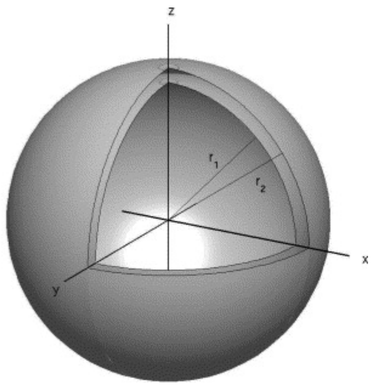

University Physics 226N/231N
Old Dominion University
Systems of Particles: Center of Mass

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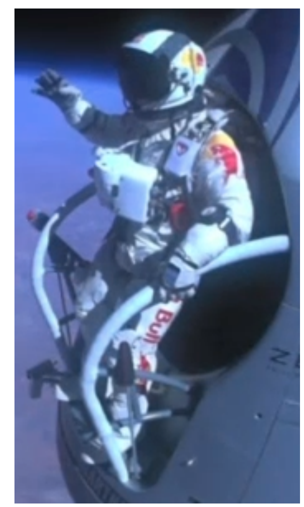
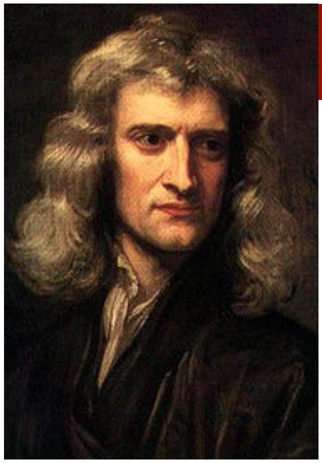
Monday, October 15 2012

Happy Birthday to Virgil, Emeril Lagasse, Michel Foucault, and P.G. Wodehouse!

Happy Bearded Presidents Day, Grouch Day, and Blind Americans Equality Day!

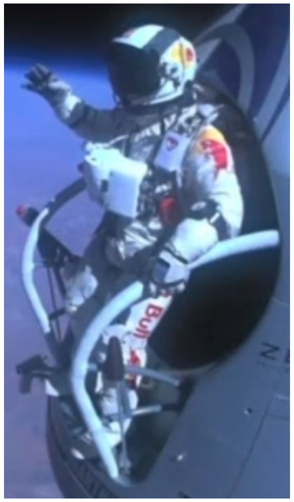


A Law of Universal Gravitation



$$\boxed{F_{\mathrm{g}} = \frac{G m_1 m_2}{r^2}}$$

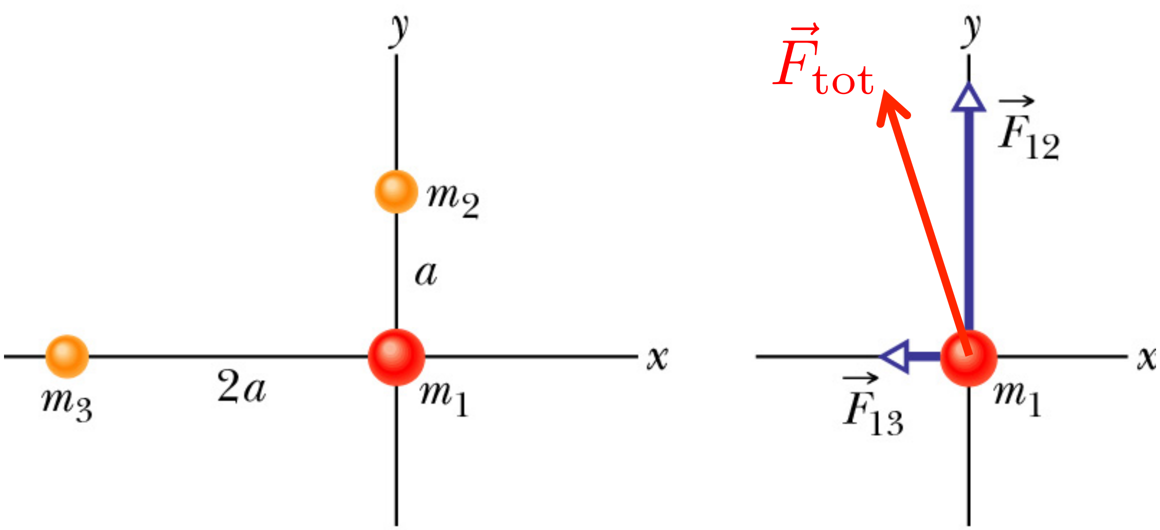
$$G = 6.67384 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$$



- This formula for the general gravitational force was developed by Newton (1687)
  - It's a much much better approximation, but again only an approximation!
  - Superceded by Einstein's general theory of relativity (1916)
  - But this equation works well enough to pretty much all everyday phenomena
- Technically for point particles, but works perfectly well for spheres where centers are distance r apart
  - Gravity is always **attractive** along the vector $\vec{r}$

Adding Forces Together (including Gravity)

- To calculate the total gravitational force from many objects, we add together the vectors of their individual forces
  - This is known as the **principle of superposition**
  - This is (mostly) a general rule for vectors that you already know

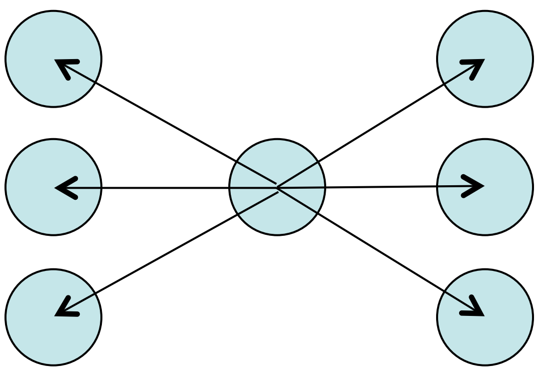


$$\vec{F}_{\text{tot}} = \vec{F}_{12} + \vec{F}_{13}$$

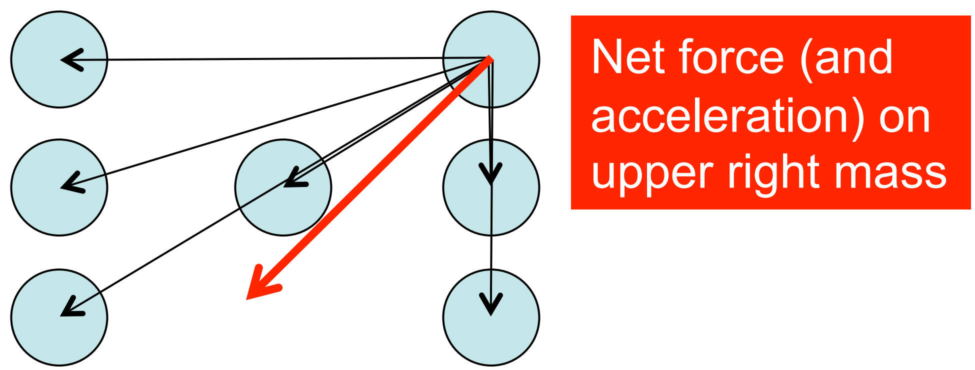
- This summation extends to any number of forces: **integral**!
  - We can calculate the net force (including gravitational forces) between any two oddly shaped objects by adding up (integrating) gravitational forces between all the combinations of all their pieces

Example

- Consider a distribution of masses pictured left and right below
  - Forces of gravity between these masses are drawn in each case
  - On the left, all gravitational forces on the center mass add up to zero
    - Newton's 2nd law: $F_{net}=ma$ The center object doesn't move from gravity
  - On the right, all gravitational forces on an object in the upper right do not add up to zero
    - Newton's 2nd law: $F_{net}=ma$ The upper right object moves towards the other masses **in a non-trivial way**



Gravitational forces on center mass



Gravitational forces on upper right mass

Gravity and Spherical Earth: Revisited

- Recall: we can treat gravity from a uniform density spherical shell (or a sphere!) as though all mass of that object is at the center of the shell (or sphere)
  - The center of the sphere is also its **center of mass**
  - From Newton's second law ($\vec{F} = m\vec{a}$) the Earth acts as if its mass is all concentrated at this center
- We will generalize the idea of a center of mass to any distribution of masses, particles, or objects

Center of Mass

- For any object or group of objects, the **center of mass** is a **unique point** where the object can be considered to be located when applying Newton's 2nd law
  - Forces directed through an object's center of mass create no **torque,** or "turning force" (or moment) around center of mass
  - Calculated as a weighted average of masses over their locations
    - Averaging involves taking a sum, so sum becomes an integral…
- For a system of discrete "particles" or spheres:

$$\boxed{\vec{r}_{\mathrm{cm}} = \frac{\sum m_i \vec{r}_i}{\sum m_i}}$$

$\sum$ stands for "sum over"

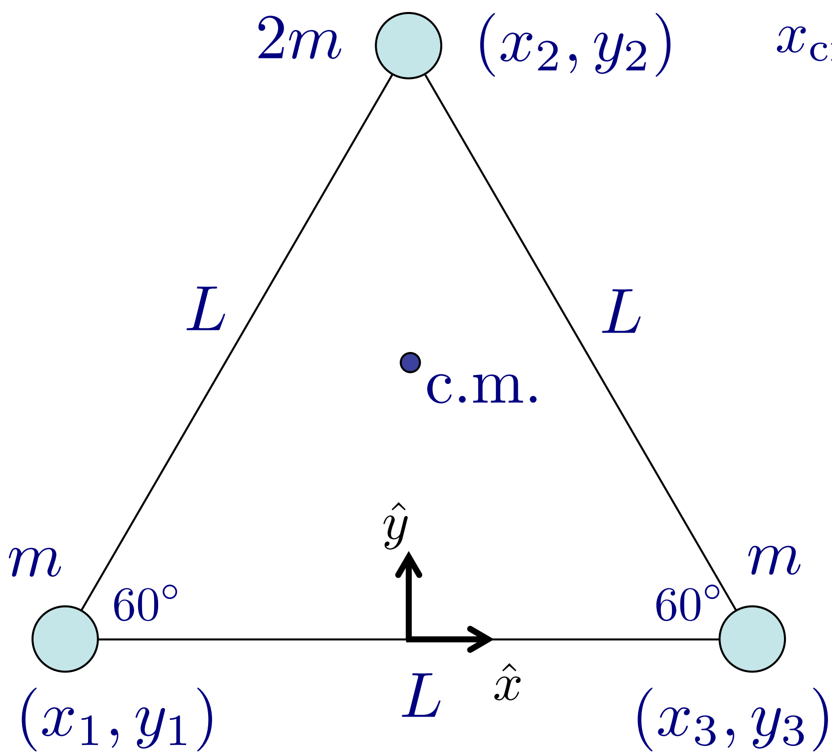
- For a continuous distribution of matter of total mass M

$$\boxed{\vec{r}_{\mathrm{cm}} = \frac{\int \vec{r}\, dm}{M}}$$

$\int$ is a limit of many small sums

Example: Finding the Center of Mass (c.m.)

- A system of three particles in an equilateral triangle



$$x_{\rm cm} = \frac{mx_1 + m_x 3}{4m} = \frac{m(L/2 - L/2)}{4m} = 0$$

$$y_{\rm cm} = \frac{(2m)y_2}{4m} = \frac{y_2}{2}$$

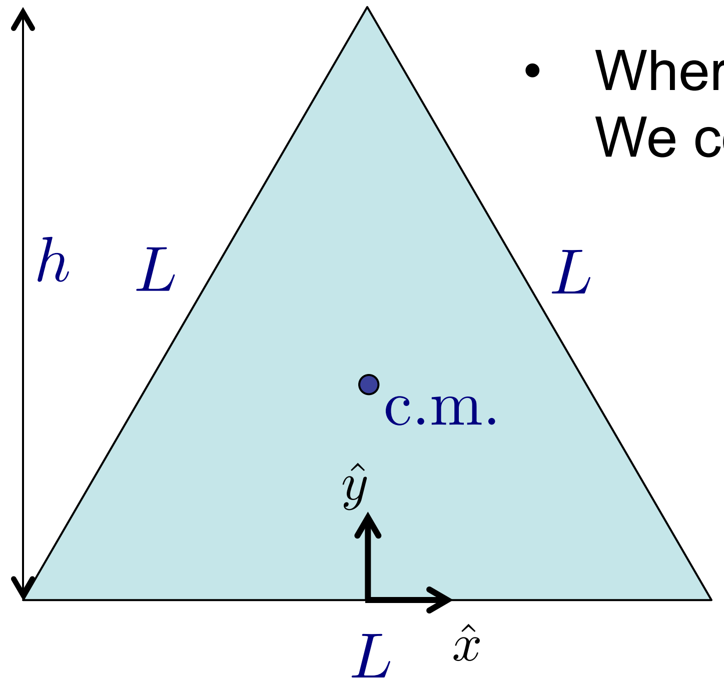
$$= \frac{L\sin(60^\circ)}{2} = 0.433\ L$$

$$\boxed{\vec{r}_{\rm cm} = (0, 0.433\ L)}$$

- Note: the center of mass (c.m.) is a location in our coordinate system
  - So its coordinates depends on where we put the coordinate origin

Calculus Example: Finding the Center of Mass

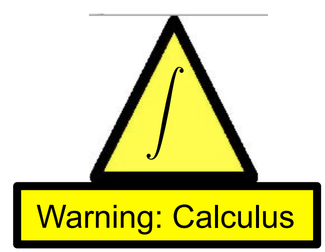
- Center of mass of a continuous uniform equilateral triangle



- Where does it balance in all directions?
  We could do an experiment to test our calculation…

Just looking tells us $x_{\mathrm{cm}} = 0$

y_{cm} is a bit more complicated…



$$y_{\mathrm{cm}} = \frac{\int y\, dm}{M}$$

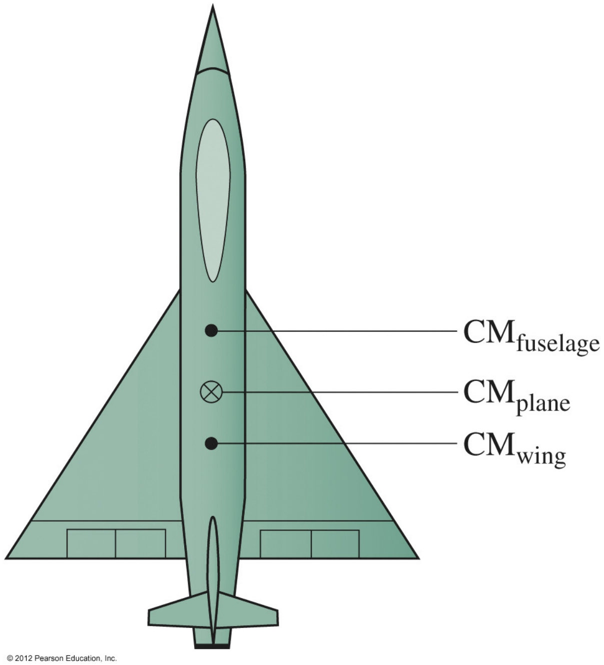
$$\frac{dm}{M} = \frac{(h-y)\frac{L}{h}}{\frac{1}{2}Lh}dy = \frac{2(h-y)\,dy}{h^2}$$

$$y_{\mathrm{cm}} = \frac{2}{h^2}\int_0^h (hy - y^2)dy = \frac{2}{h^2}\left(\frac{1}{2}hy^2 - \frac{y^3}{3}\right)\Bigg|_0^h = \frac{2}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{h}{3}$$

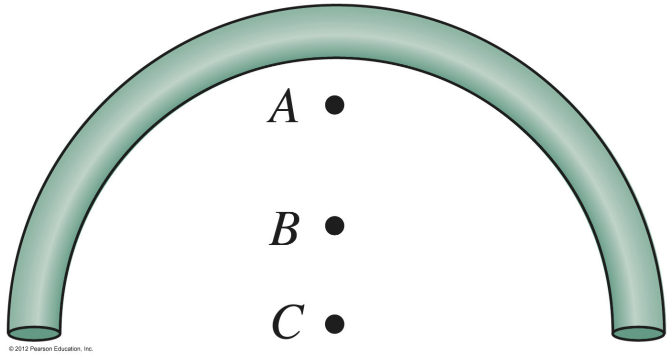
More on Center of Mass

- Just like sums can be broken into parts, center of mass calculations can be broken into parts

$$\vec{r}_{\rm cm,plane} = \vec{r}_{\rm cm,fuselage} + \vec{r}_{\rm cm,wings}$$



- Center of mass is not necessarily inside the object!
  - Which point is the center of mass?



Ponderable (10-15 minutes)

- Where is the center of mass of a meter stick?
  - You should be able to balance it there on one finger
  - “Center of mass” is what you might know as “center of gravity”
  - What about balancing the meter stick on one end? Is the center of mass still above your finger?

- Move your arm up and down while balancing the meter stick to accelerate the meter stick up and down
  - Does the orientation of the meter stick change?
  - Is this consistent with Newton’s second law?

- What happens to your answers if you do this with a broom instead of a meter stick?

Motion of the Center of Mass

- The center of mass obeys Newton's 2nd law:

$$\vec{F}_{\text{net,external}} = m\vec{a}_{\text{cm}}$$

Time-lapse photo of skier

Complicated twisting and flipping motions of her body

But her CM moves along a simple parabolic path

Motion of the Center of Mass

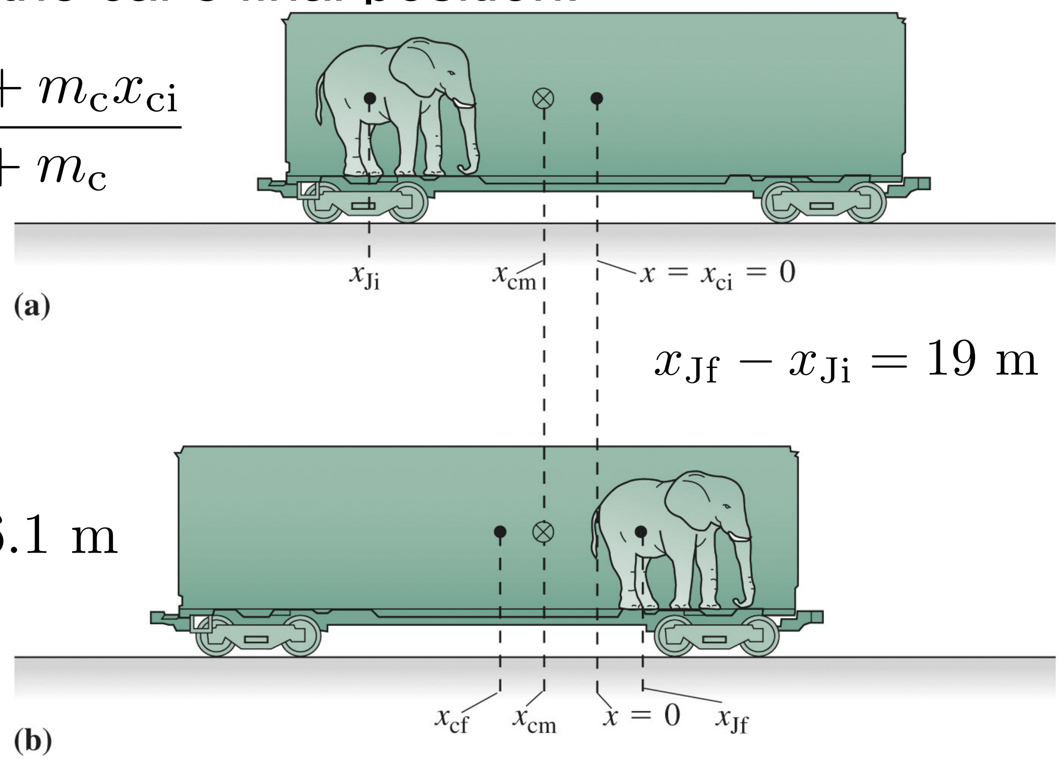
- In the absence of any *external* forces on a system, the c.m. motion remains unchanged; if it's at rest, it remains in the same place -- no matter what *internal* forces may act.

- Example: Jumbo, a 4.8-ton elephant, walks 19m toward one end of the car, but the c.m. of the 15-ton rail car plus elephant doesn't move. This allows us to find the car's final position:

$$x_{\mathrm{cm}} = \frac{m_{\mathrm{J}} x_{\mathrm{Jf}} + m_{\mathrm{c}} x_{\mathrm{cf}}}{m_{\mathrm{J}} + m_{\mathrm{c}}} = \frac{m_{\mathrm{J}} x_{\mathrm{Ji}} + m_{\mathrm{c}} x_{\mathrm{ci}}}{m_{\mathrm{J}} + m_{\mathrm{c}}}$$

$$x_{\mathrm{cf}} - x_{\mathrm{ci}} = \frac{m_{\mathrm{J}}}{m_{\mathrm{c}}}(x_{\mathrm{Ji}} - x_{\mathrm{Jf}})$$

$$= \frac{(4.8\ \mathrm{tons})}{(15\ \mathrm{tons})}(-19\ \mathrm{m}) = -6.1\ \mathrm{m}$$



Momentum and the Center of Mass

- Remember that the center of mass obey's Newton's 2nd law
  - Like the skier's c.m. traveling on a parabolic trajectory

$$\vec{F}_{\mathrm{net,external}} = m\vec{a}_{\mathrm{cm}}$$

  - What about internal forces? If all internal forces are equal and opposite (Newton's 3rd law) then the net *internal* forces are zero

$$\vec{F}_{\mathrm{net,internal}} = 0$$

If we could force as something that changes with time, then here this "something" would be constant (conserved) if we count it for all objects within our system…

Conservation of Linear Momentum

When the net external force is zero, $\vec{F}_{\rm net} = m\vec{a} = \frac{d\vec{p}}{dt} = 0$ $\quad \boxed{\vec{p} \equiv m\vec{v}}$

Linear Momentum

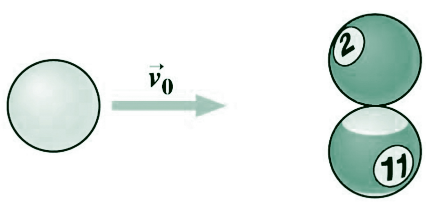
Therefore the total momentum of all objects in the system is constant:

$$\boxed{\sum \vec{p}_i = \text{constant}}$$

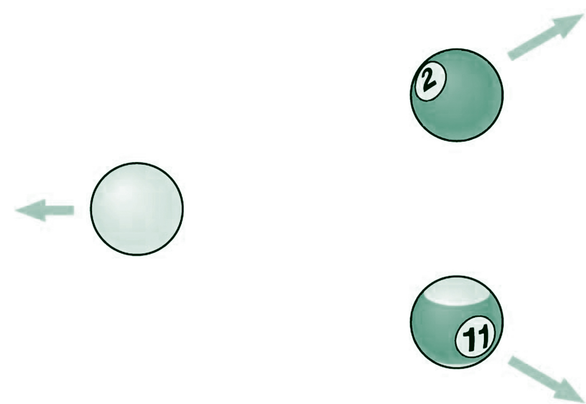
This is the **conservation of linear momentum.**

Example: A system of three billiard balls:

– Initially two are at rest; all the momentum is in the left-hand ball:



– Now they're all moving, but the **total** momentum remains the same:



Collisions

- A collision is a brief, intense interaction between objects.
  - Examples: balls on a pool table, a tennis ball and racket, baseball and bat, football and foot, an asteroid colliding with a planet.
  - The collision time is short compared with the timescale of the objects' overall motion.
  - Internal forces of the collision are so large that we can neglect any external forces acting on the system during the brief collision time.
  - Therefore linear momentum is essentially conserved during collisions.