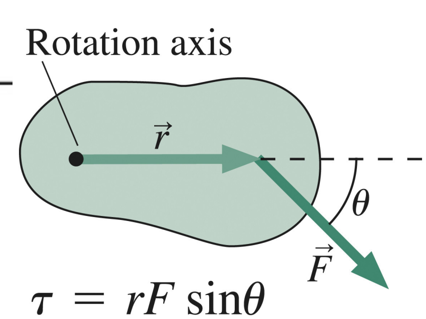
Reference and Reiteration

- Rotational motion in one dimension is exactly analogous to linear motion in one dimension.
  - Analogies between rotational and linear quantities:

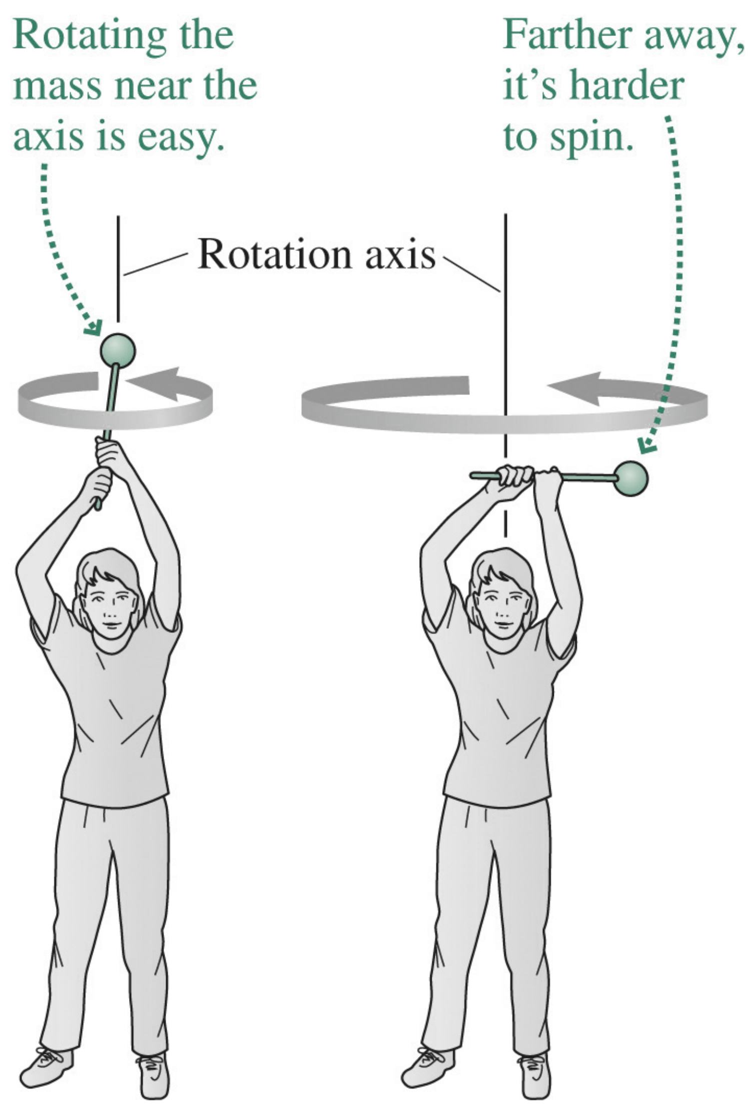
Linear Quantity or Equation	Angular Quantity or Equation	Relation Between Linear and Angular Quantities
Position x	Angular position θ	
Speed $v = dx/dt$	Angular speed $\omega = d\theta/dt$	$v = \omega r$
Acceleration a	Angular acceleration α	$a_t = \alpha r$
Mass m	Rotational inertia I	$I = \int r^2\,dm$
Force F	Torque τ	$\tau = rF\sin\theta$
Kinetic energy $K_{\text{trans}} = \frac{1}{2}mv^2$	Kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$	
Newton's second law (constant mass or rotational inertia):		
$F = ma$	$\tau = I\alpha$	

Torque, τ



Rotational Inertia and the Analog of Newton's Law

- **Rotational inertia** I (or moment of inertia) is the rotational analog of mass.
  - Rotational inertia depends on the distribution of mass and its distance from the rotation axis, similar to center of mass.



- Rotational acceleration, torque, and rotational inertia combine to give the rotational analog of Newton's second law $F = ma$

$$\boxed{\tau = I\alpha}$$

(or, more properly with vectors)

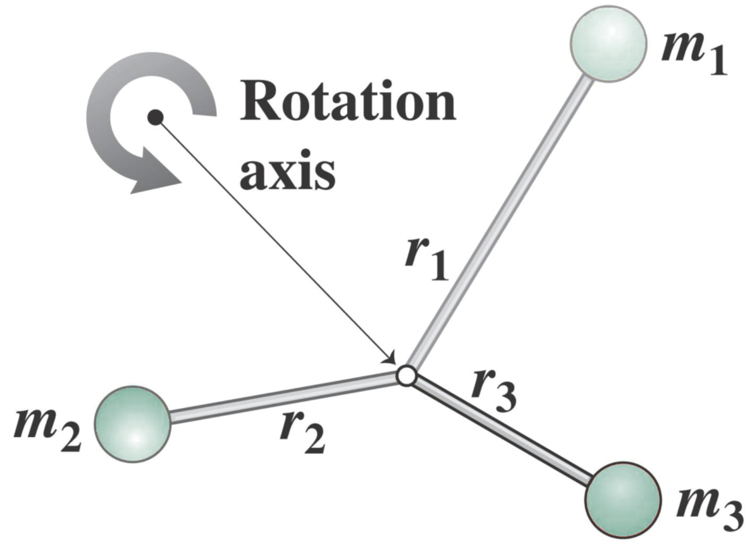
$$\vec{\tau} = I\vec{\alpha}$$

like $\vec{F} = m\vec{a}$

Calculating Rotational Inertia

- For a single point mass *m*, rotational inertia is the product of mass with the square of the distance *r* from the rotation axis:

$$I = mr^2$$

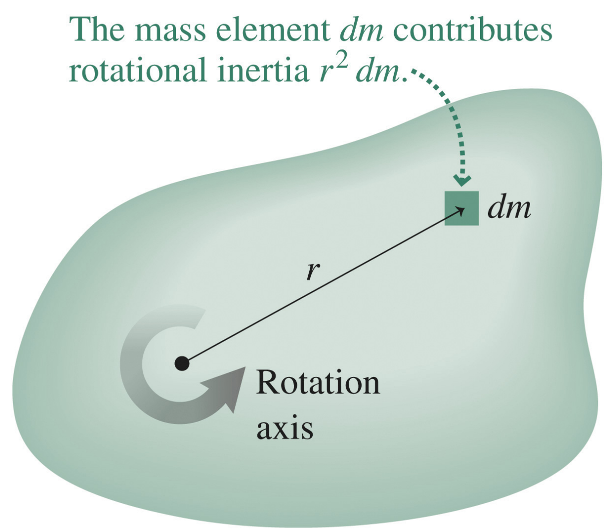


- For a system of discrete masses, the rotational inertia is the sum of the rotational inertias of the individual masses:

$$\boxed{I = \sum m_i r_i^2}$$

- For continuous matter, the rotational inertia is given by an integral over the distribution of matter:

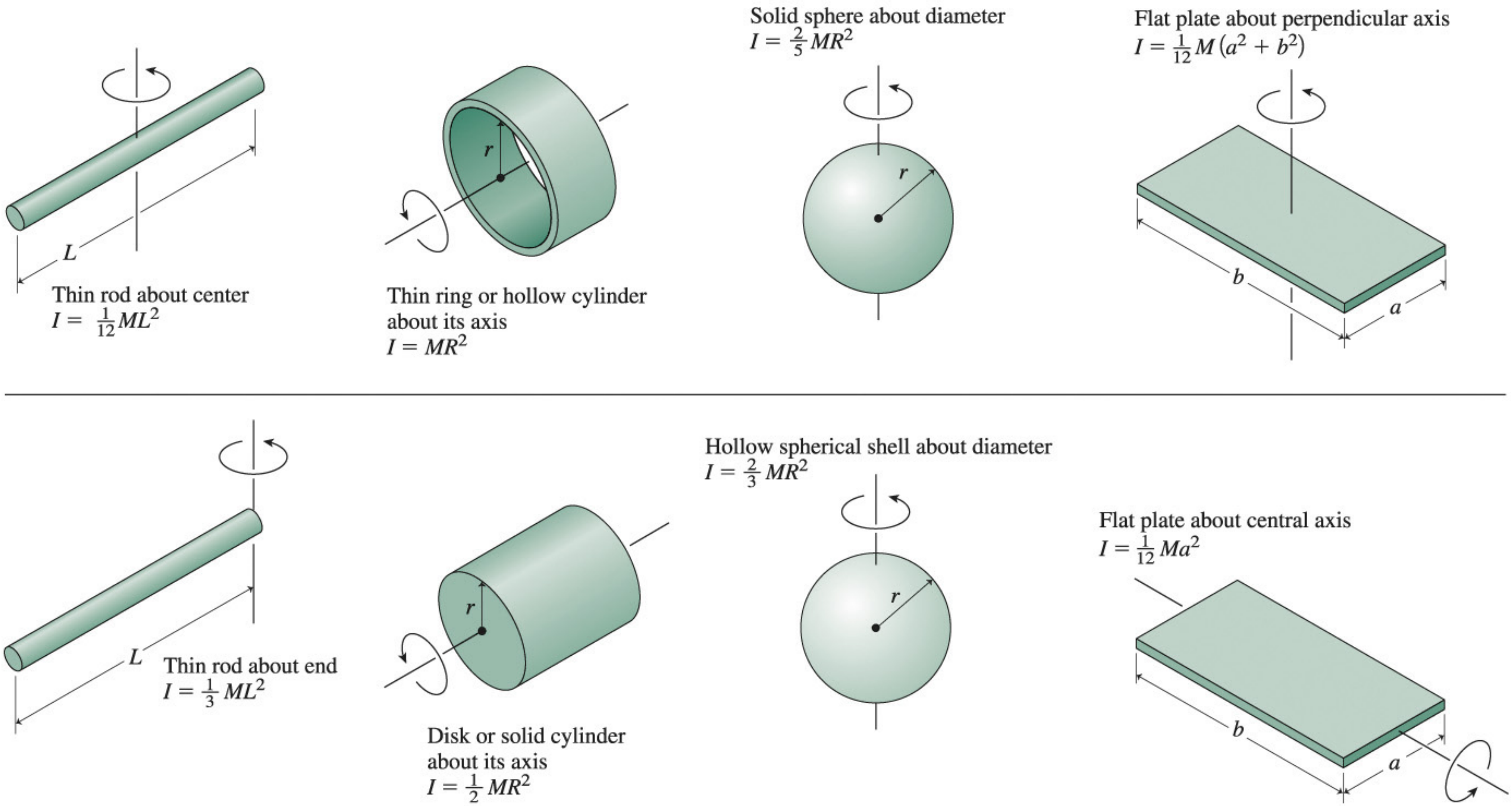
$$\boxed{I = \int r^2\, dm}$$



Similar to center of mass: $\vec{r}_{\text{cm}} = \dfrac{\int \vec{r}\, dm}{M}$

Some Rotational Inertias of Simple Objects

- We really do need to use calculus to figure out rotational inertias of most simple (three-dimensional) geometrical objects

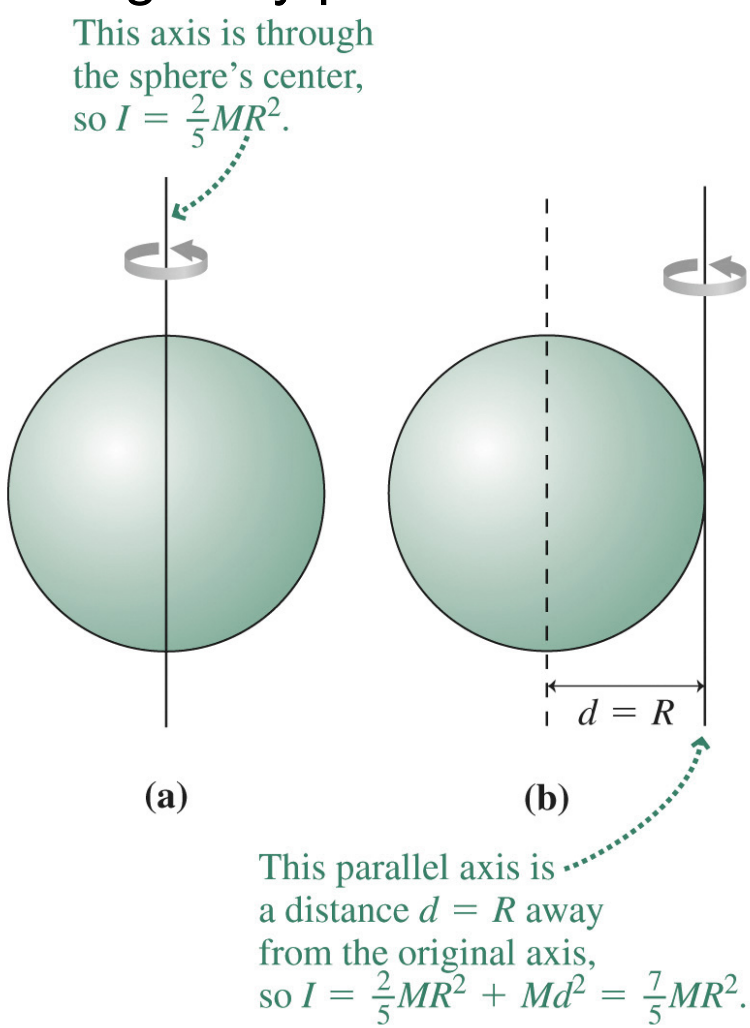


Parallel Axis Theorem

- If we know the rotational inertia I_{cm} about an axis through the center of mass of a body, the **parallel-axis theorem** allows us to calculate the rotational inertia I through any parallel axis.
- The parallel-axis theorem states that

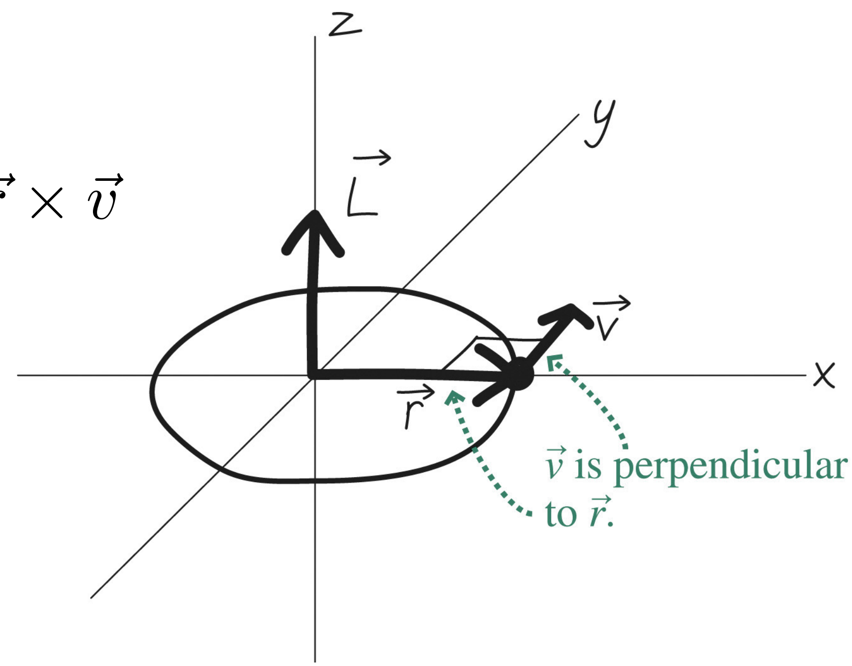
$$\boxed{I = I_{\rm cm} + Md^2}$$

where d is the distance from the center-of-mass axis to the parallel axis and M is the total mass of the object.



Angular Momentum

- For a single particle, angular momentum $\vec{L}$ is a vector given by the cross product of the displacement vector $\vec{r}$ from the rotation axis with the linear momentum of the particle: $$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

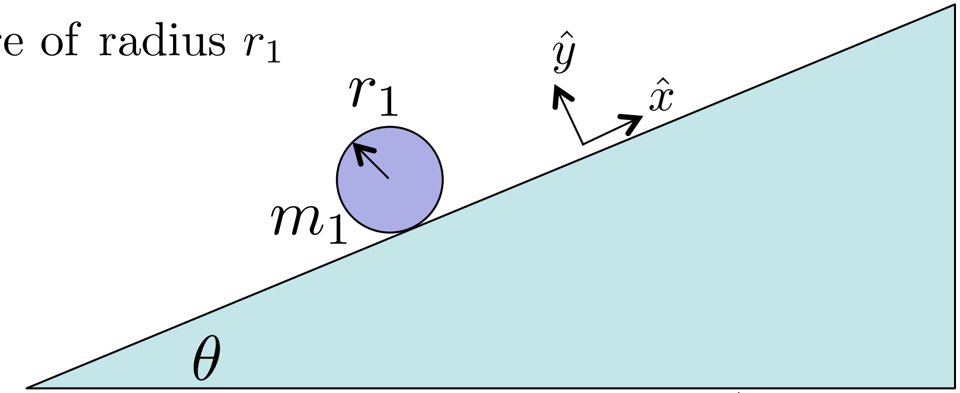


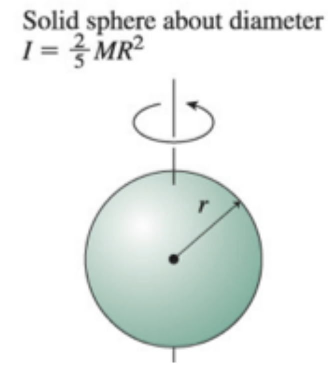
- Yes, this looks a lot like $\vec{\omega} = \vec{r} \times \vec{v}$
- For solid objects where we know the rotational inertia, we have another parallel to linear motion:

$$\vec{p} = m\vec{v} \qquad \rightarrow \qquad \boxed{\vec{L} = I\vec{\omega}}$$

Problem: Solid Sphere Rolling Down a Ramp

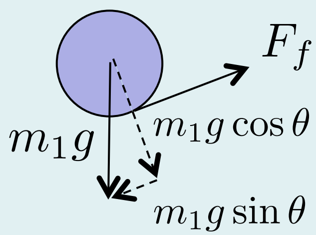
m_1 is a sphere of radius r_1





What is the linear acceleration? (no slipping)

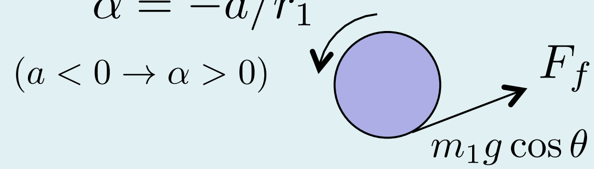
Linear motion



$$\sum F_x = F_f - m_1 g\sin\theta = m_1 a$$

Rotational motion

$$\alpha = -a/r_1$$



$$\tau = F_f r_1 \qquad I = \frac{2}{5}m_1 r_1^2$$

$$\sum \tau = F_f r_1 = I\alpha = \left(\frac{2}{5}m_1 r_1^2\right)\left(-\frac{a}{r_1}\right) = -\frac{2}{5}m_1 r_1 a$$

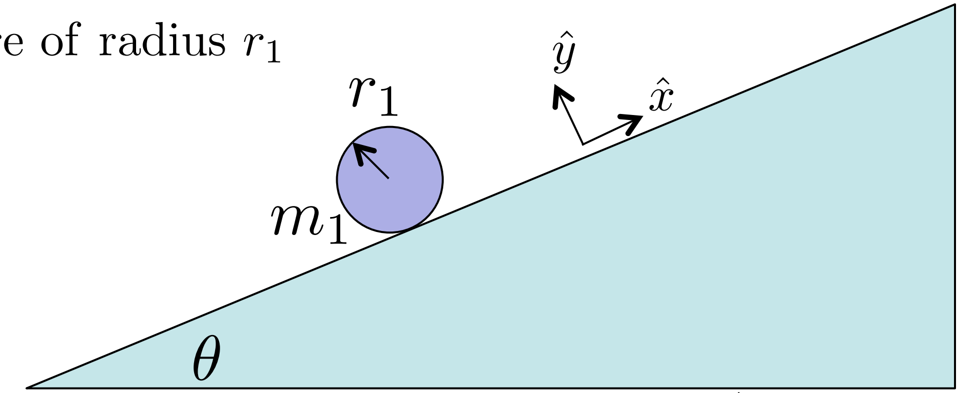
$$F_f = -\frac{2}{5}m_1 a$$

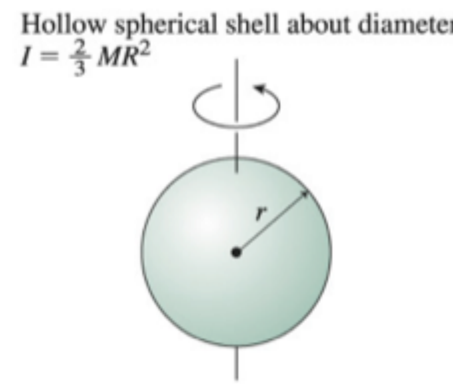
$$-\frac{2}{5}m_1 a - m_1 g\sin\theta = m_1 a$$

$$\boxed{a = -\frac{5}{7}g\sin\theta}$$

Problem 1: Hollow Sphere Rolling Down a Ramp

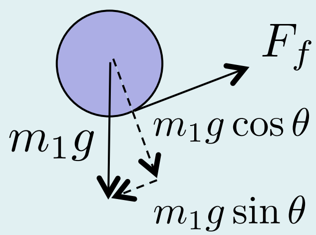
m_1 is a sphere of radius r_1





What is the linear acceleration? (no slipping)

Linear motion

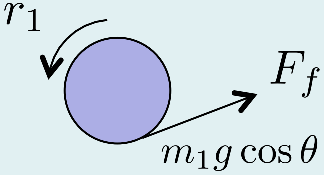


$$\sum F_x = F_f - m_1 g \sin\theta = m_1 a$$

Rotational motion

$\alpha = -a/r_1$

$(a < 0 \rightarrow \alpha > 0)$



$$\tau = F_f r_1 \qquad I = \frac{2}{3} m_1 r_1^2$$

$$\sum \tau = F_f r_1 = I\alpha = \left(\frac{2}{3} m_1 r_1^2\right)\left(-\frac{a}{r_1}\right) = -\frac{2}{3} m_1 r_1 a$$

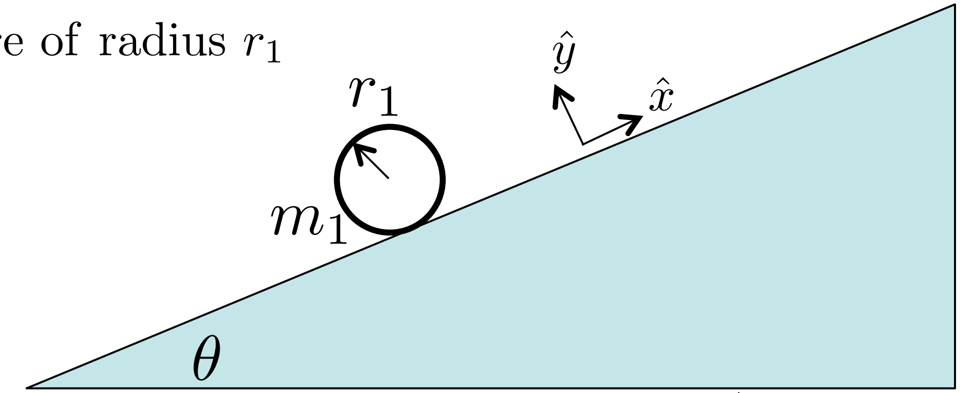
$$F_f = -\frac{2}{3} m_1 a$$

$$-\frac{2}{3} m_1 a - m_1 g \sin\theta = m_1 a$$

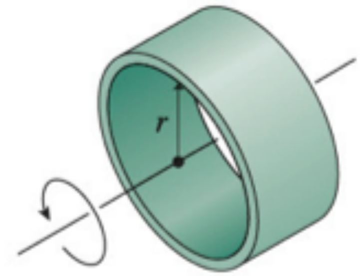
$$\boxed{a = -\frac{3}{5} g \sin\theta}$$

Problem 1: Hollow Cylinder Rolling Down a Ramp

m_1 is a sphere of radius r_1

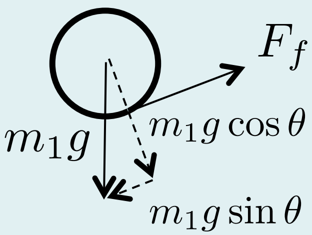


What is the linear acceleration? (no slipping)



Thin ring or hollow cylinder about its axis
$I = MR^2$

Linear motion



$$\sum F_x = F_f - m_1 g \sin\theta = m_1 a$$

Rotational motion

$\alpha = -a/r_1$

$(a < 0 \to \alpha > 0)$

$$\tau = F_f r_1 \qquad I = m_1 r_1^2$$

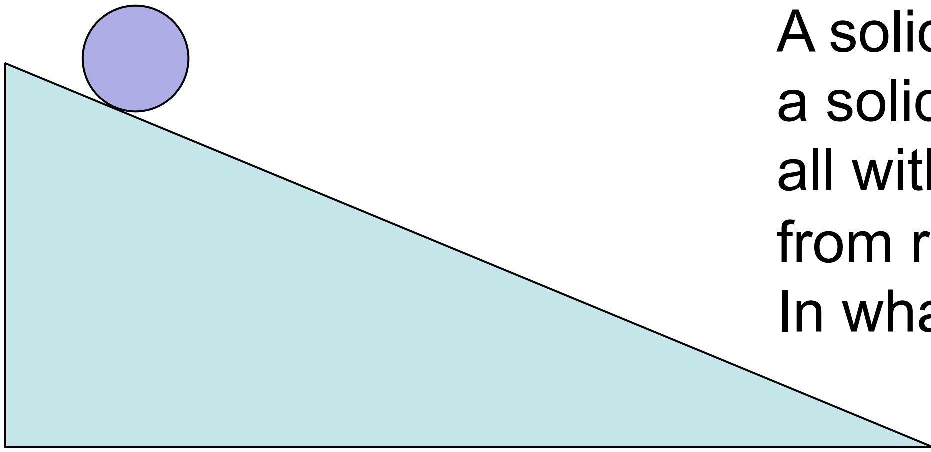
$$\sum \tau = F_f r_1 = I\alpha = (m_1 r_1^2)\left(-\frac{a}{r_1}\right) = -m_1 r_1 a$$

$$F_f = -m_1 a$$

$$-m_1 a - m_1 g \sin\theta = m_1 a$$

$$\boxed{a = -\frac{1}{2} g \sin\theta}$$

Quiz: Rolling Race

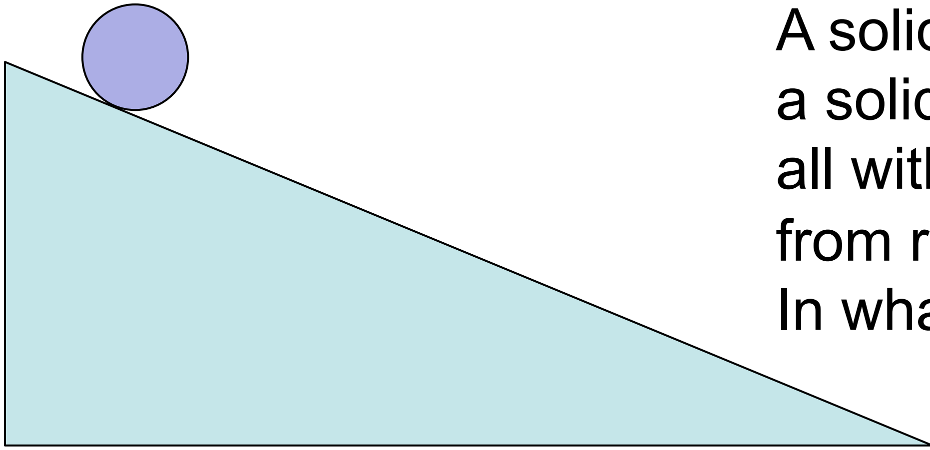


A solid sphere, a hollow sphere, a solid cylinder, and a hollow cylinder all with the same radii roll down a ramp from rest (no slipping).
In what order do they reach the bottom?

$$I_{\text{solid sphere}} = \frac{2}{5}MR^2 \qquad I_{\text{hollow sphere}} = \frac{2}{3}MR^2 \qquad I_{\text{solid cylinder}} = \frac{1}{2}MR^2 \qquad I_{\text{hollow cylinder}} = MR^2$$

A. Solid sphere, hollow sphere, solid cylinder, hollow cylinder
B. Hollow cylinder, solid cylinder, hollow sphere, solid sphere
C. It depends on their relative masses

Quiz: Rolling Race



A solid sphere, a hollow sphere, a solid cylinder, and a hollow cylinder all with the same radii roll down a ramp from rest (no slipping).
In what order do they reach the bottom?

$$I_{\text{solid sphere}} = \frac{2}{5}MR^2 \qquad I_{\text{hollow sphere}} = \frac{2}{3}MR^2 \qquad I_{\text{solid cylinder}} = \frac{1}{2}MR^2 \qquad I_{\text{hollow cylinder}} = MR^2$$

A. Solid sphere, hollow sphere, solid cylinder, hollow cylinder

We calculated $\boxed{a = -\frac{5}{7} g \sin\theta}$ for the solid sphere

$\boxed{a = -\frac{3}{5} g \sin\theta}$ for the hollow sphere (smaller)

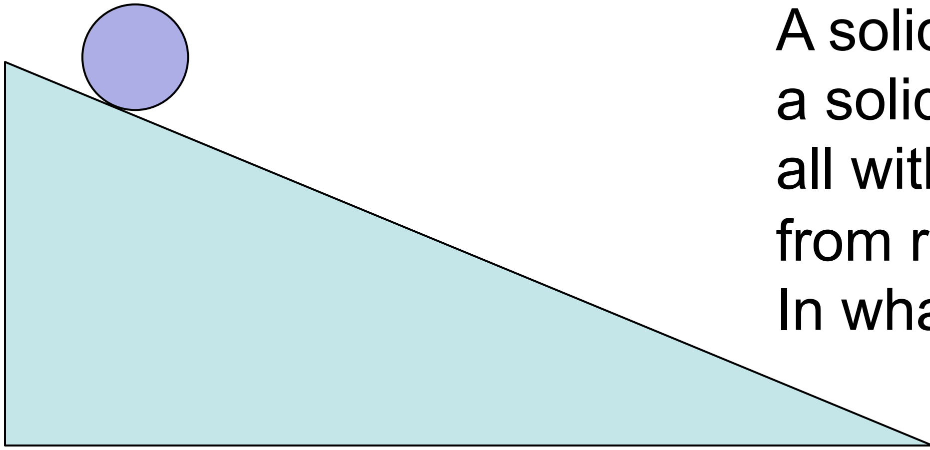
$\boxed{a = -\frac{1}{2} g \sin\theta}$ for the hollow cylinder (even smaller)

Note that the acceleration doesn't depend on the mass!

Lower moment of inertia => larger rolling acceleration

Higher moment of inertia => smaller rolling acceleration

Quiz: Rolling Race



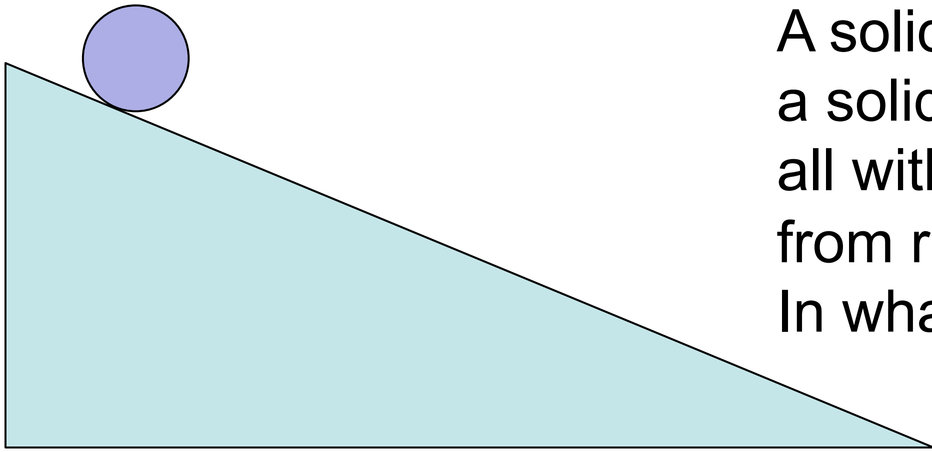
A solid sphere, a hollow sphere, a solid cylinder, and a hollow cylinder all with the same radii roll down a ramp from rest (no slipping).

In what order (fastest to slowest) are their velocities at ramp bottom?

$$I_{\text{solid sphere}} = \frac{2}{5}MR^2 \qquad I_{\text{hollow sphere}} = \frac{2}{3}MR^2 \qquad I_{\text{solid cylinder}} = \frac{1}{2}MR^2 \qquad I_{\text{hollow cylinder}} = MR^2$$

A. Solid sphere, hollow sphere, solid cylinder, hollow cylinder
B. Hollow cylinder, solid cylinder, hollow sphere, solid sphere
C. It depends on their relative masses

Quiz: Rolling Race



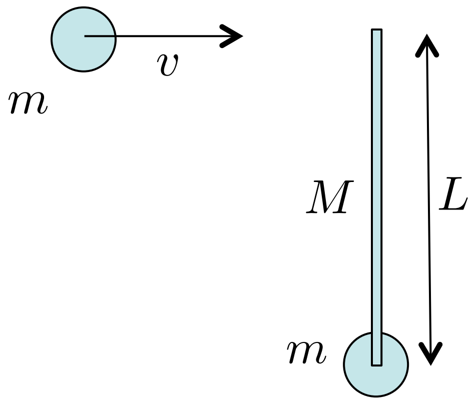
A solid sphere, a hollow sphere, a solid cylinder, and a hollow cylinder all with the same radii roll down a ramp from rest (no slipping).
In what order (fastest to slowest) are their velocities at ramp bottom?

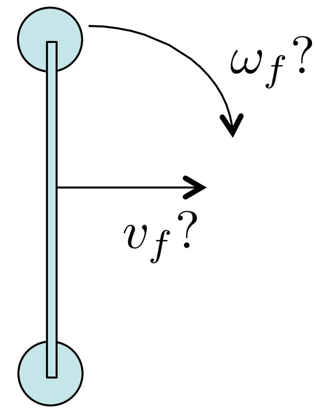
$$I_{\text{solid sphere}} = \frac{2}{5}MR^2 \qquad I_{\text{hollow sphere}} = \frac{2}{3}MR^2 \qquad I_{\text{solid cylinder}} = \frac{1}{2}MR^2 \qquad I_{\text{hollow cylinder}} = MR^2$$

A. Solid sphere, hollow sphere, solid cylinder, hollow cylinder

Remember for linear motion, $v^2 = v_0^2 + 2a(x - x_0)$
So the highest linear acceleration over the same distance also gives the largest final velocity.
The division of the (same) gravitational energy gained by each between rotational and kinetic energy is different for each object though.

Problem: Angular and Linear Momentum





A mass m with velocity v strikes perpendicularly to the end of a stick (at rest) of mass M and total length L that's already got another identical mass stuck to its other end.

Find the final linear and angular velocities of the system.

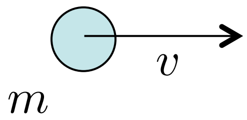
Conservation of linear momentum gives us the final velocity fairly quickly. Recall that linear momentum is defined as p=mv for each object.

$$\sum p_{\text{before}} = \sum p_{\text{after}} \qquad mv + (M+m)(0) = (M+2m)v_f$$

$$\Rightarrow \quad \boxed{v_f = \left(\frac{m}{M+2m}\right) v}$$

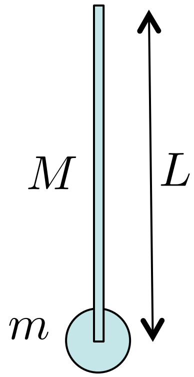
We can't use conservation of energy: this is a "splat" collision!

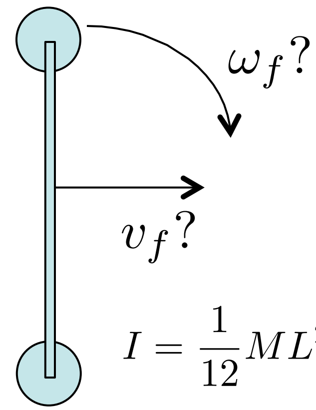
Problem: Angular and Linear Momentum



$$I = m\left(\frac{L}{2}\right)^2$$

$$\omega_0 = v/r = v/(L/2)$$





$$I = \frac{1}{12}ML^2 + 2m\left(\frac{L}{2}\right)^2$$

A mass m with velocity v strikes perpendicularly to the end of a stick (at rest) of mass M and total length L that's already got another identical mass stuck to its other end.

Find the final linear and angular velocities of the system.

The angular momentum calculation needs the rotational inertia for moving objects both before and after. They are shown above. Remember that rotational inertias can just be added together for multiple objects, and angular momentum is defined as $L = I\omega$

$$\sum L_{\text{before}} = \sum L_{\text{after}} \qquad \left[m\left(\frac{L}{2}\right)^2\right]\left[\frac{v}{(L/2)}\right] = \left[\frac{1}{12}ML^2 + 2m\left(\frac{L}{2}\right)^2\right]\omega_f$$

A little algebra gives $\boxed{\omega_f = \dfrac{6mv}{L(M+6m)}}$