Chapter 13

Periodic Motion

or

Oscillation

Oscillations – motions that repeat themselves

Examples:

- ✓ swinging chandeliers
- ✓ boats bobbing at anchor
- ✓ guitar strings
- ✓ quartz crystals in wristwatches
- ✓ atoms in solids
- ✓ air molecules that transmit sound
- ✓ …

Oscillations in real word are usually *damped*; that is, the motion dies out gradually, transferring mechanical energy to thermal energy

Periodic motion

Any motion that repeat itself at regular intervals is called periodic motion or harmonic motion.

Key characteristics

- Period (T) – time required for one cycle of periodic motion. Units: s (seconds/cycle)
- Frequency (f) – number of oscillations that are completed each second. Units: s^{-1} (cycle/second) that is 1 hertz = 1 Hz = 1 oscillation per second = 1 s^{-1}

$$\boxed{f = \frac{1}{T}}$$

Periodic motion – Simple harmonic motion

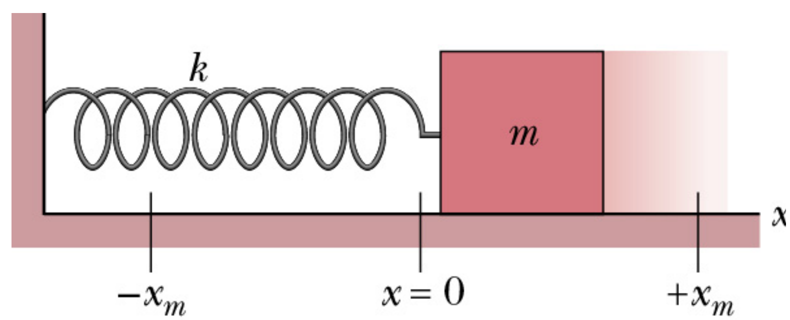
Simple harmonic motion (SHM) – a special case of a periodic motion when the displacement *x* of the particle (object) from the origin is given as a sinusoidal function of time

$$\boxed{x(t) = x_m \cos\left(\frac{2\pi}{T}t + \phi\right) = x_m \cos(\omega t + \phi)}$$

$$\overbrace{x(t)}^{\text{Displacemnt at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos\overbrace{(\underset{\text{Angular frequency}}{\omega}\,\underset{\text{Time}}{t} + \underset{\text{Phase constant or phase angle}}{\phi})}^{\text{Phase}}$$

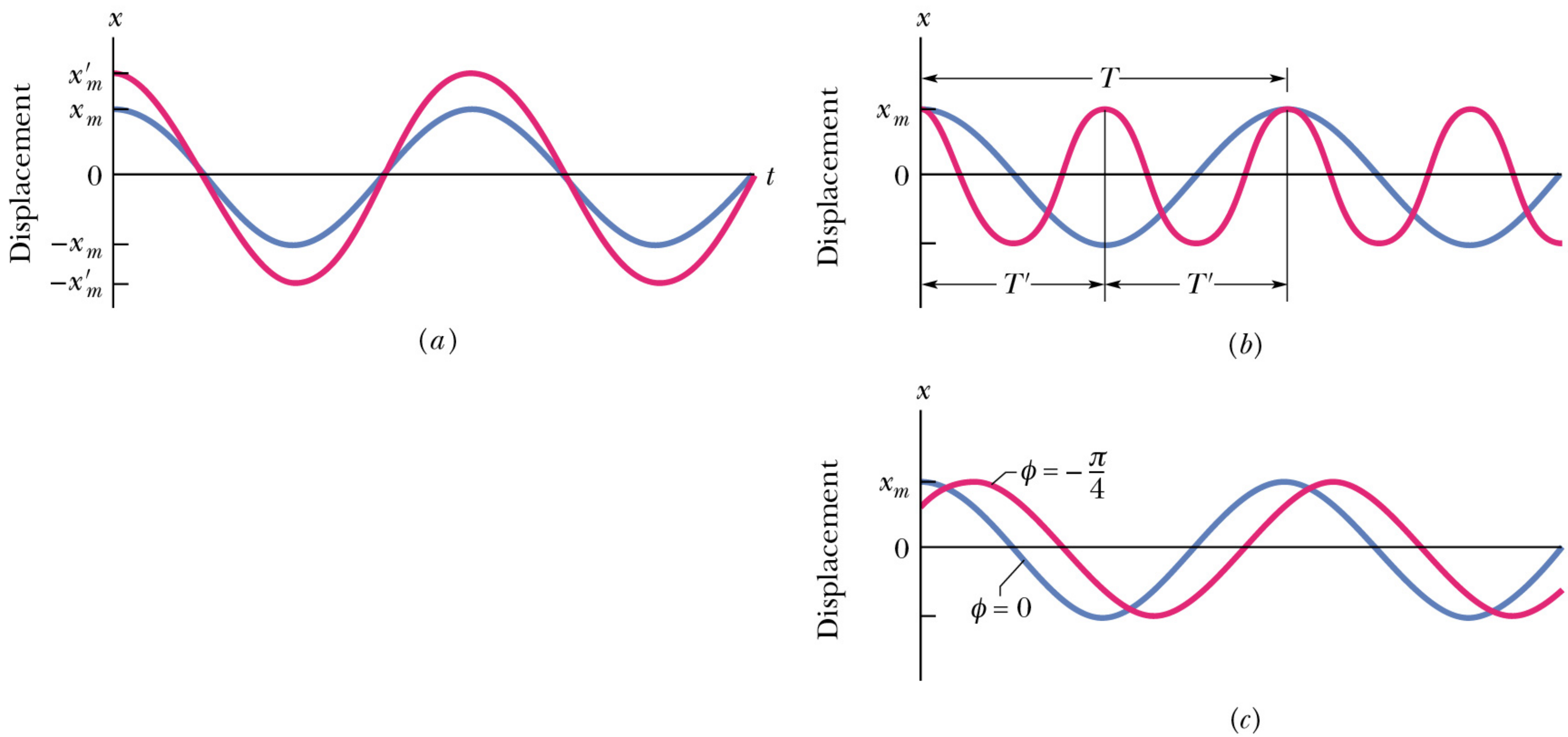
$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$$

SI unit for ω is the radian per second



Examples

$$x(t) = x_m \cos\left(\frac{2\pi}{T}t + \phi\right) = x_m \cos(\omega t + \phi)$$



The velocity and acceleration of SHM

The velocity of SHM

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

$$\boxed{v(t) = -\omega\, x_m \sin(\omega t + \phi)}$$

The acceleration of SHM

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[-\omega\, x_m \sin(\omega t + \phi)\right]$$

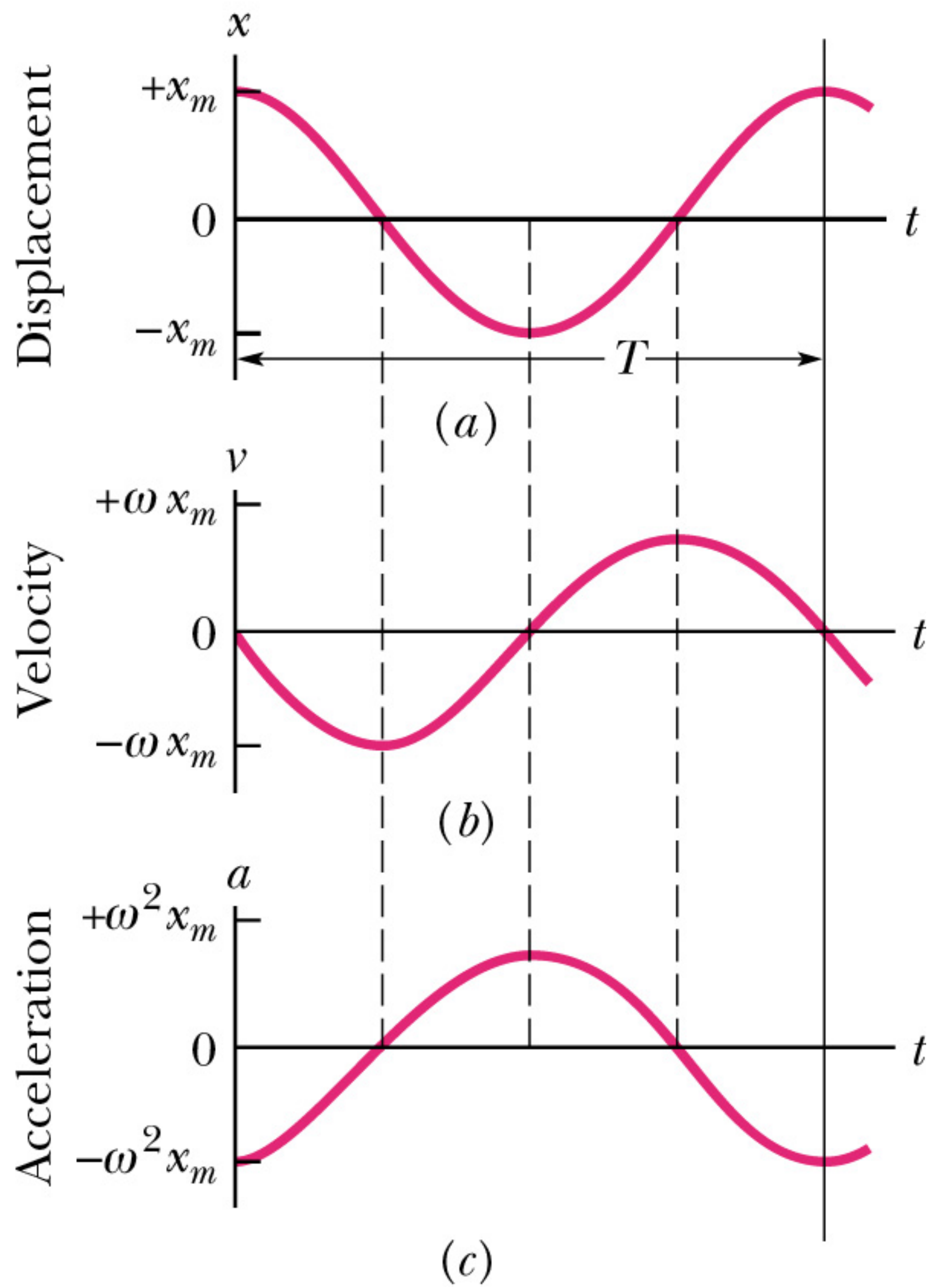
$$\boxed{a(t) = -\omega^2\, x_m \cos(\omega t + \phi) = -\omega^2 x(t)}$$

Examples

$$x(t) = x_m \cos(\omega t + \phi)$$

$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$



Checkpoint

In simple harmonic motion, the magnitude of the acceleration is:

A) constant

B) proportional to the displacement

C) inversely proportional to the displacement

D) greatest when the velocity is greatest

E) never greater than g

$$a(t) = -\omega^2 \, x_m \cos(\omega t + \phi)$$

The force law for simple harmonic motion

From Newton's second law

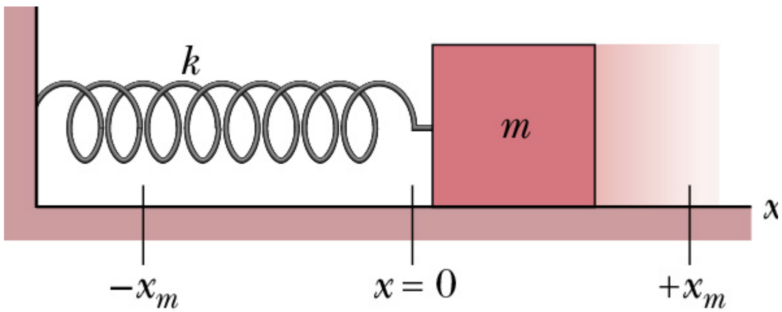
$$F = ma = -m(\omega^2 x) \qquad\qquad a(t) = -\omega^2\, x_m \cos(\omega t + \phi)$$

For a SHM – a restoring force that is proportional to the displacement but opposite in sign

Example: Hook's law for a spring

$$F = -kx \quad \text{then} \quad k = m\omega^2$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \qquad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$



example

Car springs

When a family of four people with a total mass of 200 kg step into their 1200-kg car, the car springs compress 3.0 cm

(a) What is the spring constant for the car springs, assuming that they act as a single spring?

(b) How far will the car lower if loaded with 300 kg?

(c) What are the period and frequency of the car after hitting a bump? Assume the shock absorbers are poor, so the car really oscillates up and down.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

example

Car springs (cont.)

$$F = kx \text{ then } k = \frac{F}{x} = \frac{(200.0\,\text{kg})\cdot(9.8\,\text{m/s}^2)}{0.03\,\text{m}} = 6.5\cdot 10^4\,\text{N/m}$$

if the car loaded with 300.0 kg

$$x = \frac{F}{k} = 4.5\cdot 10^{-2}\ \text{m}$$

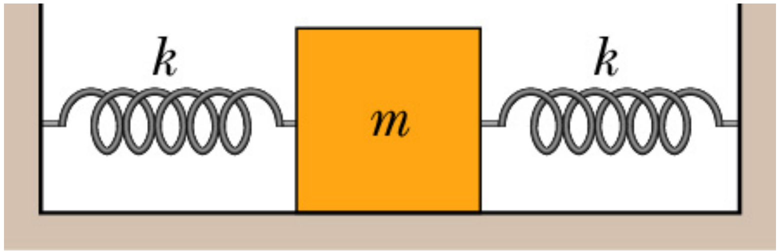
The period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{(1200.0 + 300.0)\text{kg}}{6.5\cdot 10^4\,\text{N/m}}} = 0.92s$

The frequency $f = \frac{1}{T} = 1.09Hz$

example

Two springs

Suppose that two springs in figure have different spring constant k_1 and k_2. What is the frequency of oscillations of the block?



$$F = -kx \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} \qquad f_2 = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}$$

$$F_{net} = -k_1 x - k_2 x = -(k_1 + k_2)x$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} \quad \text{and} \quad f = \sqrt{f_1^2 + f_2^2}$$

Energy in simple harmonic motion

The potential energy

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

The kinetic energy

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

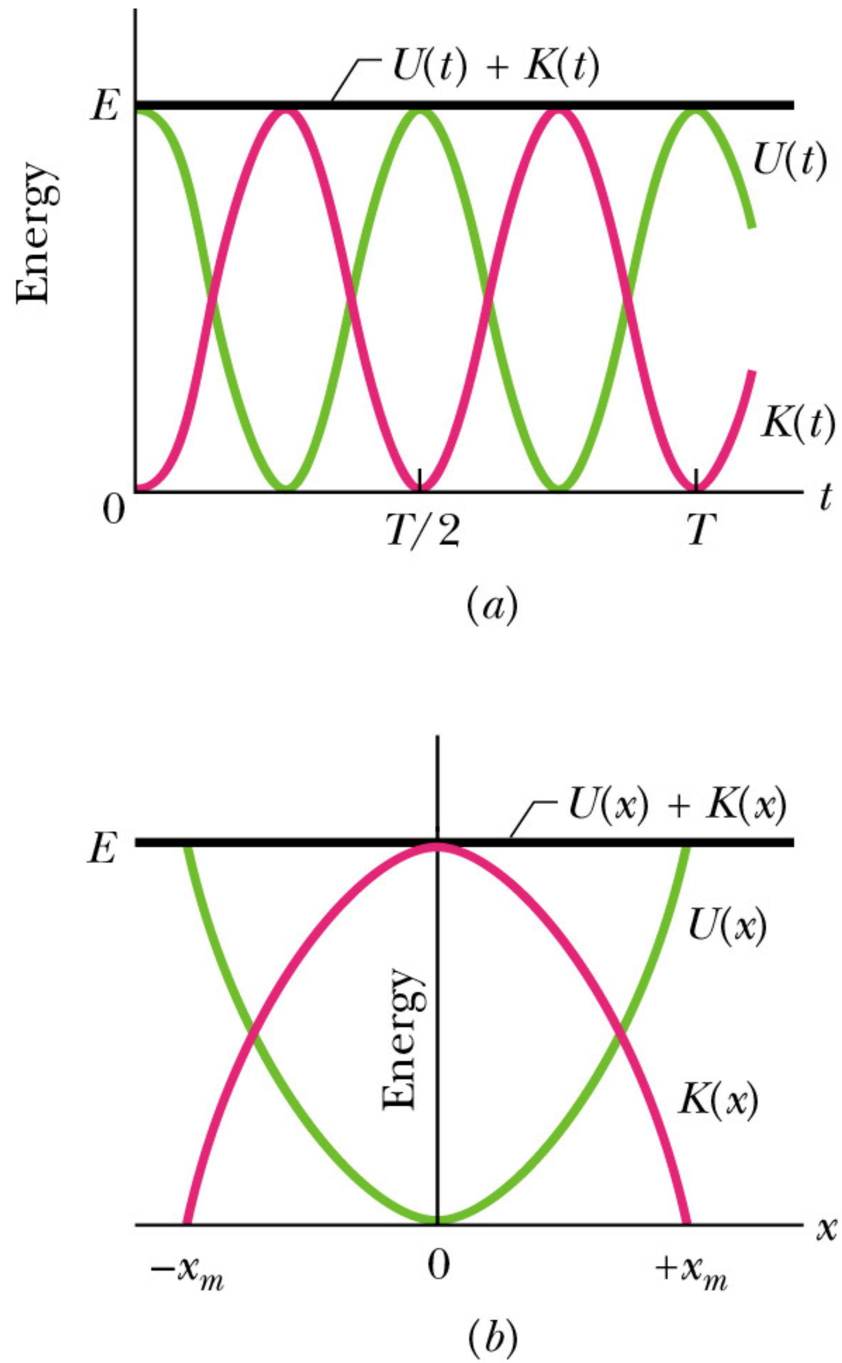
The total mechanical energy

$$E = U + K = \frac{1}{2}kx_m^2$$

Examples

$$U(t) = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$$

$$K(t) = \frac{1}{2}kx_m^2\sin^2(\omega t + \phi)$$



example

Problem: energy in SHM

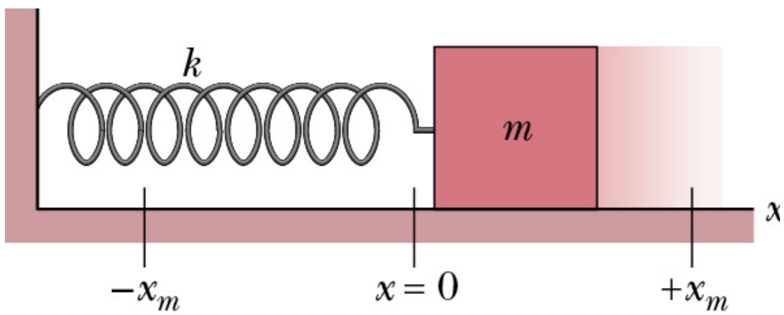
An object of mass *m* on a a horizontal frictionless surface is attached to a spring with spring constant *k*. The object is displaced from equilibrium of x_0 horizontally and given an initial velocity of v_0 back toward the equilibrium position.

(a) What is the frequency of the motion?

(b) What is the initial potential energy?

(c) What is the initial kinetic energy?

(d) What is the amplitude of the oscillation?



example

Problem: energy in SHM

Given : m, k, x_0, v_0

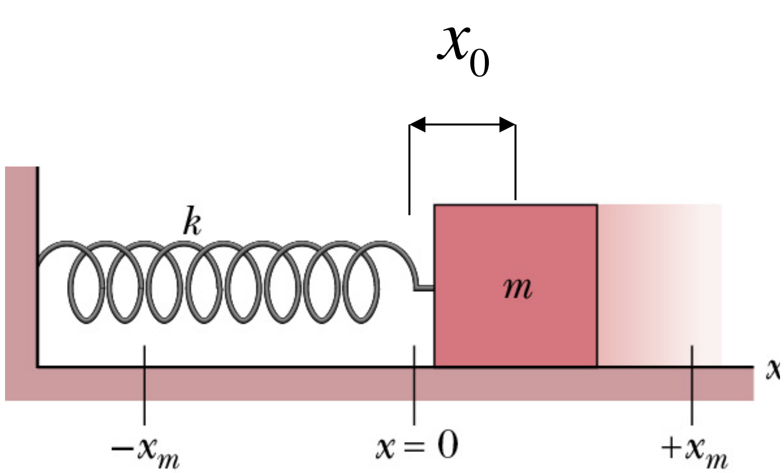
Find : f, U_0, K_0, x_m

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad f = \frac{1}{T}$$

$$U_0 = \frac{kx_0^2}{2}$$

$$K_0 = \frac{mv_0^2}{2}$$

$$E = U_0 + K_0 = \frac{kx_m^2}{2}$$



The simple pendulum

The restoring force

$$F = mg\sin(\theta)$$

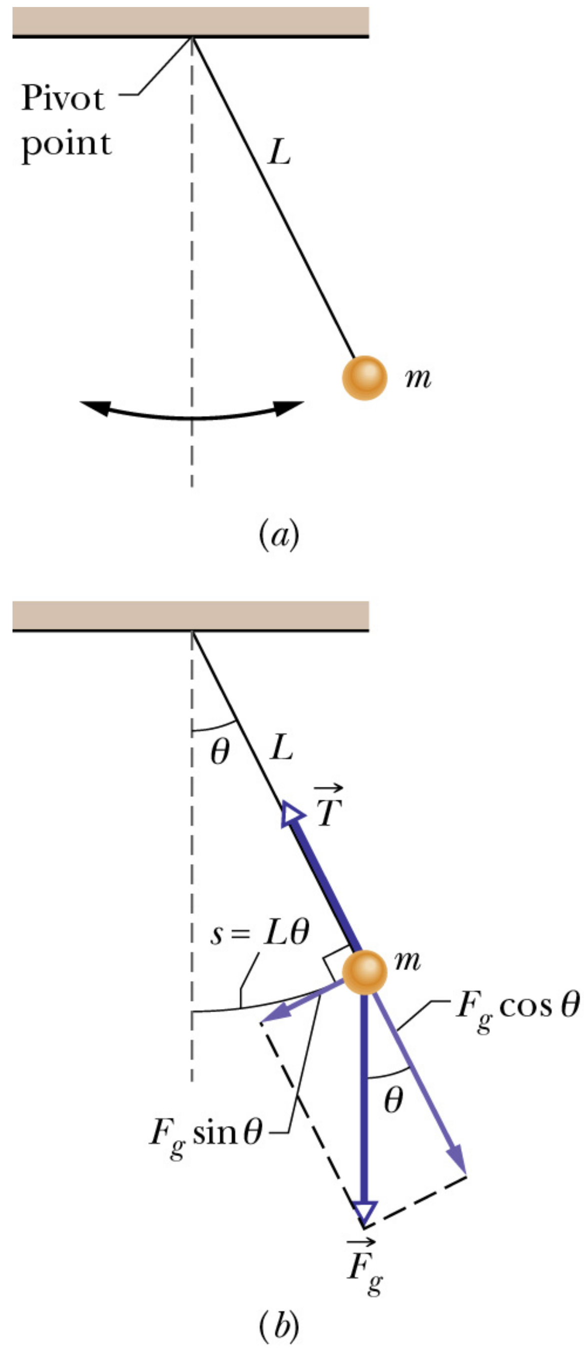
For small angles

$$F \approx mg\theta = mg\frac{s}{L} = \frac{mg}{L}s$$

Comparing to $F = -kx$

$$\boxed{k = \frac{mg}{L}}$$

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{L}{g}}}$$



example

An experiment to measure g

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

$$g = (2\pi)^2 \frac{L}{T^2}$$

for $L = 1.0\ \mathrm{m}$

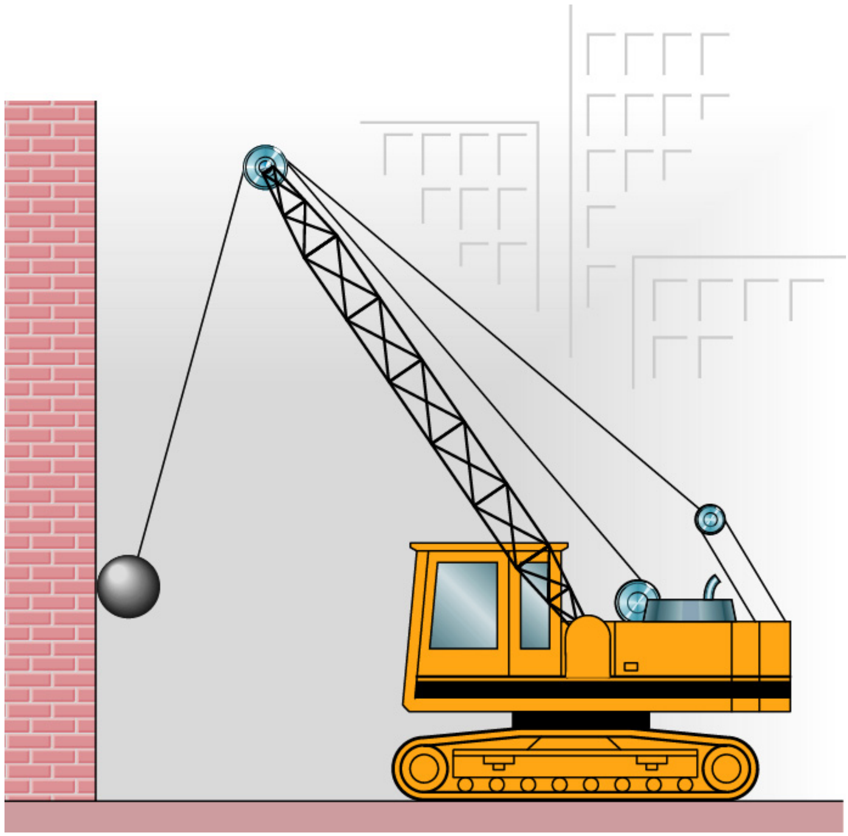
$$g = \frac{39.5}{T^2} m/s^2$$

Problem: Pendulum

A 2500 kg demolition ball sings from the end of the crane. The length of the swinging segment of cable is 17 m.

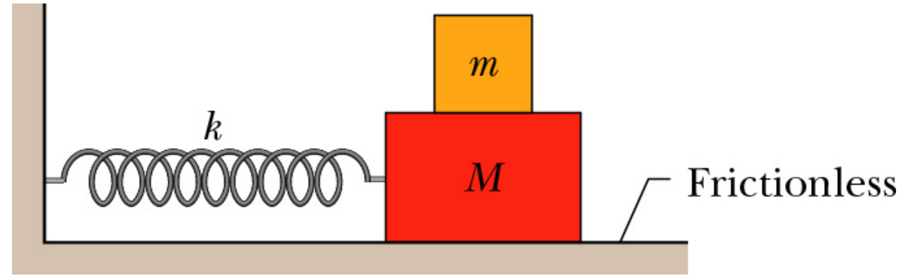
(a) Find the period of the swinging, assuming that the system can be treated as a simple pendulum

(b) Does the period depends on the ball's mass?



$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{17.0m}{9.8m/s^2}} = 8.3s$$

Example



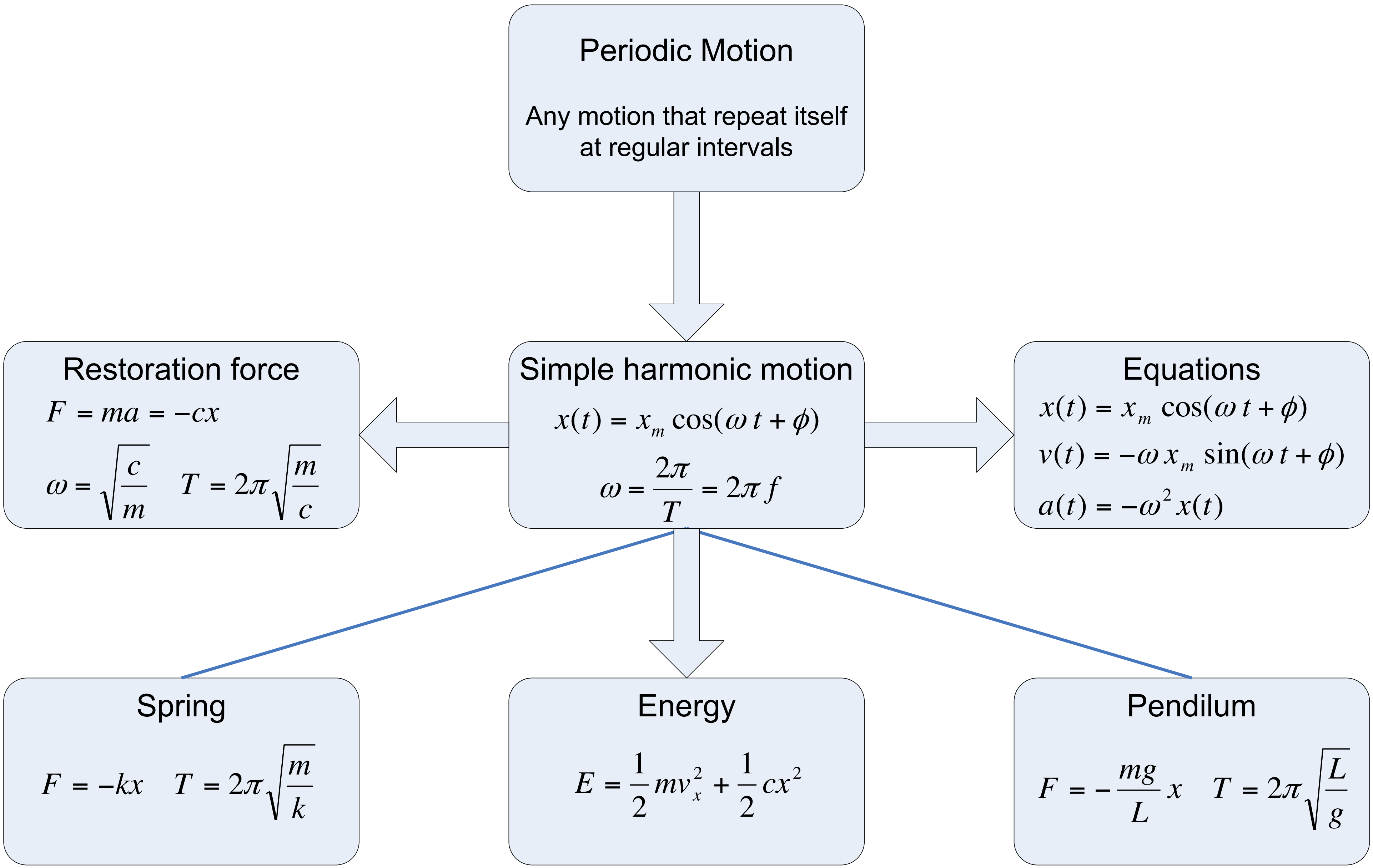
Two blocks (m=1.0 kg and M=10 kg) and a spring (k=200 N/m) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the spring-block's system puts the smaller block on the verge of slipping over the larger block?

$$F_{\max} = ma_{\max} = \mu_s mg$$

$$a_{\max} = \omega^2 x_{\max}$$

then $\omega^2 x_{\max} = \mu_s g$

and $x_{\max} = \dfrac{\mu_s g}{\omega^2} = \dfrac{\mu_s g(m+M)}{k} = 0.59m$


Periodic Motion
Any motion that repeat itself at regular intervals
Simple harmonic motion
$x(t) = x_m \cos(\omega t + \phi)$
$\omega = \frac{2\pi}{T} = 2\pi f$
Restoration force
$F = ma = -cx$
$\omega = \sqrt{\frac{c}{m}}$ $T = 2\pi\sqrt{\frac{m}{c}}$
Equations
$x(t) = x_m \cos(\omega t + \phi)$
$v(t) = -\omega x_m \sin(\omega t + \phi)$
$a(t) = -\omega^2 x(t)$
Spring
$F = -kx$ $T = 2\pi\sqrt{\frac{m}{k}}$
Energy
$E = \frac{1}{2} m v_x^2 + \frac{1}{2} c x^2$
Pendilum
$F = -\frac{mg}{L} x$ $T = 2\pi\sqrt{\frac{L}{g}}$

Damped Oscillations

Real-world always have some dissipative (frictional) forces.

The decrease in amplitude caused by dissipative forces is called by damping.

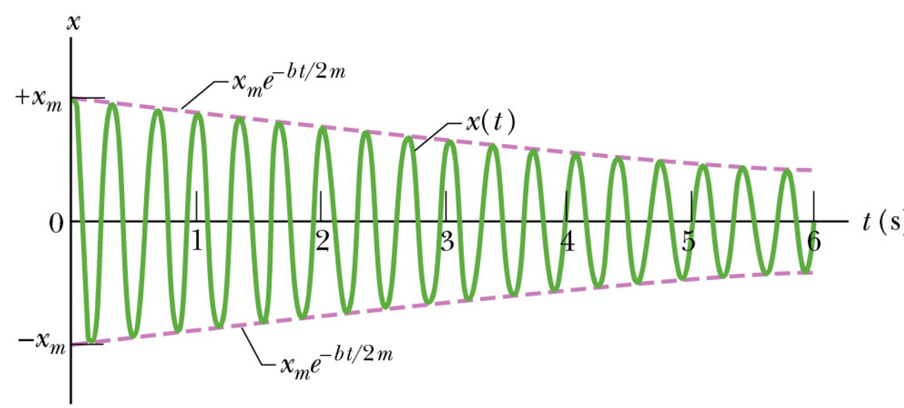
Quite often damping forces are proportional to the velocity of the oscillating object.

$$\boxed{\begin{aligned} F_x &= -kx - bv_x \\ m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx &= 0 \end{aligned}}$$

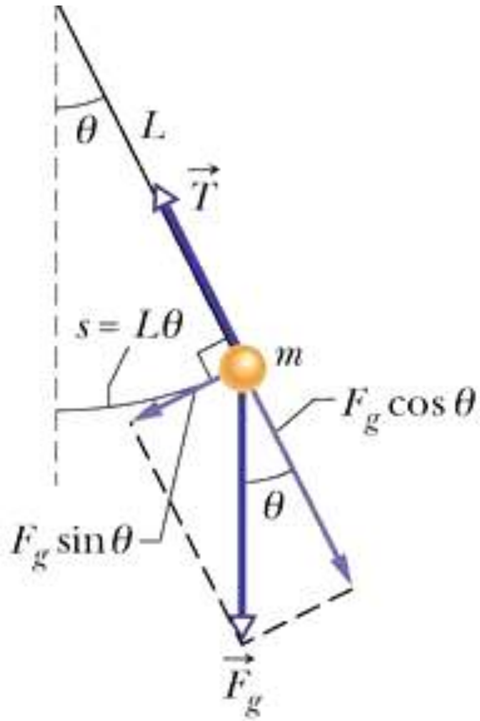
Second-order differential equation (we need two initial conditions)

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



A pendulum + periodic external force



Model: 3 forces

- gravitational force
- frictional force is proportional to velocity
- periodic external force

$$I\frac{d^2\theta}{dt^2} = \tau_g + \tau_f + \tau_{ext}$$

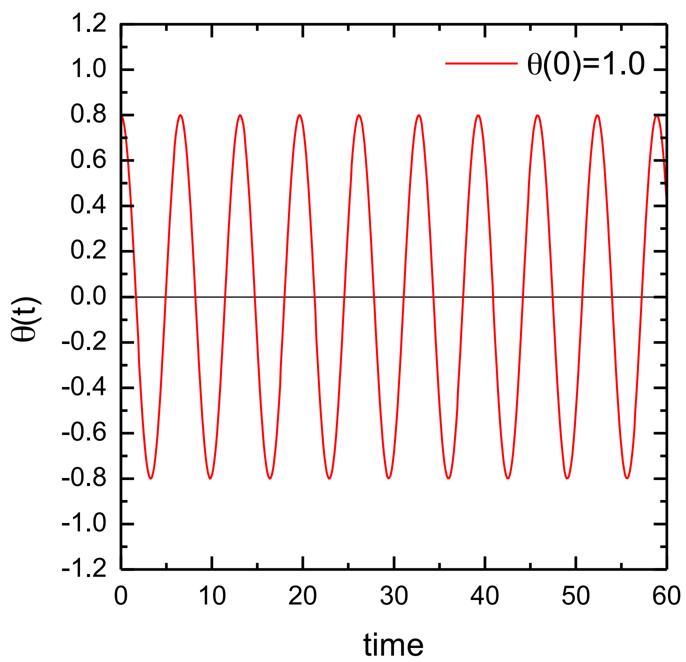
$$\tau_g = -mgL\sin(\theta), \quad \tau_f = -\beta\frac{d\theta}{dt},. \quad \tau_{ext} = F\cos(\omega t)$$

example 1

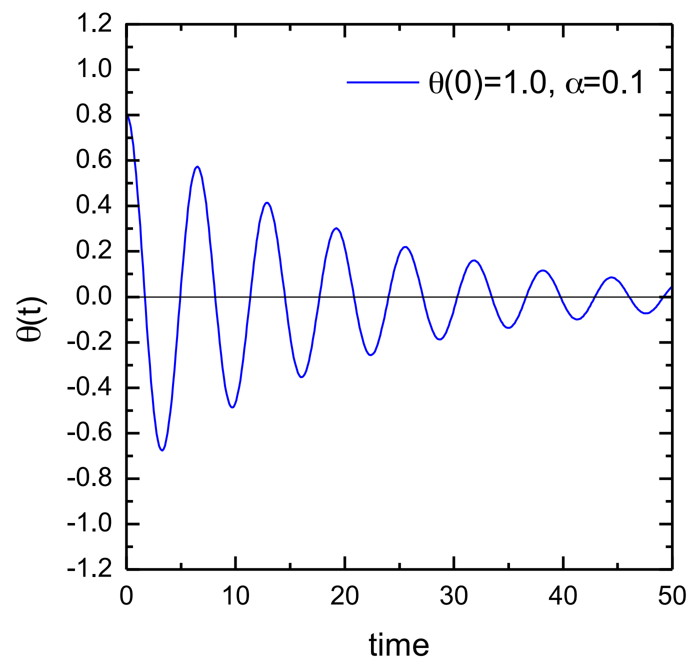
$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt} + f\cos(\omega t)$$

$$\omega_0^2 = \frac{mgL}{I} = \frac{g}{L}, \quad \alpha = \frac{\beta}{mL^2}, \quad f = \frac{F}{mL^2}$$

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta)$$



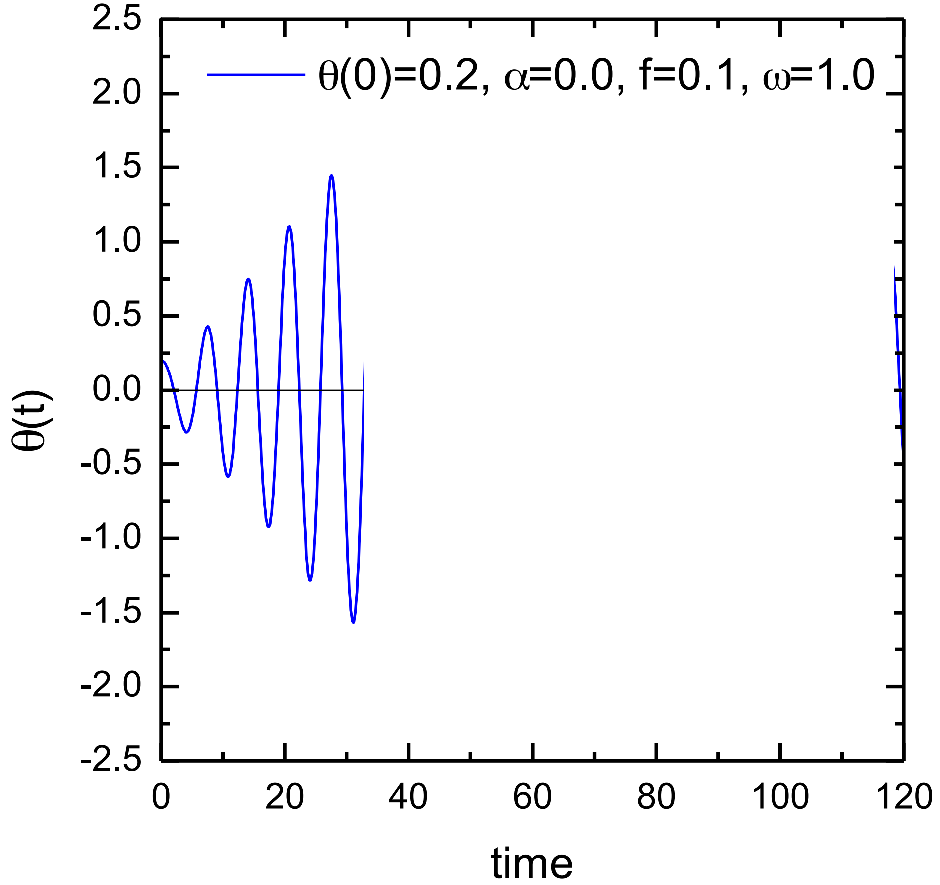
$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt}$$



example 3: resonance

When the frequency of an external force is close to a natural frequency

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) + f\cos(\omega t)$$



Oscillations: The Tacoma bridge (1940)

example 2: beats

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) + f\cos(\omega t)$$

- When the magnitude of the force is very large – the system is overwhelmed by the driven force (*mode locking*) and the are no beats
- When the magnitude of the force is comparable with the magnitude of the natural restoring force the beats may occur