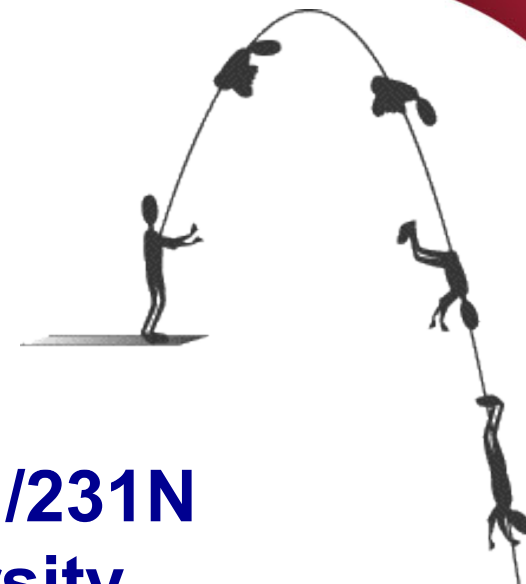
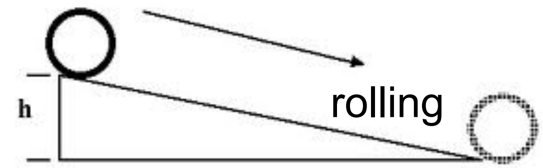


University Physics 226N/231N
Old Dominion University
More Rotational Motion



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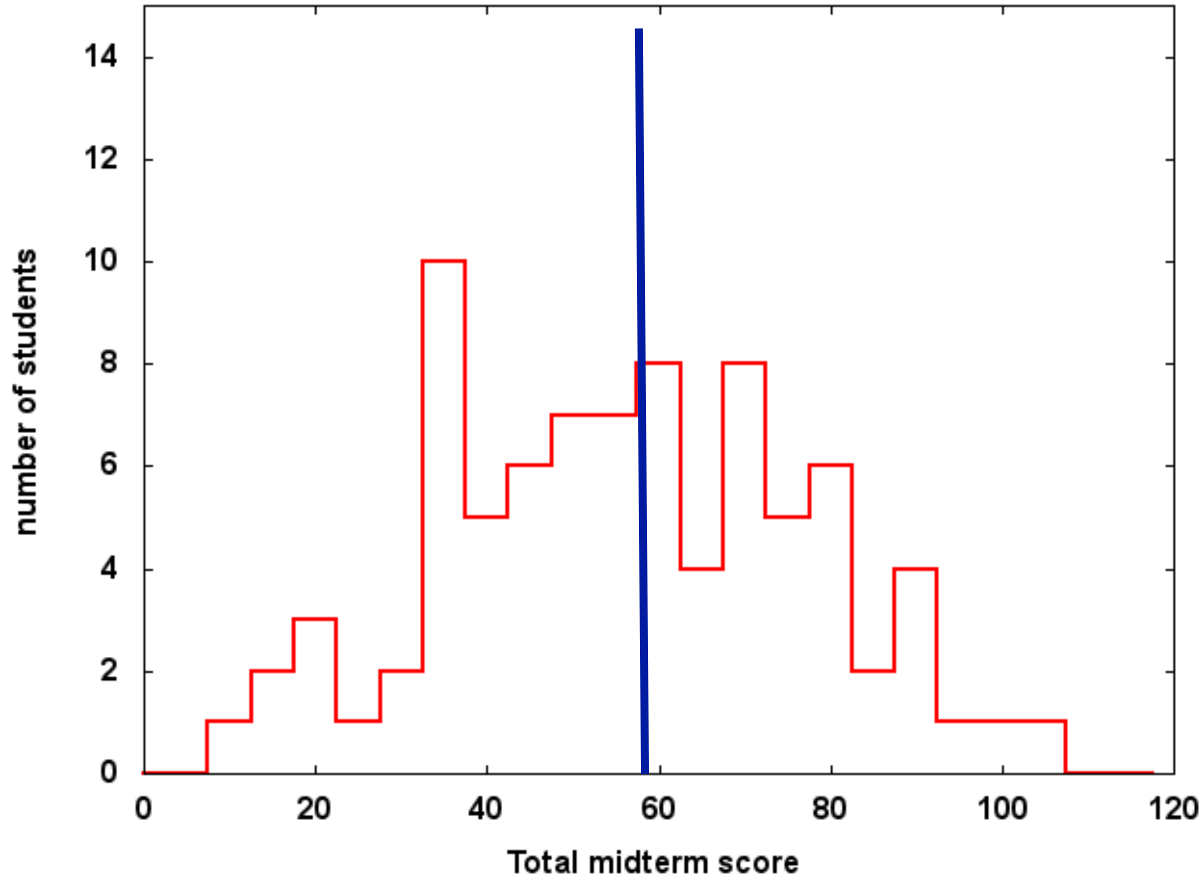
Wednesday October 24, 2012

Happy Birthday to Pierre-Gilles de Gennes (1991 Nobel), Wilhelm Weber, and Anton van Leeuwenhoek!

Happy Food Day, World Development Information Day, and **Get Midterm 2 Back Day**!

Midterm 2 Grade Distribution

- Midterm 2 class average score: 58.2 (+/- 21.2 std deviation)



- Full statistics will be in solutions on class website later today
- Remember that I drop one midterm grade for final grading!

Quick One-Dimensional Kinematics Review

- We're going to draw explicit analogies between angular motion quantities and our old friends **position, velocity,** and **acceleration** from one-dimensional kinematics
- Time for a bit of review
  - Definitions of velocity and acceleration

$$\text{velocity} \quad v \equiv \frac{dx}{dt}$$

$$\text{acceleration} \quad a \equiv \frac{dv}{dt}$$

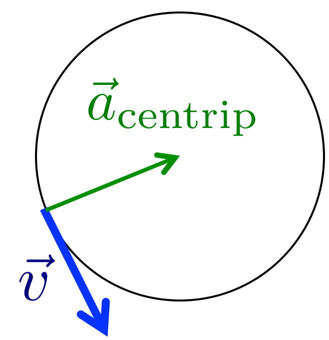
  - Constant acceleration motion

$$x - x_0 = v_0 t + \frac{1}{2} a t^2$$

$$v = v_0 + at$$

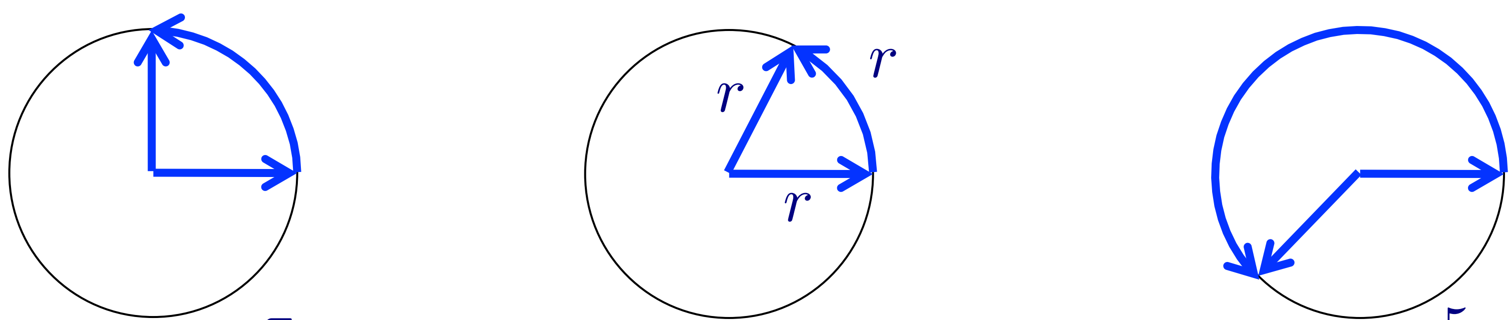
  - Centripetal acceleration related to tangential velocity

$$a_{\text{centrip}} = \frac{v^2}{r}$$



Angular Position

- What do we use for position in angular motion problems?
  - The angle that an object is from a reference angle $\theta = 0$
  - The sign convention is **usually that clockwise is positive**
  - The $\theta = 0$ location, like x=0, is usually defined by the problem
    - We care more about **angular distances**, $\Delta\theta = \theta(t_2) - \theta(t_1)$
  - We also always use **radians** where $\boxed{2\pi \text{ rad} = 360^\circ}$
    - 1 rad is the angle where the arc length is equal to the circle radius



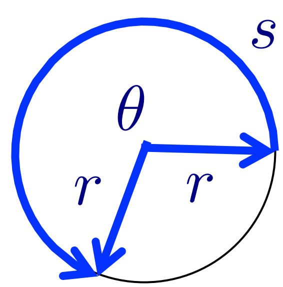
$$\Delta\theta = 90^\circ = \frac{\pi}{2} \text{ rad} \qquad \Delta\theta = 57.3^\circ = 1 \text{ rad} \qquad \Delta\theta = 225^\circ = \frac{5\pi}{4} \text{ rad}$$

- A conversion example: $225^\circ \left(\frac{2\pi \text{ rad}}{360^\circ}\right) = \frac{5}{8}(2\pi) \text{ rad} = \frac{5\pi}{4} \text{ rad}$

Angular Position: Importance of Radians

- That radians thing? Yeah, that's important…
  - If we write angles in radians, we can write a **tremendously useful equation** that relates the actual distance around the arc s to angles and radii:

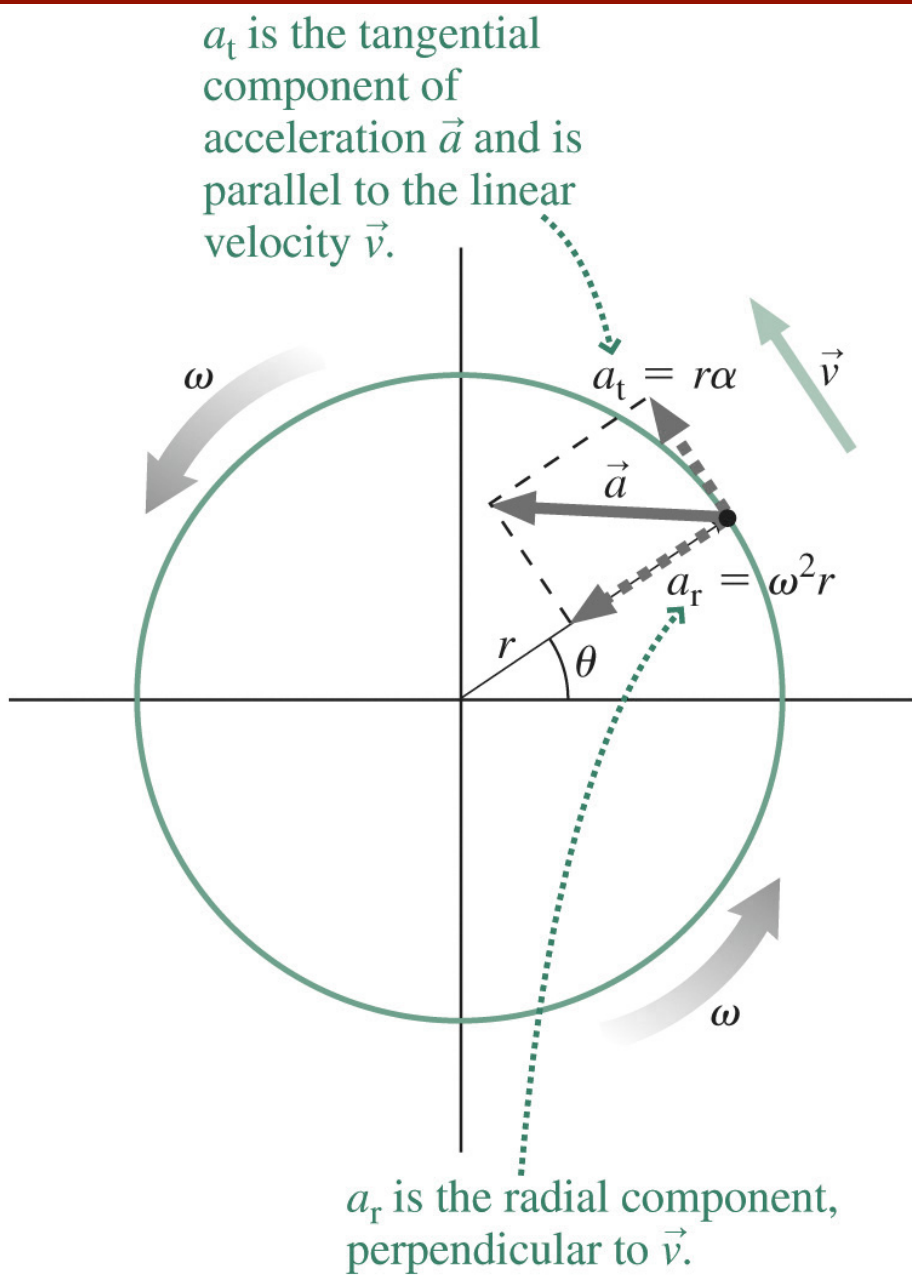
$$\boxed{s = r\theta}$$



  - Here s is the distance around the arc. This formula lets us switch between real distances (like s and r, which are in distance units like meters) and angular distances (which are in radians)
  - Example: $\theta = 30^\circ = \frac{\pi}{6}\ \mathrm{rad}, \quad r = 2\ \mathrm{m} \quad \Rightarrow \quad s = \frac{\pi}{3}\ \mathrm{m} \approx 1.05\ \mathrm{m}$

- **Warning**: This equation (and most others we'll derive from now on) only work if the angle θ is in radians!
  - Since a radian is a ratio, it really is technically "dimensionless"

Angular Acceleration Picture (from text)



$$a_t = \alpha\, r$$

$$a_r = a_{\rm centrip} = \frac{v^2}{r} = \omega^2 r$$

Constant Angular Acceleration

- Problems with constant angular acceleration are exactly analogous to similar problems involving linear motion in one dimension.
  - The **exact** same equations apply, with $x \to \theta, \quad v \to \omega, \quad a \to \alpha$

Table 10.1 Angular and Linear Position, Velocity, and Acceleration

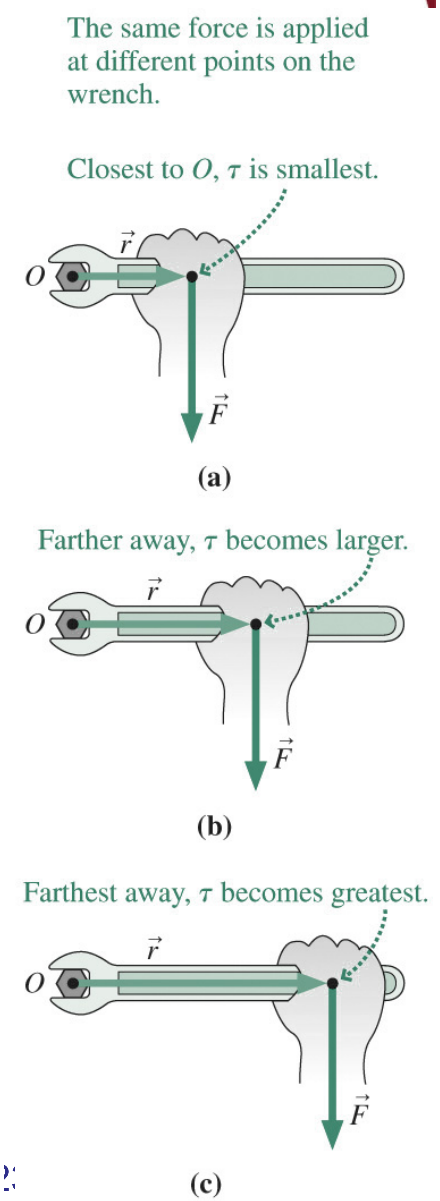
Linear Quantity	**Angular Quantity**
Position x	Angular position θ
Velocity $v = \dfrac{dx}{dt}$	Angular velocity $\omega = \dfrac{d\theta}{dt}$
Acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$	Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

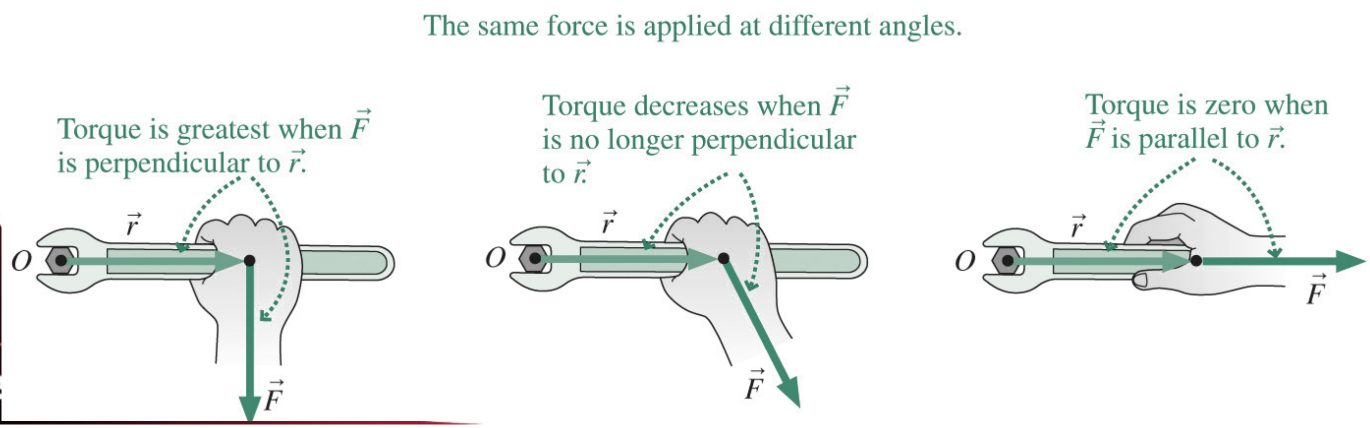
Equations for Constant Linear Acceleration		**Equations for Constant Angular Acceleration**	
$\bar{v} = \frac{1}{2}(v_0 + v)$	(2.8)	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$	(10.6)
$v = v_0 + at$	(2.7)	$\omega = \omega_0 + \alpha t$	(10.7)
$x = x_0 + v_0 t + \frac{1}{2}at^2$	(2.10)	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(10.8)
$v^2 = v_0^2 + 2a(x - x_0)$	(2.11)	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10.9)

Torque

- Torque τ is the rotational analog of force, and results from the application of one or more forces.
- Torque is relative to a chosen rotation axis.
- Torque depends on:
  - the distance from the rotation axis to the force application point.
  - the magnitude of the force $\vec{F}$
  - the orientation of the force relative to the displacement $\vec{r}$ from axis to force application point:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \tau = rF\sin\theta$$

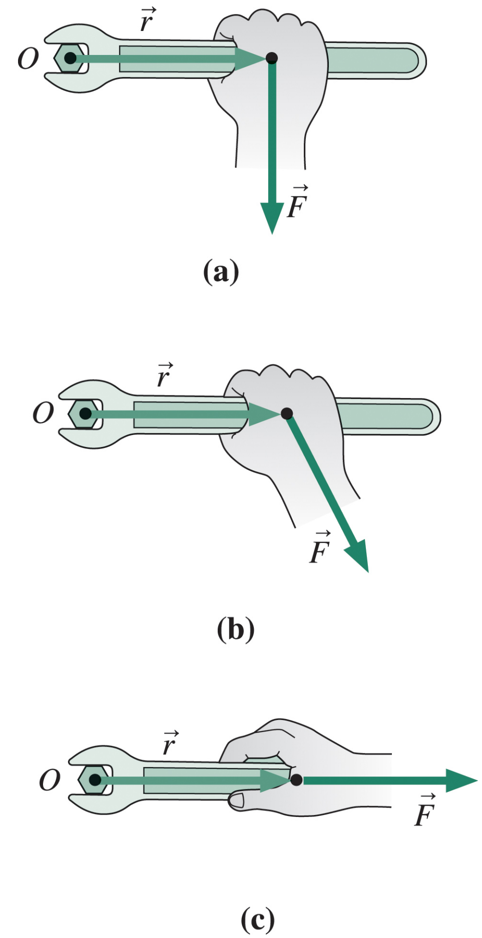




Clicker Question 1

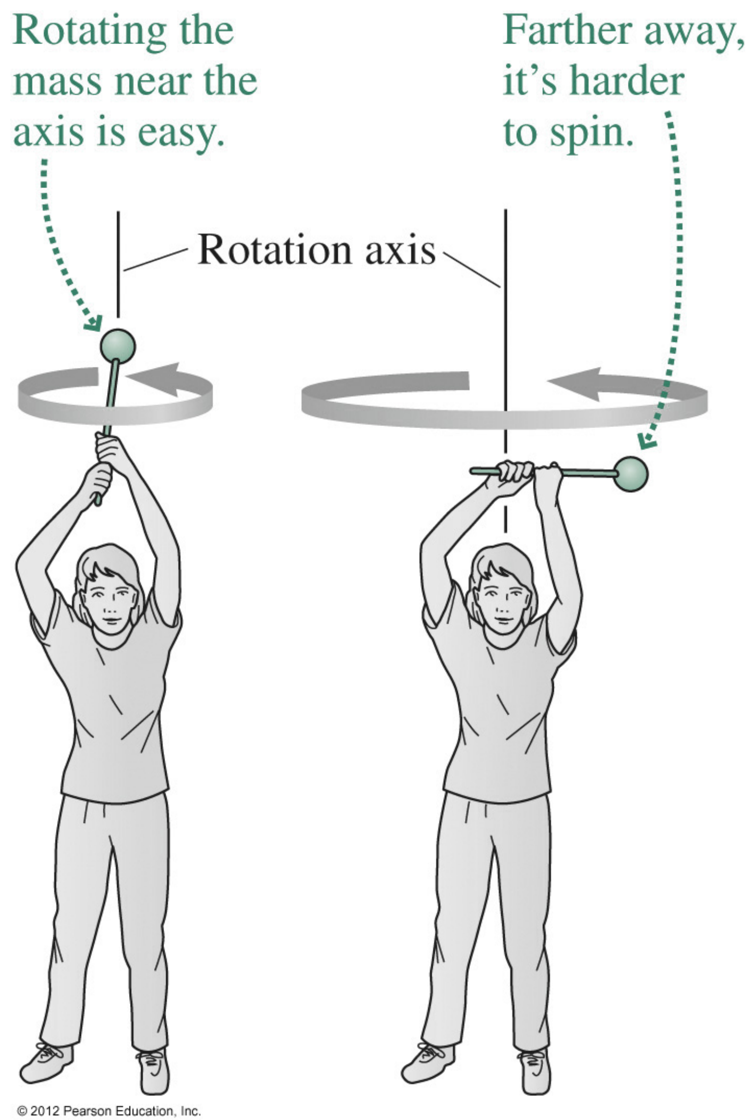
- The forces in the figures all have the same magnitude. Which force produces zero torque?

A. The force in figure (a)
B. The force in figure (b)
C. The force in figure (c)
D. All of the forces produce torque.



Rotational Inertia and the Analog of Newton's Law

- **Rotational inertia** I (or moment of inertia) is the rotational analog of mass.
  - Rotational inertia depends on the distribution of mass and its distance from the rotation axis, similar to center of mass.



- Rotational acceleration, torque, and rotational inertia combine to give the rotational analog of Newton's second law $F = ma$

$$\boxed{\tau = I\alpha}$$

(or, more properly with vectors)

$$\vec{\tau} = I\vec{\alpha}$$

like $\vec{F} = m\vec{a}$

Connections Between Newton's Laws

- Rotational Newton's 2nd law: $\tau = I\alpha$
- What is I ? We've already defined torque and angular acceleration:

$$\tau = rF \ (\text{when } \vec{r},\ \vec{F} \text{ are } \perp) \qquad a = r\alpha \quad \Rightarrow \quad \alpha = a/r$$

- Substituting gives

$$rF = I(a/r)$$

$$F = \left(\frac{I}{r^2}\right) a$$

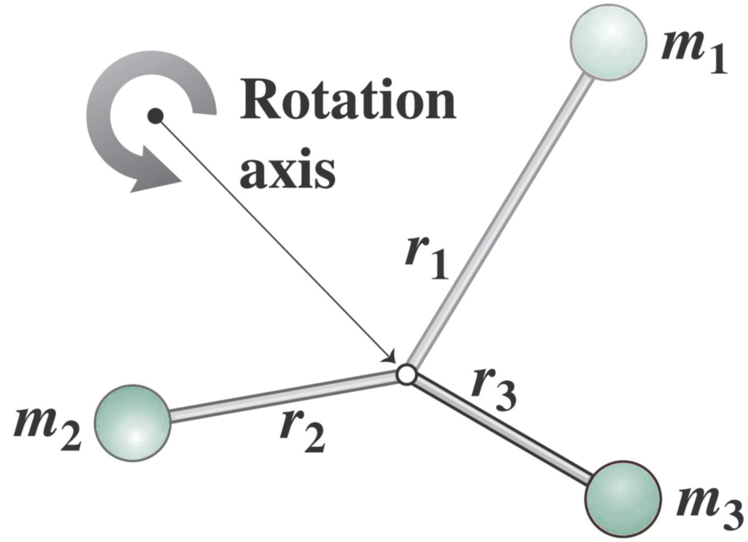
- This looks remarkably like $F = ma$ when we make the association

$$\left(\frac{I}{r^2}\right) = m \quad \Rightarrow \quad \boxed{I = mr^2}$$

Calculating Rotational Inertia

- For a single point mass *m*, rotational inertia is the product of mass with the square of the distance *r* from the rotation axis:

$$I = mr^2$$

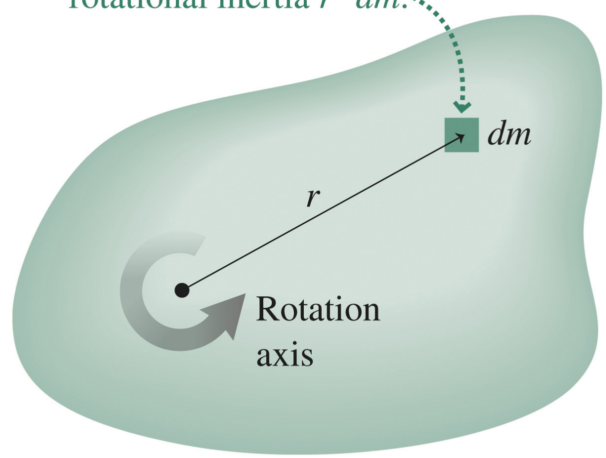


- For a system of discrete masses, the rotational inertia is the sum of the rotational inertias of the individual masses:

$$\boxed{I = \sum m_i r_i^2}$$

- For continuous matter, the rotational inertia is given by an integral over the distribution of matter:

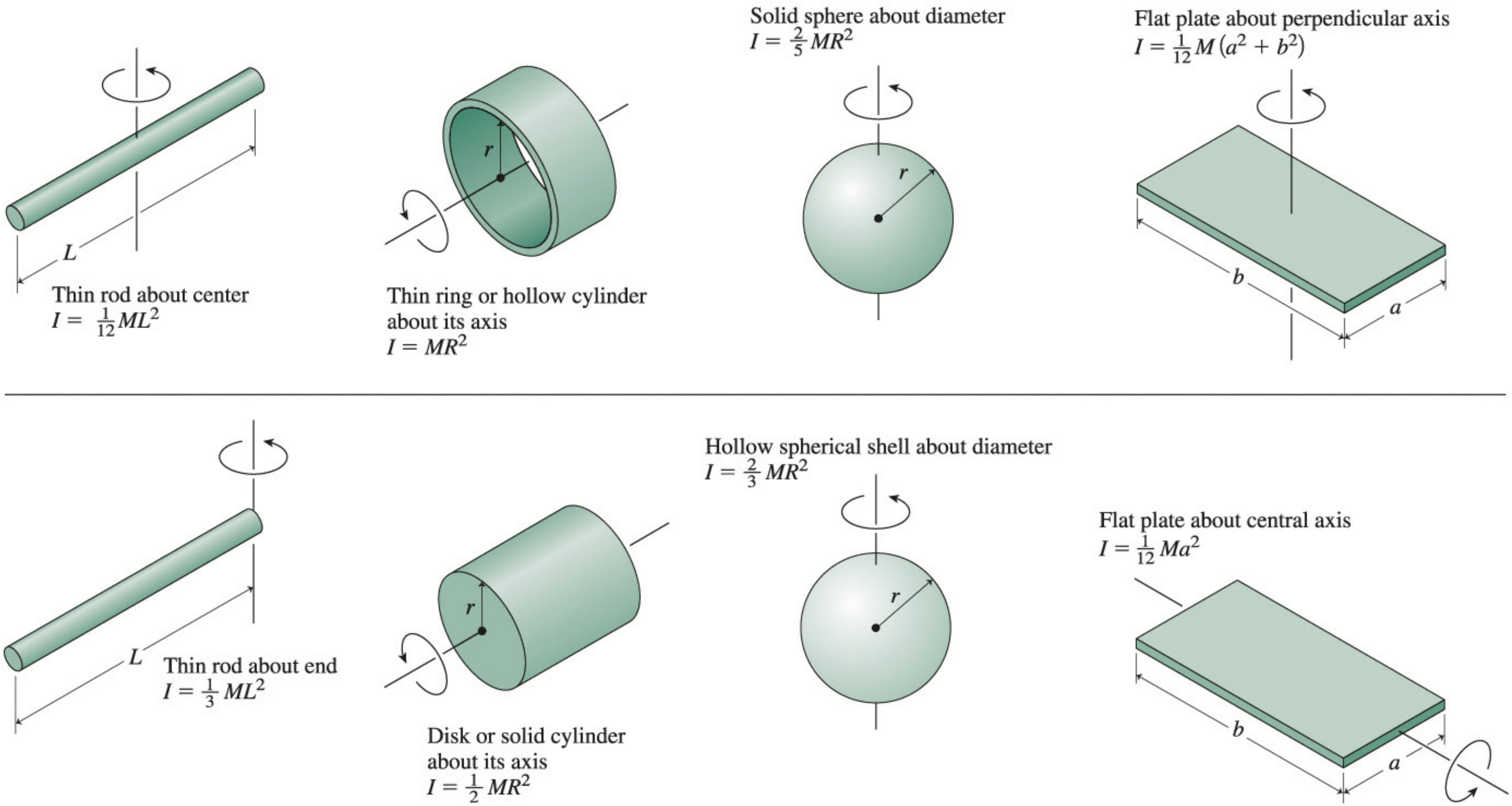
$$\boxed{I = \int r^2 \, dm}$$



Similar to center of mass: $\vec{r}_{\mathrm{cm}} = \dfrac{\int \vec{r}\,dm}{M}$

Some Rotational Inertias of Simple Objects

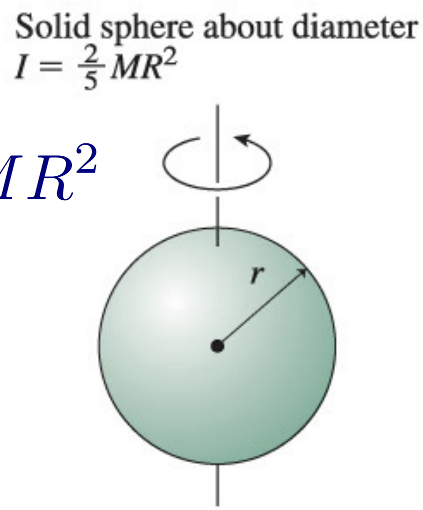
- We really do need to use calculus to figure out rotational inertias of most simple (three-dimensional) geometrical objects

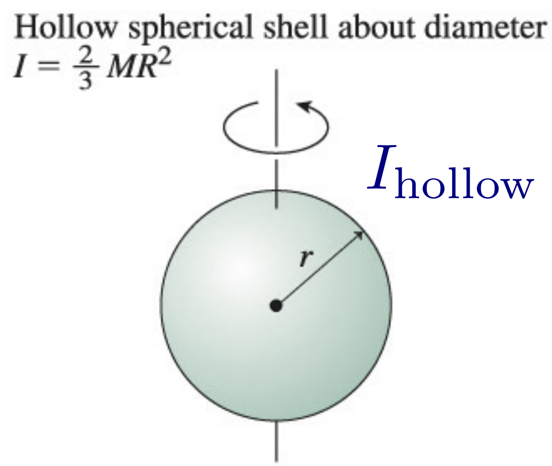


Quick Question

- A hollow ball and a solid ball roll without slipping down an inclined plane. Which ball reaches the bottom of the incline first?
  A. The solid ball reaches the bottom first.
  B. The hollow ball reaches the bottom first.
  C. Both balls reach the bottom at the same time.
  D. We can't determine this without information about the mass.

$$I_{\text{solid}} = \frac{2}{5} M R^2$$



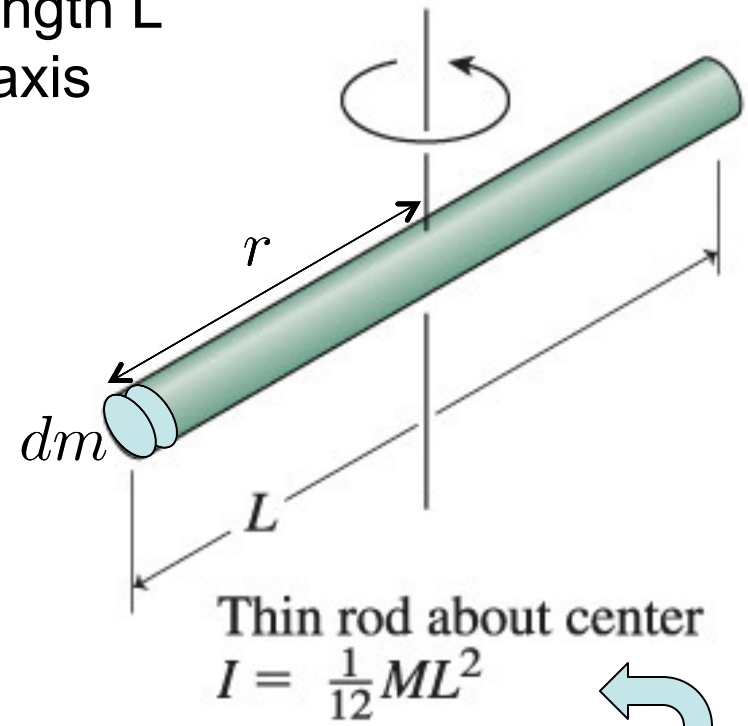


$$I_{\text{hollow}} = \frac{2}{3} M R^2$$

(Example Rotational Inertia Calculation)

- Rotational inertia of a thin rod of length L and total mass M about its center axis
  - Take small slices along the length of the rod
  - Mass of each little slice is

$$dm = \left(\frac{dr}{L}\right) M$$



  - Rotational inertia:

$$I = \int r^2\, dm$$

Warning: Calculus

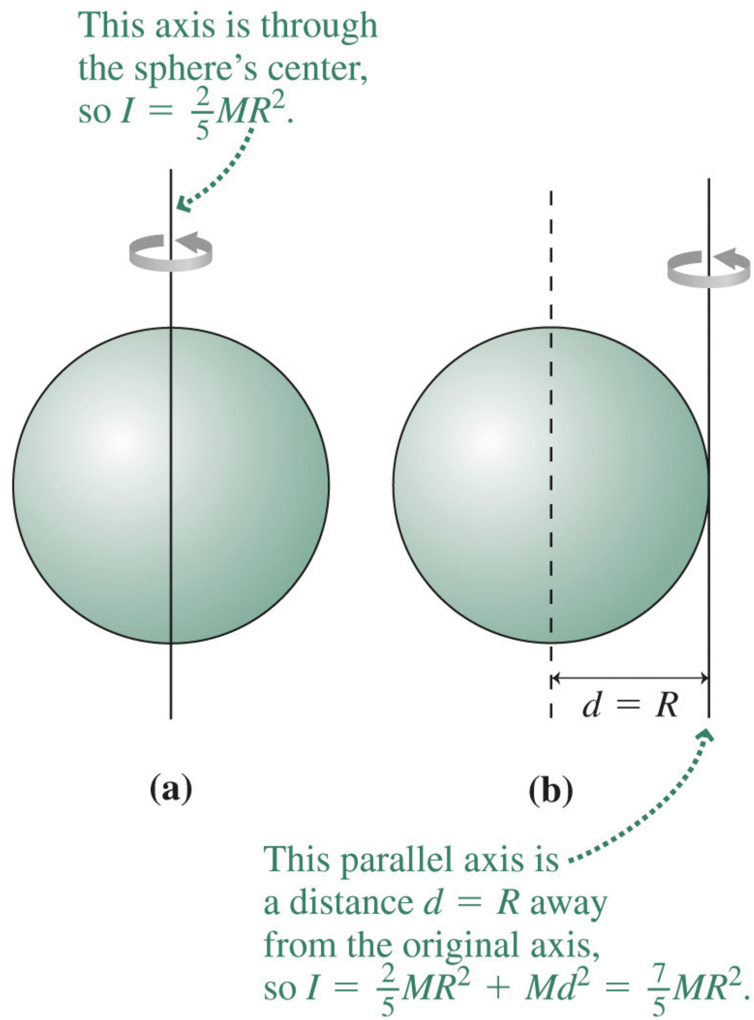
$$I = \int_{-L/2}^{L/2} r^2 \left(\frac{dr}{L}\right) M = \frac{M}{L}\left(\frac{r^3}{3}\right)\Bigg|_{-L/2}^{L/2} = \frac{M}{L}\left[\frac{L^3}{24} - \left(-\frac{L^3}{24}\right)\right] = \frac{M}{L}\left(\frac{L^3}{12}\right) = \frac{ML^2}{12}$$

Parallel Axis Theorem

- If we know the rotational inertia I_{cm} about an axis through the center of mass of a body, the **parallel-axis theorem** allows us to calculate the rotational inertia I through any parallel axis.
- The parallel-axis theorem states that

$$\boxed{I = I_{\rm cm} + Md^2}$$

where d is the distance from the center-of-mass axis to the parallel axis and M is the total mass of the object.



Example: Hula Hoop Rotational Inertia

- What is the rotational inertia of a hula hoop of radius r and mass M around its edge?

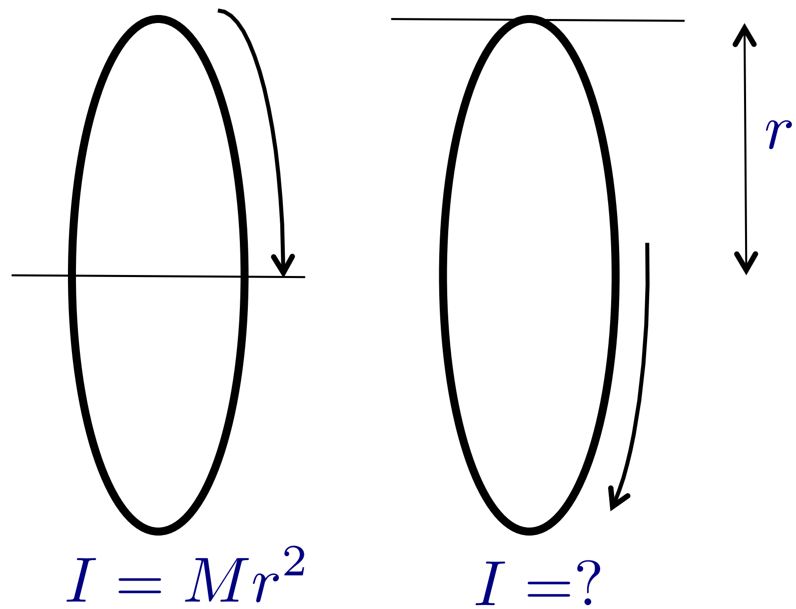
- Its center of mass is (obviously) at the center of the circle and all of its mass is at the same radius

$$I_{\rm cm} = Mr^2$$

- The parallel axis theorem gives

$$I_{\rm edge} = Mr^2 + Mr^2 = 2Mr^2$$

Interesting… It takes twice as much torque to turn a ring around its edge as it takes to turn around its center.

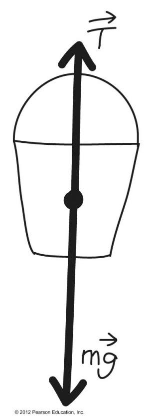


Combining Linear and Rotational Motion

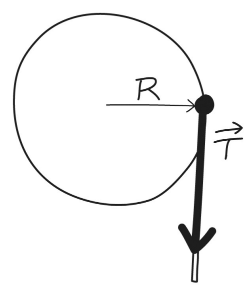
- We now have the tools to do some interesting problems…
  - Bucket of mass m unrolls from a cylinder of mass M and radius R into a well.
  - What is the bucket's (linear) acceleration?

$\hat{y}$

Be careful about signs!



bucket



cylinder

$$\text{Cylinder}: \sum \tau = I\alpha$$

$$I_{\text{cylinder}} = \frac{1}{2}MR^2$$

$$\tau = TR\sin\theta = TR$$

$$\alpha = a/R$$

$$\Rightarrow TR = \left(\frac{1}{2}MR^2\right)\frac{a}{R}$$

$$\Rightarrow \quad T = Ma/2$$

$$\text{Bucket}: \quad \sum F = mg - T = ma$$

$$mg - Ma/2 = ma$$

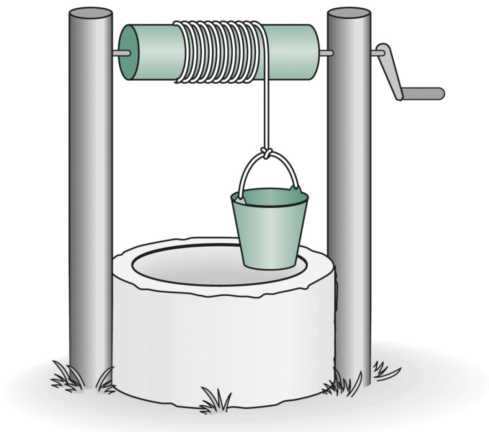
$$a = \frac{mg}{M/2 + m}$$

Combining Rotational and Linear Dynamics

- In problems involving both linear and rotational motion:
  - **IDENTIFY** the objects and forces or torques acting.
  - **DEVELOP** your solution with drawings and by writing Newton's law and its rotational analog. Note physical connections between the objects.
  - **EVALUATE** to find the solution.
  - **ASSESS** to be sure your answer makes sense.

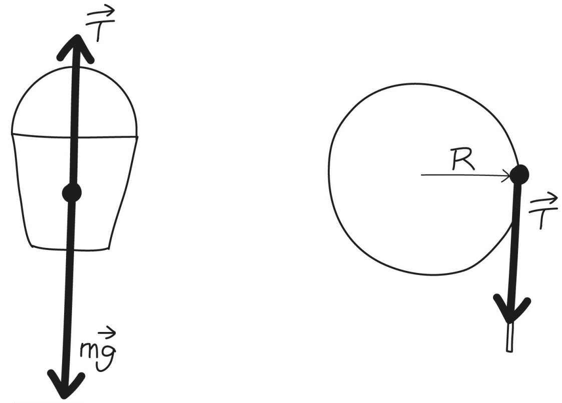
A bucket of mass *m* drops into a well, its rope unrolling from a cylinder of mass *M* and radius *R.*

What's its acceleration?



Free-body diagrams for bucket and cylinder

Rope tension $\vec{T}$ provides the connection



Newton's law, bucket:

$$F_{net} = mg - T = ma$$

Rotational analogy of Newton's law, cylinder:

$$RT = I a/R$$

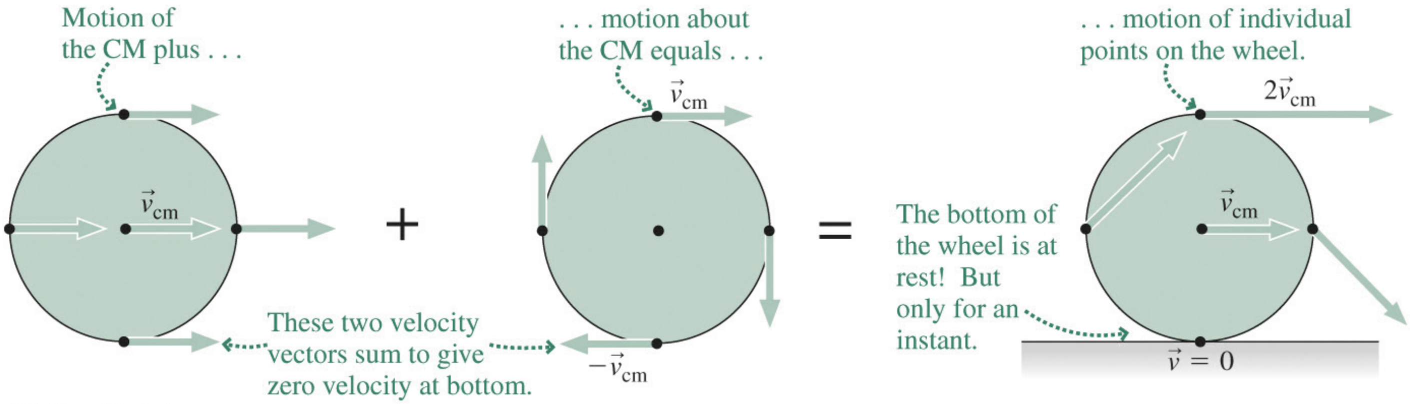
Here $I = \frac{1}{2}MR^2$

Solve the two equations to get

$$a = \frac{mg}{m + \frac{1}{2}M}$$

Rolling Motion

- Rolling motion combines translational (linear) motion and rotational motion.
  - The rolling object's center of mass undergoes translational motion.
  - The object itself rotates about the center of mass.
  - In true rolling motion, the object moves without slipping and its point of contact with the ground is instantaneously at rest.
  - Then the rotational speed ω and linear speed v are related by $v = \omega R$, where R is the object's radius.

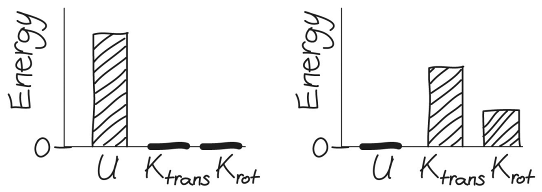


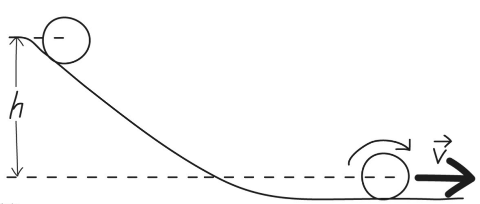
Rotational Energy

- A rotating object has kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$ associated with its rotational motion alone.
  - It may also have translational kinetic energy: $K_{\text{trans}} = \frac{1}{2}Mv^2$.
- In problems involving energy conservation with rotating objects, both forms of kinetic energy must be considered.
  - For rolling objects, the two are related:
    - The relation depends on the rotational inertia.

Example: A solid ball rolls down a hill. How fast is it moving at the bottom?

Energy bar graphs





Equation for energy conservation

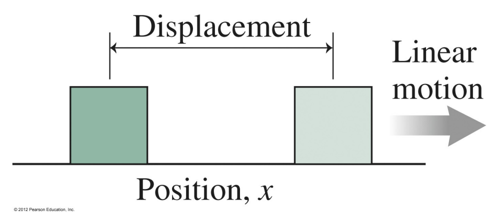
$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}Mv^2$$

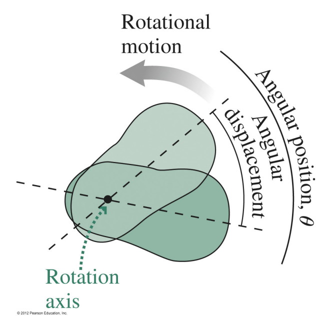
Solution:

$$v = \sqrt{\frac{10}{7}gh}$$

Summary

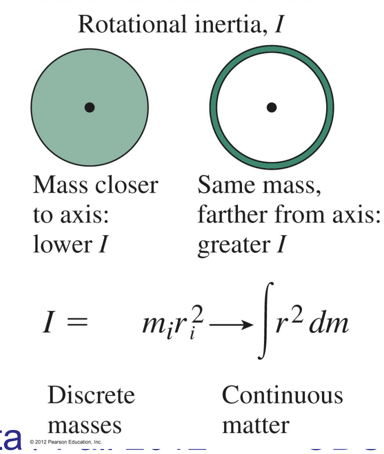
- Rotational motion in one dimension is exactly analogous to linear motion in one dimension.
  - Linear and angular motion:

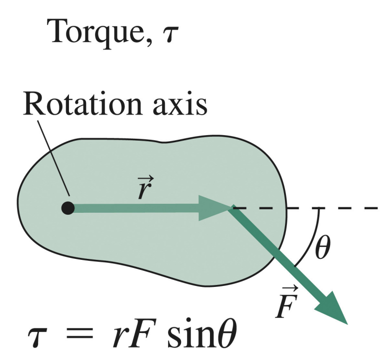




  - Analogies between rotational and linear quantities:

Linear Quantity or Equation	Angular Quantity or Equation	Relation Between Linear and Angular Quantities
Position x	Angular position θ	
Speed $v = dx/dt$	Angular speed $\omega = d\theta/dt$	$v = \omega r$
Acceleration a	Angular acceleration α	$a_t = \alpha r$
Mass m	Rotational inertia I	$I = \int r^2\,dm$
Force F	Torque τ	$\tau = rF\sin\theta$
Kinetic energy $K_{\text{trans}} = \frac{1}{2}mv^2$	Kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$	
Newton's second law (constant mass or rotational inertia):		
$F = ma$	$\tau = I\alpha$	

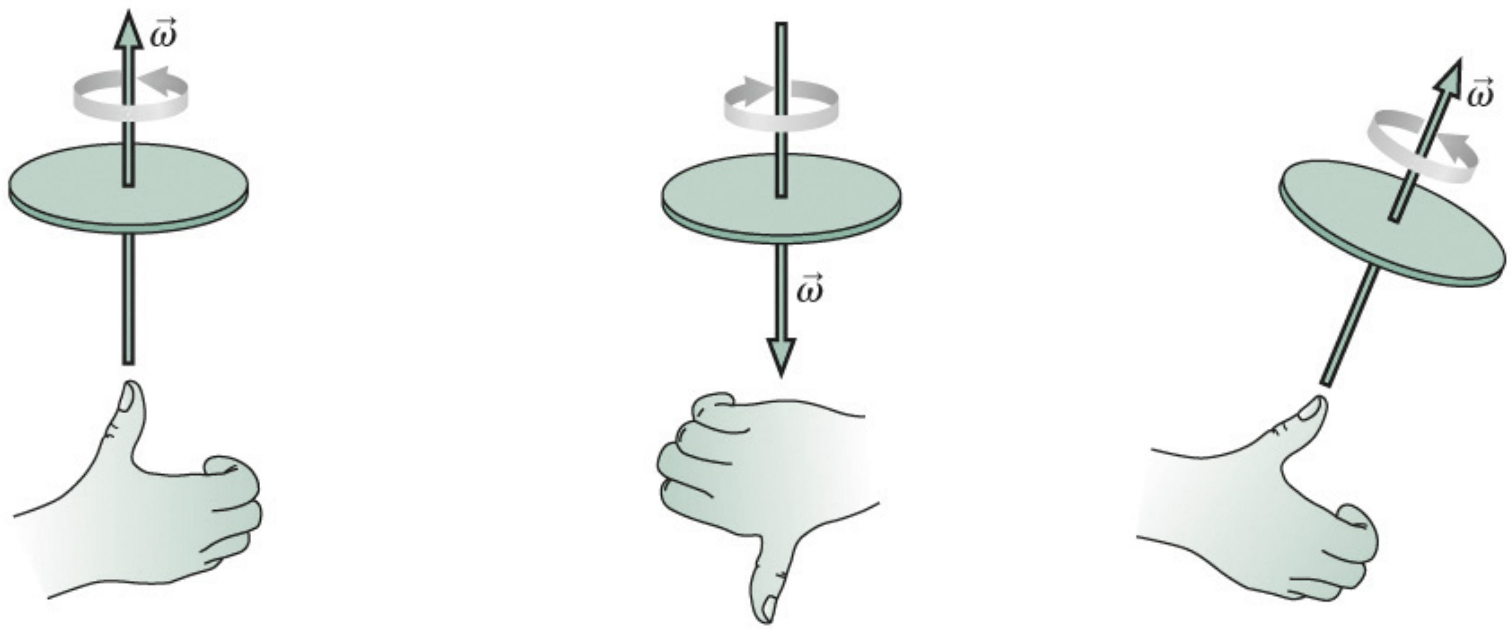




Direction of the Angular Velocity Vector

- The direction of angular velocity is given by the **right-hand rule.**
  - Curl the fingers of your right hand in the direction of rotation, and your thumb points in the direction of the angular velocity vector

$$\vec{\omega}.$$

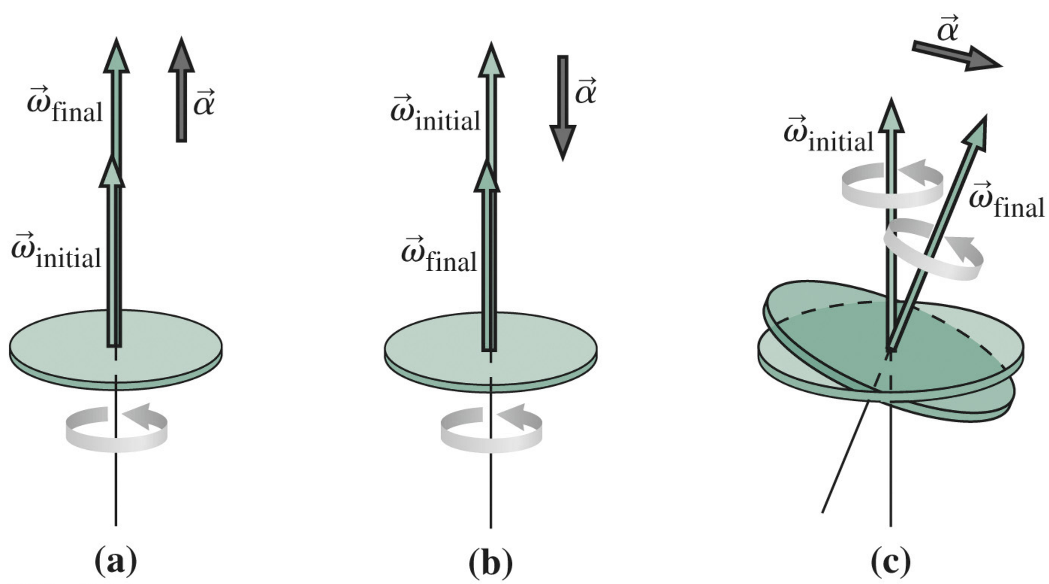


Direction of the Angular Acceleration

- Angular acceleration points in the direction of the change in the angular velocity $\Delta\vec{\omega}$:

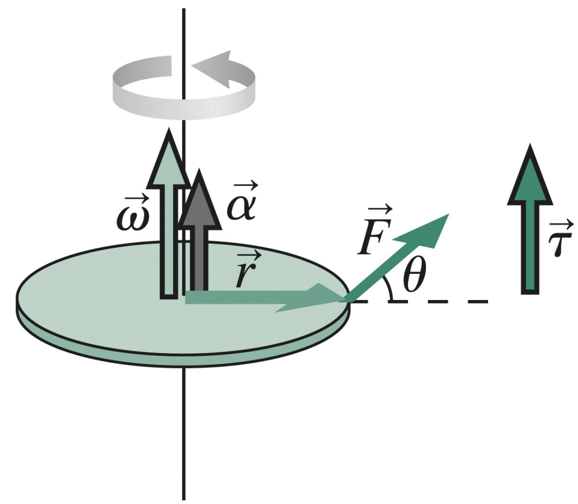
$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$$

  - The change can be in the same direction as the angular velocity, increasing the angular speed.
  - The change can be opposite the angular velocity, decreasing the angular speed.
  - Or it can be in an arbitrary direction, changing the direction and speed as well.



Direction of the Torque Vector

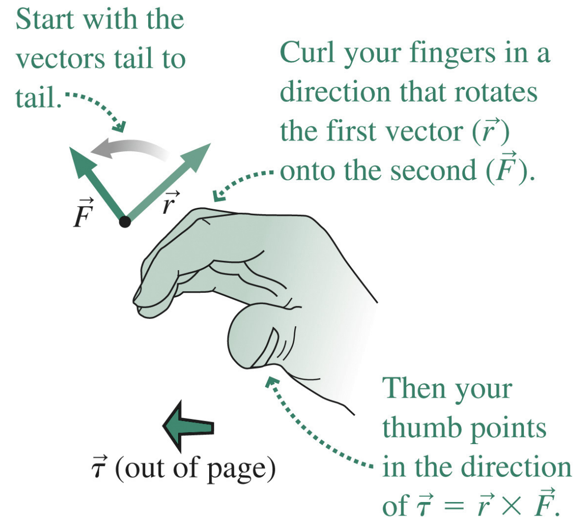
- The torque vector is perpendicular to both the force vector and the displacement vector from the rotation axis to the force application point.
  - The magnitude of the torque is $\tau = rF\sin\theta$.



– Of the two possible directions perpendicular to $\vec{r}$ and $\vec{F}$, the correct direction is given by the right-hand rule.

– Torque is compactly expressed using the **vector cross product**:

$$\vec{\tau} = \vec{r} \times \vec{F}$$



The Cross Product

- Forming from two vectors $\vec{A}$ and $\vec{B}$ a third vector $\vec{C}$ of magnitude $C = AB\sin\theta$ and direction given by the right-hand rule is called the **cross product**:

The cross product $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ is written

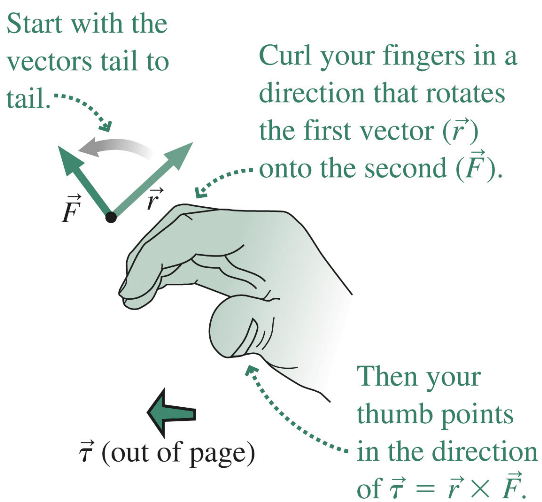
$$\vec{C} = \vec{A} \times \vec{B}$$

and is a vector with magnitude $AB \sin\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$, and where the direction of $\vec{C}$ is given by the right-hand rule of Fig. 11.4.

- Some properties of cross products:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$



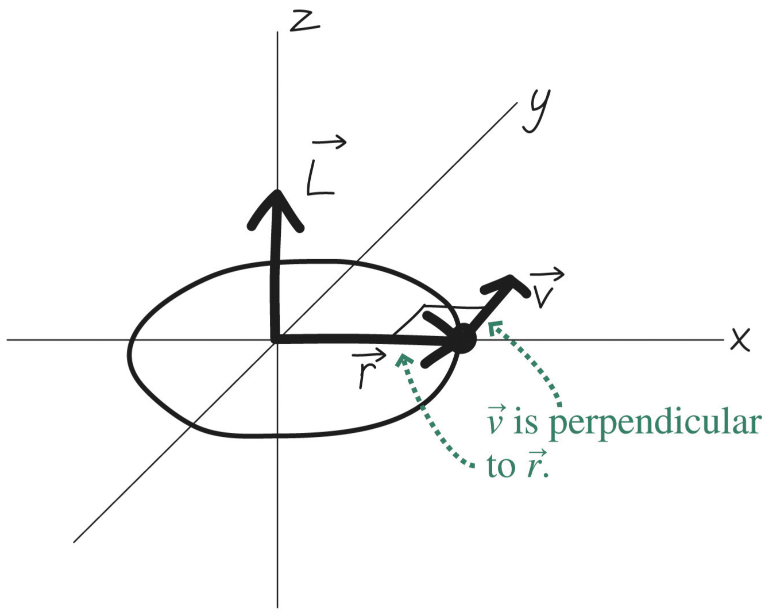
Angular Momentum

- For a single particle, angular momentum $\vec{L}$ is a vector given by the cross product of the displacement vector from the rotation axis with the linear momentum of the particle:

$$\vec{L} = \vec{r} \times \vec{p}$$

  - For the case of a particle in a circular path, $L = mvr$, and $\vec{L}$ is upward, perpendicular to the circle.
  - For sufficiently symmetric objects, $\vec{L}$ is the product of rotational inertia and angular velocity:

$$\vec{L} = I\vec{\omega}$$

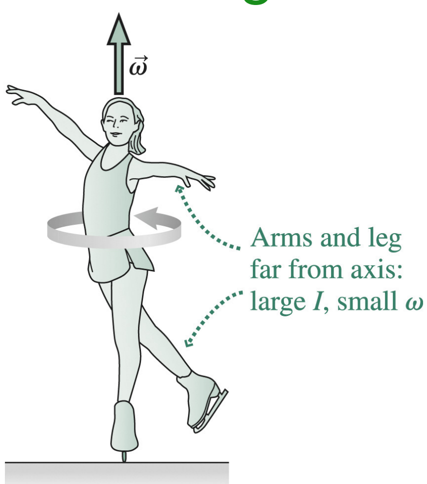


Newton's Law and Angular Momentum

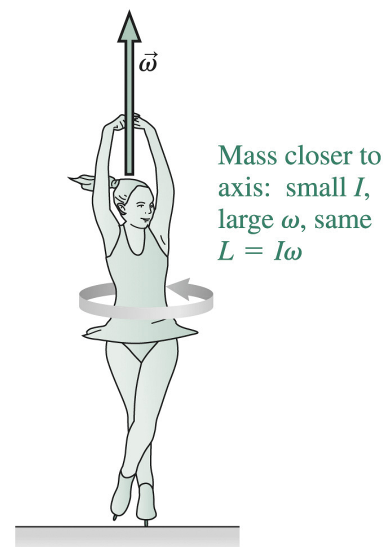
- In terms of angular momentum, the rotational analog of Newton's second law is

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

  - Therefore a system's angular momentum changes only if there's a non-zero net torque acting on the system.
  - If the net torque is zero, then angular momentum is conserved.
    - Changes in rotational inertia then result in changes in angular speed:

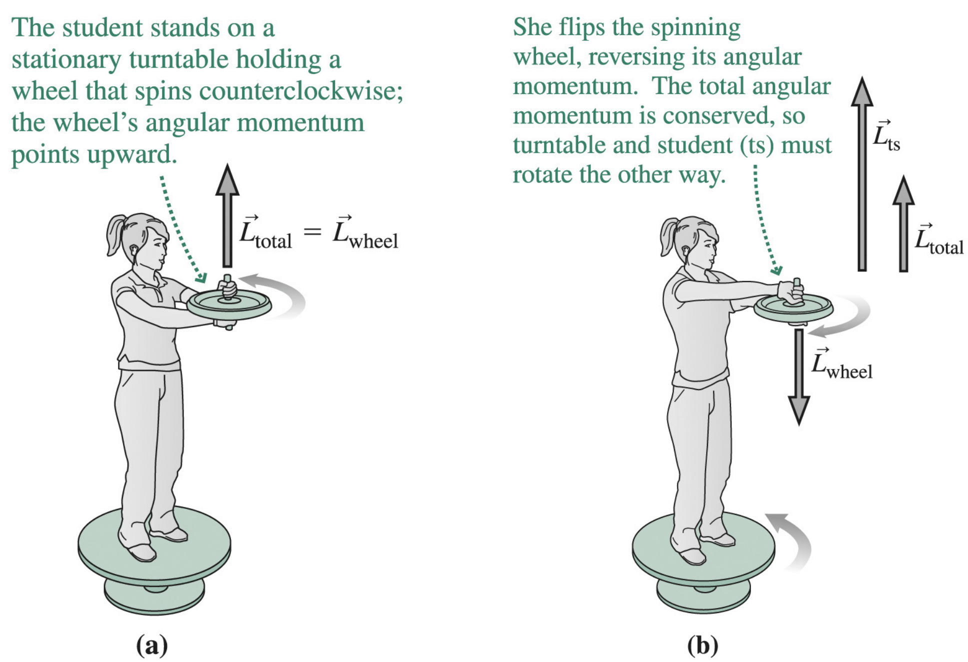


The skater's angular momentum is conserved, so her angular speed increases when she reduces her rotational inertia.



Conservation of Angular Momentum

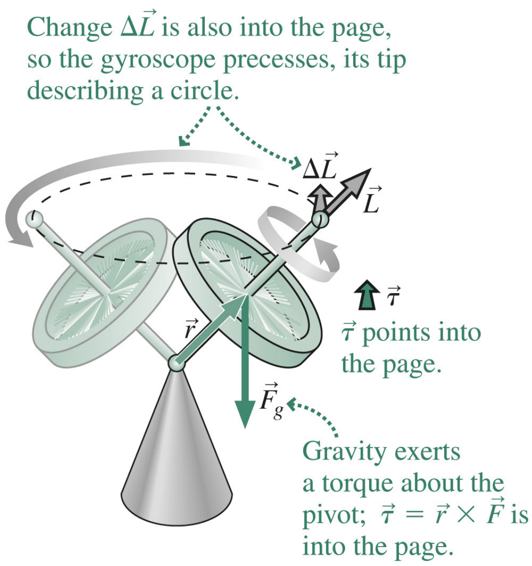
- The spinning wheel initially contains all the system's angular momentum.
- When the student turns the wheel upside down, she changes the direction of its angular momentum vector.
- Student and turntable rotate the other way to keep the total angular momentum unchanged.



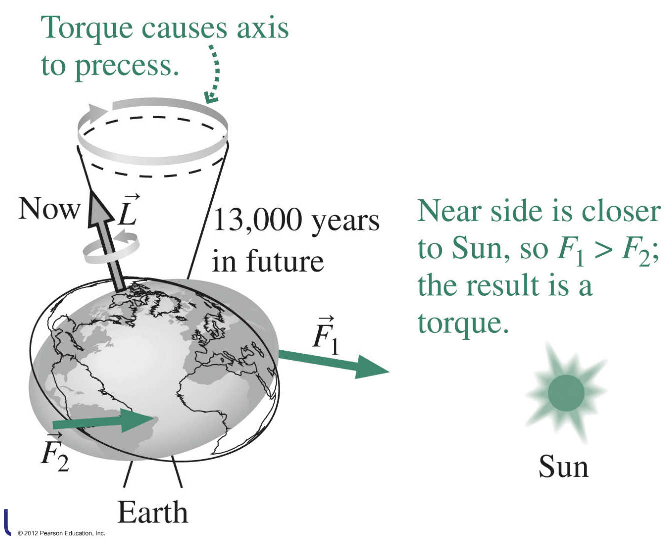
Precession

- Precession is a three-dimensional phenomenon involving rotational motion.
  - Precession occurs when a torque acts on a rotating object, changing the direction but not the magnitude of its angular momentum vector.
  - As a result the rotation axis undergoes circular motion:

Precession of a gyroscope

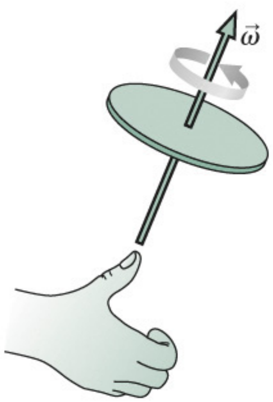


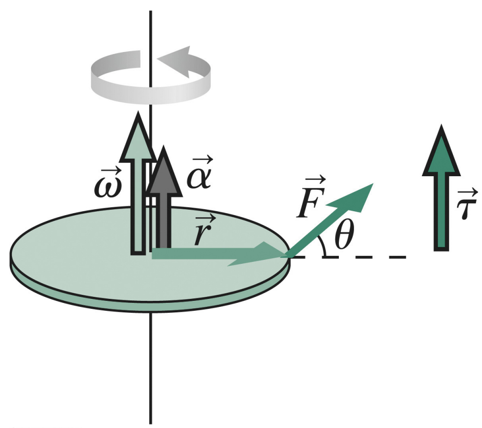
Precession slowly changes the direction of Earth's rotation axis

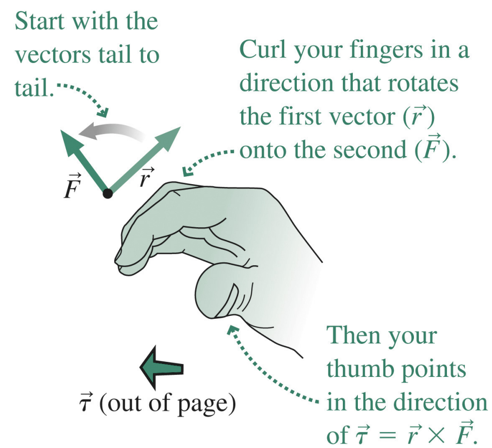


Summary

- Angular quantities are vectors whose direction is generally associated with the direction of the rotation axis.
  - Specifically, direction is given by the right-hand rule.
  - The vector cross product provides a compact representation for torque and angular momentum.





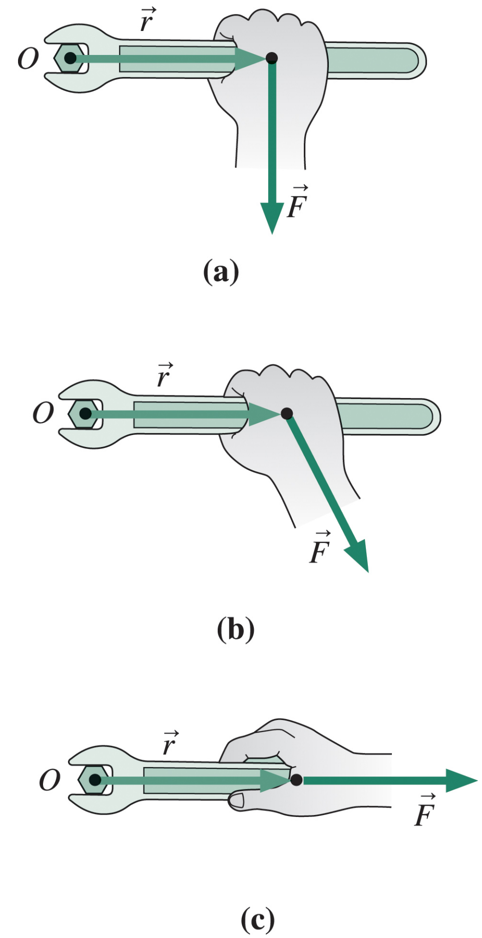


  - Angular momentum is the rotational analog of linear momentum: $\vec{L} = \vec{r} \times \vec{p}$; with symmetry, $\vec{L} = I\vec{\omega}$.
  - In the absence of a net external torque, a system's angular momentum is conserved.

Clicker Question 1

- The forces in the figures all have the same magnitude. Which force produces zero torque?

A. The force in figure (a)
B. The force in figure (b)
C. The force in figure (c)
D. All of the forces produce torque.



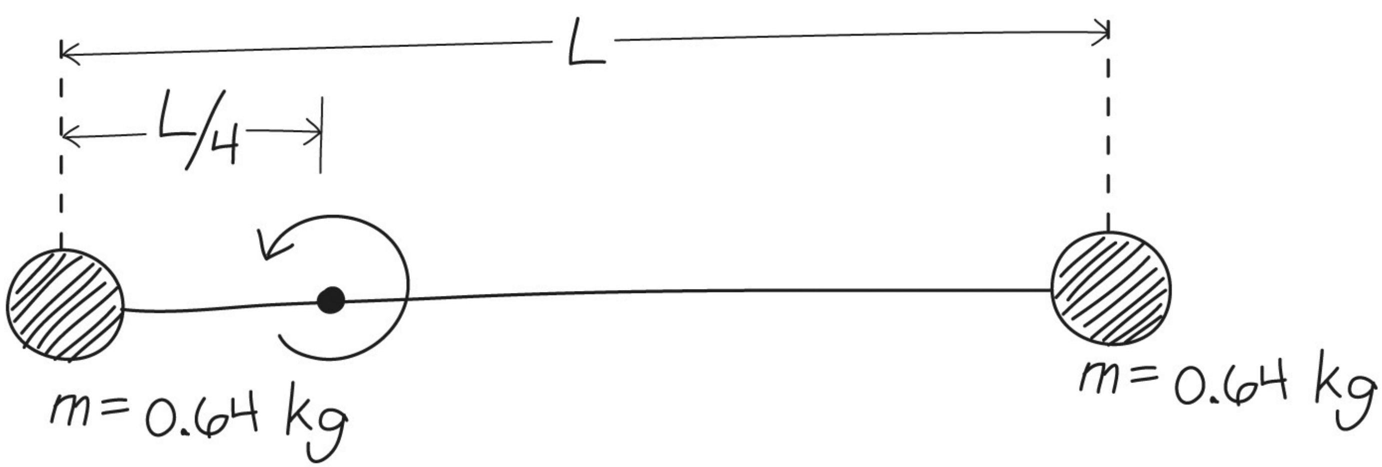
Clicker Question 2

- Consider the dumbbell in the figure. How would its rotational inertia change if the rotation axis were at the center of the rod?

A. *I* would increase.

B. *I* would decrease.

C. *I* would remain the same.



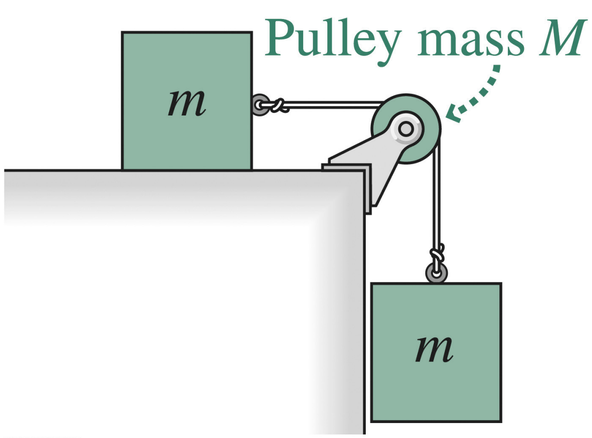
Clicker Question 3

- The figure shows two identical masses *m* connected by a string that passes over a frictionless pulley whose mass is *not* negligible. One mass rests on a frictionless table while the other hangs vertically, as shown. Compare the force of tension in the horizontal and vertical sections of the string.

A. The tension in the horizontal section is greater.

B. The tension in the vertical section is greater.

C. The tensions in the two sections are equal.



Clicker Question 5

- Two forces of equal magnitude are applied to a door at the doorknob. The first force is perpendicular to the door, while the second force is applied at an angle 20° to the plane of the door. Which force produces the greater torque?

A. The first force
B. The second force
C. Both forces produce the same non-zero torque.
D. Both forces produce zero torque.

Clicker Question 6

- A solid sphere subjected to a net torque will experience

A. an angular acceleration.
B. a linear acceleration.
C. a constant angular velocity.
D. a changing moment of inertia.
E. a linear acceleration and an angular acceleration.

Clicker Question 8

- Two uniform solid spheres have the same radius, but one is three times more massive than the other. What is the ratio of the larger moment of inertia to that of the smaller moment of inertia?

A. 3
B. 6
C. 9
D. 16
E. 27