pretty general result for circular motion
  - It's not easy to derive, so it's kind of a fundamental equation
  - To keep an object moving with velocity v in a circle of radius r, we need a **constant acceleration towards the center of the circle of**

$$\boxed{a_{\text{centrip}} = \frac{v^2}{r}}$$

  - This is usually called "centripetal acceleration"
  - We'll derive the "falling distance" equation I gave you and start covering circular motion next time
    - Radii and arc lengths, angles in radians, angular velocity…

Example: Olympic Archery Accuracy

- 90m to target, 122cm target
- Each ring is 6.1cm diameter