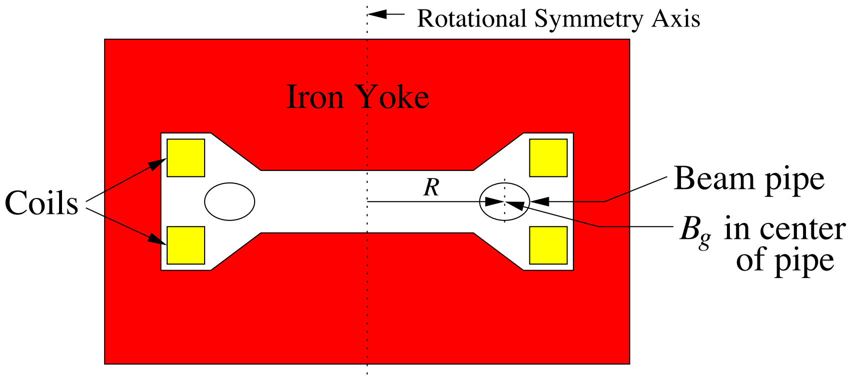
# Weak Focusing

Not to be confused with "weak folk cussing".

- Lawrence originally thought that the cyclotron needed to have a uniform (vertical) field.
  - Actually unstable: protons with $p_{\text{vert}} \neq 0$ would crash into poles.
  - First cyclotron required empirical shimming to eliminate sparking.
- Radial falloff of field to provide some focusing.
  - Understood after second "improved" cyclotron did not work. (probably involved some cussing.)

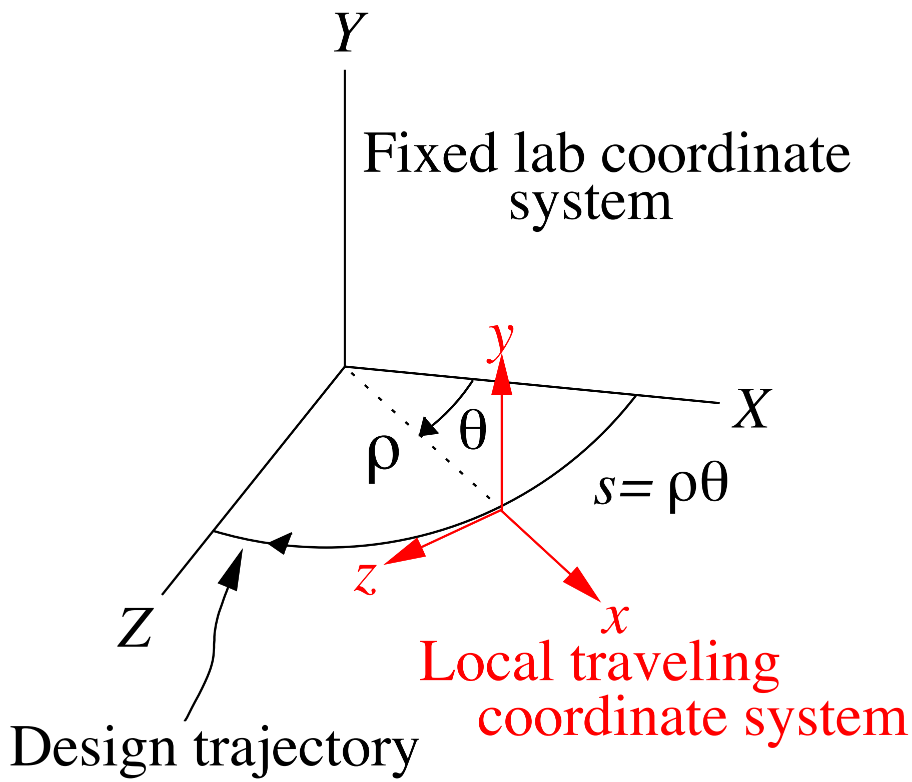
# The Betatron




- Works like a tranformer.
  - "Primary winding": coils.
  - "Secondary winding": beam.
- Focusing from beveled gap.

- Flux inside loop: $\phi = \pi R^2 \bar{B}$. (1)
- Force: $F = qE = (-e)\frac{V}{2\pi R} = \frac{-e}{2\pi R}\left(-\frac{d\phi}{dt}\right) = \frac{1}{2} eR\frac{d\bar{B}}{dt}$. (2)
- Cyclotron radius: $R = \frac{p}{eB_g}$. (3)
- $F = \frac{dp}{dt} = eR\frac{dB_g}{dt}$. (4)
- Wideröe condition: $B_g = \frac{1}{2}\bar{B}$, by comparing Eqs. (1&2). (5)

# Coordinates and Design Particle




- $\rho$ is radius of design orbit or trajectory.
- Red local system moves with design particle ($z = 0$).
- Another particle at $(x, y, z)$
  - $R = \rho + x$

# Equations of Motion

- Assume $\vec{E} = 0$ and static $\vec{B} = (B_x, B_y, 0)$.
  i. e., no longitudinal magnetic field component.

- Also assume midplane symmetry $B_x(x, 0, \theta) = 0$, i. e. only vertical field in midplane at $y = 0$.

$$
\begin{aligned}
\vec{F} &= \frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B}, \qquad (\vec{E} = 0), \\
F_x &= \frac{d}{dt}(\gamma m \dot{x}) = -qvB_y, \\
F_y &= \frac{d}{dt}(\gamma m \dot{y}) = \quad qvB_y;
\end{aligned}
$$

- For static $\vec{B}$, $\dot{\gamma} = 0$, so

$$\frac{d}{dt}(\gamma m \dot{x}) \simeq \gamma m \left( \ddot{x} - \frac{v^2}{R} \right) = \gamma m \left( \frac{v^2}{R^2} \frac{d^2 x}{d\theta^2} - \frac{v^2}{R} \right) = -qvB_y,$$

since $\theta = \dfrac{s}{R} = \dfrac{vt}{R}$, and $dt = \dfrac{R}{v}\, d\theta$.

Recall that for circular motion, the centripital acceleration is $\frac{v^2}{R}$.

Rearranging:

$$\frac{d^2 x}{d\theta^2} + \left( \frac{qB_y}{p} R - 1 \right) R = 0$$

Similarly in the vertical plane:

$$\frac{d^2 y}{d\theta^2} - \frac{qB_x}{p} R^2 = 0.$$

- Paraxial approximation, i. e. small oscillations about design trajectory $|x|, |y| \ll \rho, R$
- implies small slopes of the trajectory in the local coordinates

$$\left|\frac{dx}{ds}\right| = \left|\frac{1}{R}\,\frac{dx}{d\theta}\right| \ll 1, \qquad \text{and} \qquad \left|\frac{dy}{ds}\right| = \left|\frac{1}{R}\,\frac{dx}{d\theta}\right| \ll 1.$$

- Define subscript "0" for values of design particle:
  - $v_0 = \beta_0 c$ for velocity,
  - $B_0$ for vertical magnetic field on design trajectory.

$$\frac{1}{\rho} = \frac{qB_0}{p_0} = \frac{qB_0}{\gamma_0\beta_0 mc},$$

  - and $p = p_0 + \delta p$.
- Expand magnetic field in Taylor series:

$$\left.\begin{aligned} B_y &= B_0 + \left(\frac{\partial B_y}{\partial x}\right)_0 x + \mathcal{O}(2), \\ B_x &= \left(\frac{\partial B_y}{\partial x}\right)_0 y + \mathcal{O}(2). \end{aligned}\right\} \text{From Maxwell:} \quad \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}.$$

$$\frac{d^2x}{d\theta^2} + \left[\frac{1}{\left(1+\frac{\delta p}{p_0}\right)}\left(1+\frac{1}{B_0}\left.\frac{\partial B_y}{\partial x}\right|_0 x\right)\left(1+\frac{x}{\rho}\right) - 1\right]\rho\left(1+\frac{x}{\rho}\right) \simeq 0,$$

$$\frac{d^2x}{d\theta^2} + \left(1+\frac{\rho}{B_0}\left.\frac{\partial B_y}{\partial x}\right|_0\right)x \simeq \rho\frac{\delta p}{p_0}.$$

or by defining the *field index*

$$n = -\frac{\rho}{B_0}\left.\frac{\partial B_y}{\partial x}\right|_0,$$

we have to first order:

$$\frac{d^2x}{d\theta^2} + (1-n)x = \rho\,\frac{\delta p}{p_0}.$$

The inhomogeneous term $\rho\frac{\delta p}{p_0}$ is called the *dispersion term*.

For vertical, recall (from about 4 pages back)

$$\frac{d^2y}{d\theta^2} - \frac{qB_x}{p}\,R^2 = 0,$$

This expands to

$$\frac{d^2y}{d\theta^2} - \left(1 - \frac{\delta p}{p_0}\right)\frac{q}{p_0}\;\frac{\partial B_y}{\partial x}\bigg|_0 y(\rho + x)^2 = 0,$$

or

$$\frac{d^2y}{d\theta^2} + ny = 0,$$

with the dispersion and horizontal effects coming in at higher order.

# Stability

Ignoring the dispersion term, for on-momentum ($\delta p = 0$) particles, we have

$$\frac{d^2x}{d\theta^2} + (1-n)x = 0, \qquad \text{(H)}$$

$$\frac{d^2y}{d\theta^2} + ny = 0. \qquad \text{(V)}$$

Clearly for constant $n$, these can have either sinusoidal or exponential solutions.

Stable oscillations will happen

in the horizontal for $n < 1$, and

in the vertical when $n > 0$.

For stability in both planes $0 < n < 1$, or

$$-\frac{B_0}{\rho} < \frac{\partial B_y}{\partial x} < 0.$$

# Solutions

$$
\begin{aligned}
x(\theta) &= \quad A\cos\sqrt{1-n}\theta \qquad\qquad + B\sin\sqrt{1-n}\theta, \\
\frac{dx}{d\theta}(\theta) &= -A\sqrt{1-n}\sin\sqrt{1-n}\theta + B\sqrt{1-n}\cos\sqrt{1-n}\theta.
\end{aligned}
$$

- Definition: a prime subscript indicates a derivative with respect to the $s$ coordinate of the design trajectory:

$$x' = \frac{dx}{ds} = \frac{1}{\rho}\frac{dx}{d\theta}.$$

- Note that we will generally parameterize trajectories in terms of the arc length variable $s$ of the design orbit rather than in terms of $\theta$.
  - This is because we will have drift spaces and other magnets inserted between the bending magnets.

In terms of the initial $x_0$ and $x'_0$ at $\theta = 0$, our constants $A$ and $B$ are then

$$A = x_0, \quad \text{and} \quad B = \frac{\rho}{\sqrt{1-n}}x'_0.$$

# Matrices

In the horizontal plane this may be written in matrix form as

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \mathbf{M}_{\mathrm{H}} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{1-n}\,\theta & \frac{\rho}{\sqrt{1-n}} \sin\sqrt{1-n}\,\theta \\ -\frac{\sqrt{1-n}}{\rho} \sin\sqrt{1-n}\,\theta & \cos\sqrt{1-n}\,\theta \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}.$$

For the vertical matrix equation — swap $n$ of Eq. (V) for $1-n$ of Eq. (H):

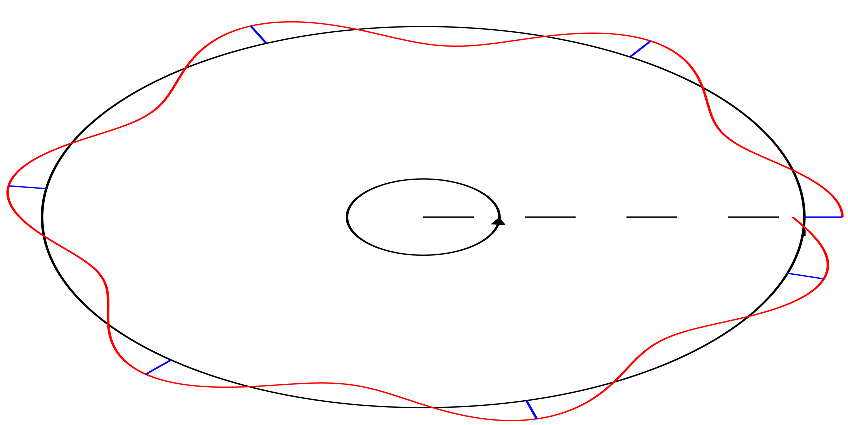
$$\begin{pmatrix} y \\ y' \end{pmatrix} = \mathbf{M}_{\mathrm{V}} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{n}\,\theta & \frac{\rho}{\sqrt{n}} \sin\sqrt{n}\,\theta \\ -\frac{\sqrt{n}}{\rho} \sin\sqrt{n}\,\theta & \cos\sqrt{n}\,\theta \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}.$$

When $n$ is outside the region of stability for either plane, then we may apply

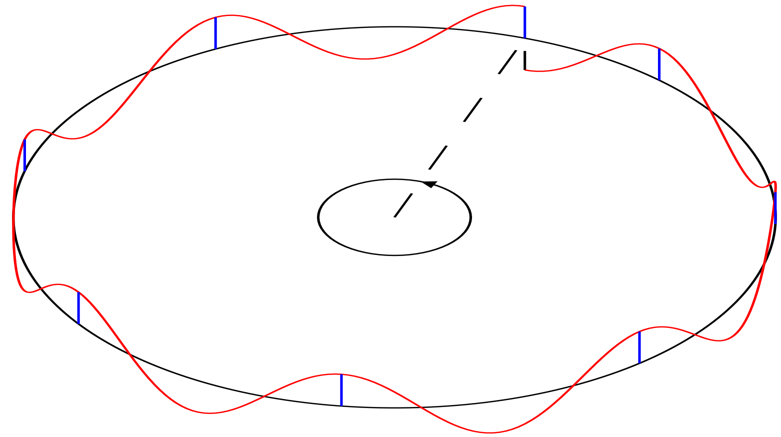
$$\cos(iz) = \frac{e^{-z}+e^{z}}{2} = \cosh(z), \qquad \text{and} \qquad \sin(iz) = \frac{e^{-z}-e^{z}}{2i} = -\sinh(z),$$

to obtain the exponential solutions.

# Betatron tunes



Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation
with tune: $Q_v = 7.5$,
i.e., 7.5 oscillations per turn.

Definition: The tune is the number of wavelengths of the oscillation in one complete orbit around the ring.

Note that Weak focusing refers to betatron tunes less than 1: $Q_{\rm H} < 1$, $Q_{\rm V} < 1$.

$$(\text{Betatron: } Q_{\rm H} = \frac{1}{2\pi}\sqrt{1-n}2\pi = \sqrt{1-n}, \text{ and } Q_{\rm V} = \sqrt{n}.)$$

# Dispersion

Recall first-order diff. eq. for horiz. motion for betatron (from p. 7):

$$\frac{d^2x}{d\theta^2} + (1-n)x = \rho\,\frac{\delta p}{p_0}.$$

This has an inhomogeneous solution of

$$x_\delta = \frac{\rho}{1-n}\,\delta, \qquad \text{with the definition: } \delta = \tfrac{\delta p}{p_0}.$$

For off-momentum particles, we find the total horiz. position is shifted by $x_\delta$:

$$\begin{aligned} x &= x_\beta \qquad\qquad\qquad\qquad\qquad\qquad\qquad + x_\delta \\ &= A\cos\left(\sqrt{1-n}\theta\right) + B\sin\left(\sqrt{1-n}\theta\right) + \frac{\rho}{1-n}\,\delta. \\ x' &= \frac{dx}{ds} = \frac{1}{\rho}\,\frac{dx}{d\theta} = \frac{\sqrt{1-n}}{\rho}\left(-A\sin(\sqrt{1-n}\theta) + B\cos(\sqrt{1-n}\theta)\right). \end{aligned}$$

For the initial condition with $\theta = 0$ we have:

$$\begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{\rho}{1-n} \\ 0 & \frac{\sqrt{1-n}}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ \delta_0 \end{pmatrix}$$

or after inverting

$$\begin{pmatrix} A \\ B \\ \delta_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{\rho}{1-n} \\ 0 & \frac{\rho}{\sqrt{1-n}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

whereas at a later $\theta$:

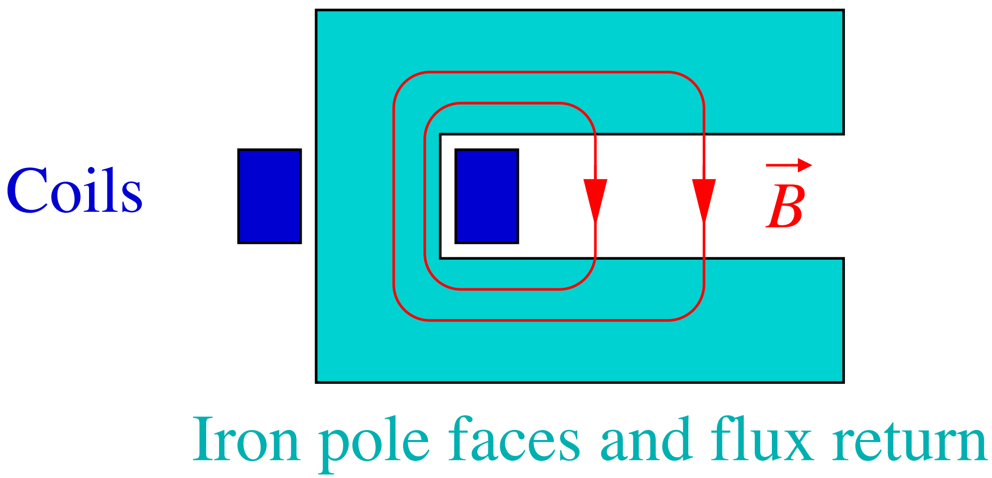
$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} \cos\sqrt{1-n}\theta & \sin\sqrt{1-n}\theta & \frac{\rho}{1-n} \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{1-n}\theta & \frac{\sqrt{1-n}}{\rho}\cos\sqrt{1-n}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ \delta_0 \end{pmatrix} = \mathbf{M}_{\mathrm{H}} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix},$$

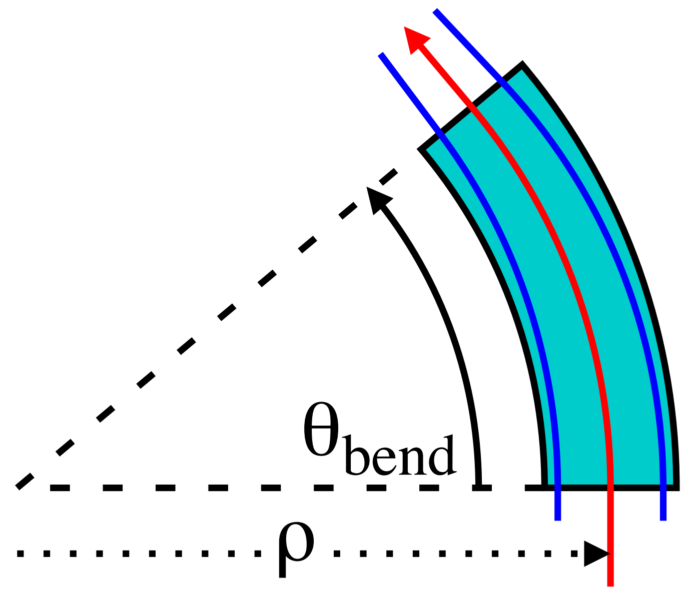
where we now have added a third row and column to $\mathbf{M}_{\mathrm{H}}$:

$$\mathbf{M}_{\mathrm{H}} = \begin{pmatrix} \cos(\sqrt{1-n}\theta) & \frac{\rho}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) & \frac{\rho}{1-n}\left(\cos(\sqrt{1-n}\theta) - 1\right) \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{(1-n\theta)} & \cos(\sqrt{1-n}\theta) & \frac{1}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) \\ 0 & 0 & 1 \end{pmatrix}.$$

# Dipole bend magnet

# Sector Dipole Bend Magnet




- Wedge segment of magnet with $n = 0$ from $\theta = 0$ to $\theta_{\text{bend}}$
- Uniform vertical field.
- Horizontal parallel incoming trajectories get focused.
  - Unless $\theta_{\text{bend}} \geq 180°$. (Think about it.)
- No vertical focusing for a sector bend. (More discussion later on.)

# Sector bend matrices with no gradient

For $n = 0$

$$\mathbf{M}_{\mathrm{H}} = \lim_{n\to 0} \begin{pmatrix} \cos(\sqrt{1-n}\theta) & \frac{\rho}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) & \frac{\rho}{1-n}\left(\cos(\sqrt{1-n}\theta) - 1\right) \\ -\frac{\sqrt{1-n}}{\rho}\sin(\sqrt{1-n}\theta) & \cos(\sqrt{1-n}\theta) & \frac{1}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho\,(\cos\theta - 1) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

Notice the $M_{21} = -\frac{1}{\rho}\sin\theta$ term produces horiz focusing.

$$\mathbf{M}_{\mathrm{V}} = \lim_{n\to 0} \begin{pmatrix} \cos\sqrt{n}\theta & \frac{\rho}{\sqrt{n}}\sin\sqrt{n}\theta \\ -\frac{\sqrt{n}}{\rho}\sin\sqrt{n}\theta & \cos\sqrt{n}\theta \end{pmatrix} = \begin{pmatrix} 1 & \rho\theta \\ 0 & 1 \end{pmatrix}.$$

Here the $M_{21}$ is zero. (Looks like a drift element.)

# Quadrupole matrix

$$
\begin{aligned}
n &= -\frac{\rho}{B_0}\frac{\partial B_y}{\partial x} = -\frac{\rho}{B_0}\,g, \qquad\qquad (g = \text{gradient}). \\
\frac{1}{B_0} &= \frac{q\rho}{p_0} \\
n &= -\frac{qg}{p_0}\,\rho^2 = -k\rho^2, \qquad\qquad k = \frac{qg}{p_0}. \\
\sqrt{1-n}\,\theta &= \sqrt{1+k\rho^2}\,\frac{s}{\rho} \\
&= \sqrt{\frac{1}{\rho^2}+k}\,s,
\end{aligned}
$$

Keep $s$, $q$, and $p_0$ constant, let $B_0 \to 0$, so $\rho \to \infty$.

$$
\begin{aligned}
\sqrt{1-n}\,\theta &\to \sqrt{k}\,\theta. \\
\sqrt{n}\,\theta &\to \sqrt{k}\,s.
\end{aligned}
$$

$$\frac{\rho}{\sqrt{1-n}} = \frac{1}{\sqrt{\rho^{-2}+k}} \to \frac{1}{\sqrt{k}}.$$

$$\frac{\sqrt{1-n}}{\rho} \to \sqrt{k}.$$

$$\frac{\rho}{1-n}(1-\cos\sqrt{1-n}\theta) = \frac{\rho}{1+\rho^2 k}(1-\cos\sqrt{k}\,s) \to 0.$$

$$\frac{1}{\sqrt{1-n}}\sin\sqrt{1-n}\theta = \frac{1}{\sqrt{1+k\rho^2}}\sin\sqrt{\frac{1}{\rho^2}+k}\,s \to 0.$$

$$\mathbf{M}_{\mathrm{H}} = \begin{pmatrix} \cos(\sqrt{1-n}\theta) & \frac{\rho}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) & \frac{\rho}{1-n}\left(\cos(\sqrt{1-n}\theta)-1\right) \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{(1-n\theta)} & \cos(\sqrt{1-n}\theta) & \frac{1}{\sqrt{1-n}}\sin(\sqrt{1-n}\theta) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} \cos\sqrt{k}\,s & \frac{1}{\sqrt{k}}\sin\sqrt{k}\,s & 0 \\ -\sqrt{k}\sin\sqrt{k}\,s & \cos\sqrt{k}\,s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly

$$\mathbf{M}_{\mathrm{V}} \to \begin{pmatrix} \cosh\sqrt{k}\,s & \frac{1}{\sqrt{k}}\sinh\sqrt{k}\,s \\ \sqrt{k}\sinh\sqrt{k}\,s & \cosh\sqrt{k}\,s \end{pmatrix}.$$

So we find

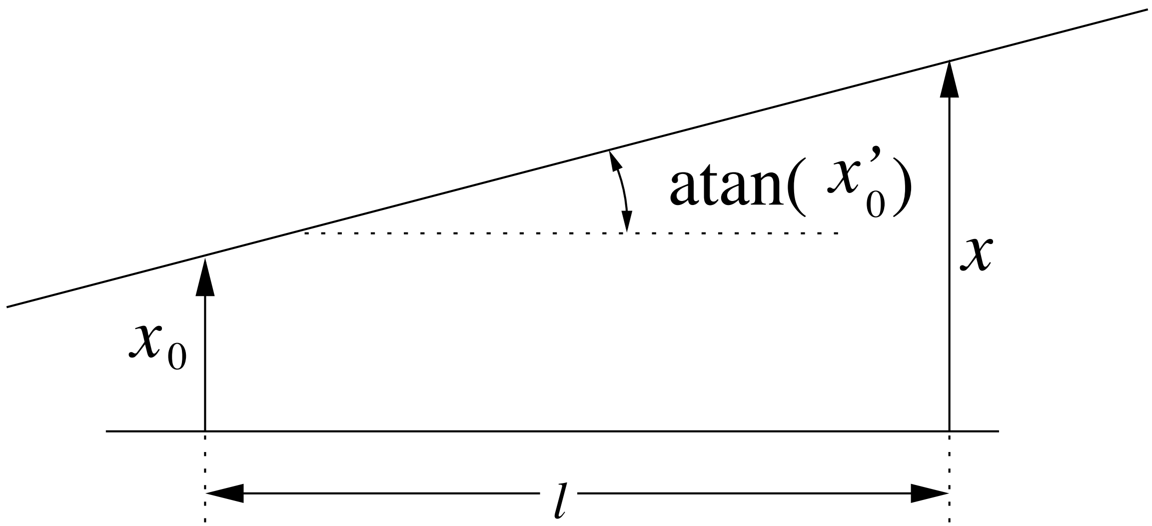
$$\mathbf{M}_{\mathrm{quad}} = \begin{pmatrix} \cos\sqrt{k}\,s & \frac{1}{\sqrt{k}}\sin\sqrt{k}\,s & 0 & 0 \\ -\sqrt{k}\sin\sqrt{k}\,s & \cos\sqrt{k}\,s & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k}\,s & \frac{1}{\sqrt{k}}\sinh\sqrt{k}\,s \\ 0 & 0 & \sqrt{k}\sinh\sqrt{k}\,s & \cosh\sqrt{k}\,s \end{pmatrix},$$

for a horizontally focusing quadrupole magnet.

For a horizontally defocusing quad, just swap the two $2\times 2$ diagonal blocks.


SEXTANT
MAGNET

# Drift matrices




- In a field free space our trajectories will be straight lines:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \mathbf{M}_{\mathrm{H,drift}} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix},$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \mathbf{M}_{\mathrm{V,drift}} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix},$$

# Forshadowing of Liouville's Theorem

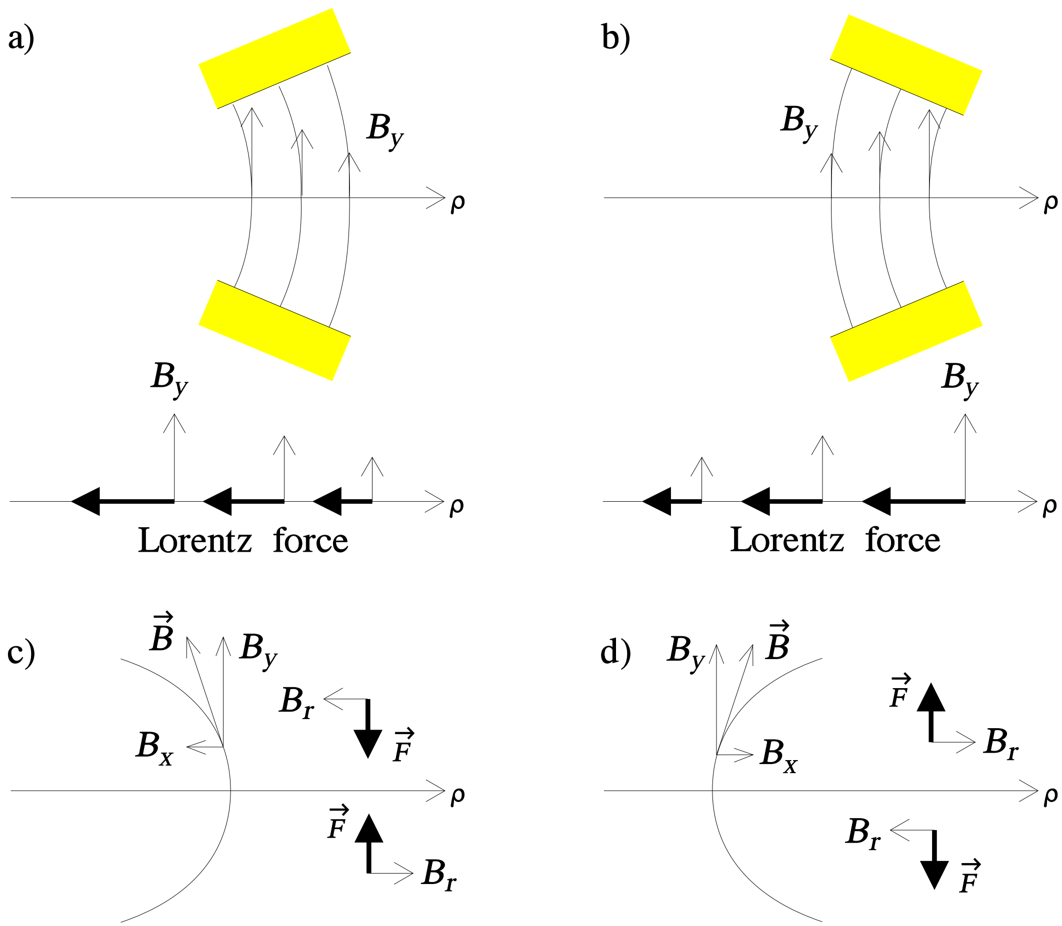
- Note that the determinants of all these matrices is identically 1.

$$
\begin{aligned}
|\mathbf{M}_{\rm drift}| &= \left| \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \right| = 1, \\
|\mathbf{M}_{\rm H}| &= \left| \begin{pmatrix} \cos\sqrt{1-n}\,\theta & \frac{\rho}{\sqrt{1-n}}\sin\sqrt{1-n}\,\theta \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{1-n}\,\theta & \cos\sqrt{1-n}\,\theta \end{pmatrix} \right| = 1, \\
|\mathbf{M}_{\rm V}| &= \left| \begin{pmatrix} \cos\sqrt{n}\,\theta & \frac{\rho}{\sqrt{n}}\sin\sqrt{n}\,\theta \\ -\frac{\sqrt{n}}{\rho}\sin\sqrt{n}\,\theta & \cos\sqrt{n}\,\theta \end{pmatrix} \right| = 1. \\
|\mathbf{M}_{\rm quad}| &= 1.
\end{aligned}
$$

- These transformations preserve area in $(x, x')$ and $(y, y')$ spaces.
- $x' = dx/ds = \frac{p_x}{p_z} \simeq \frac{p_x}{p}$ for small angles.
  This small-angle approximation is known as the *paraxial approximation.*

  Much more about Liouville's theorem to come later.

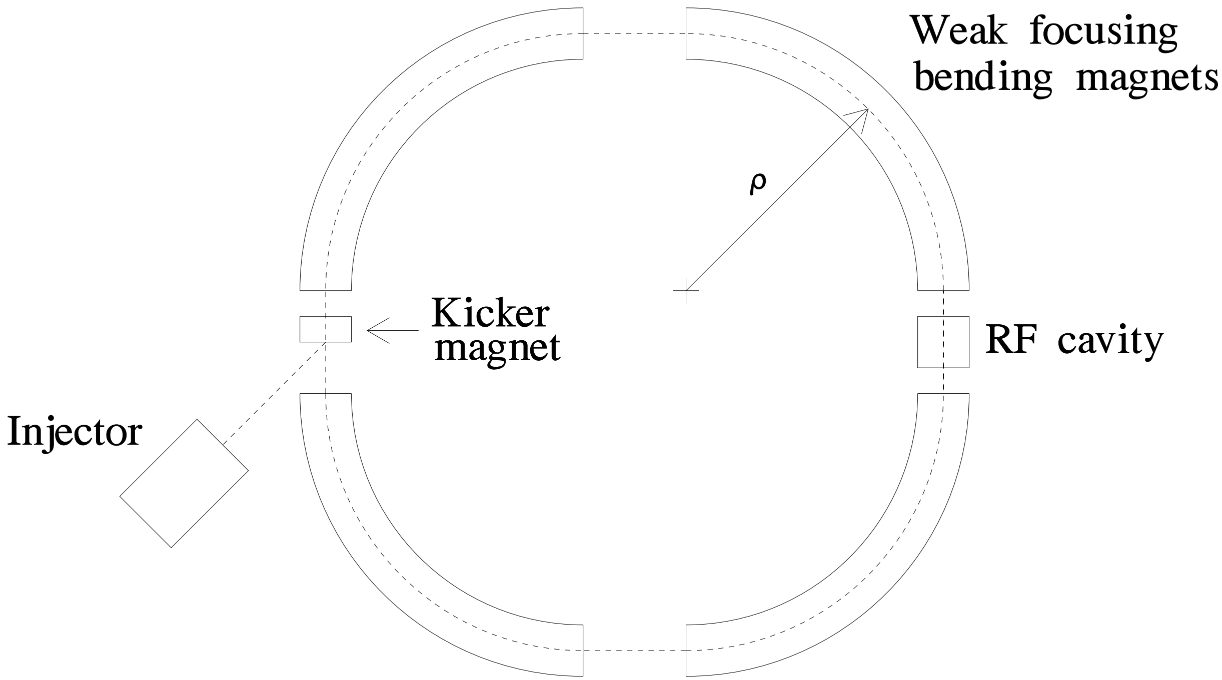
# Gradient Fields and Forces




For a dipole bend:

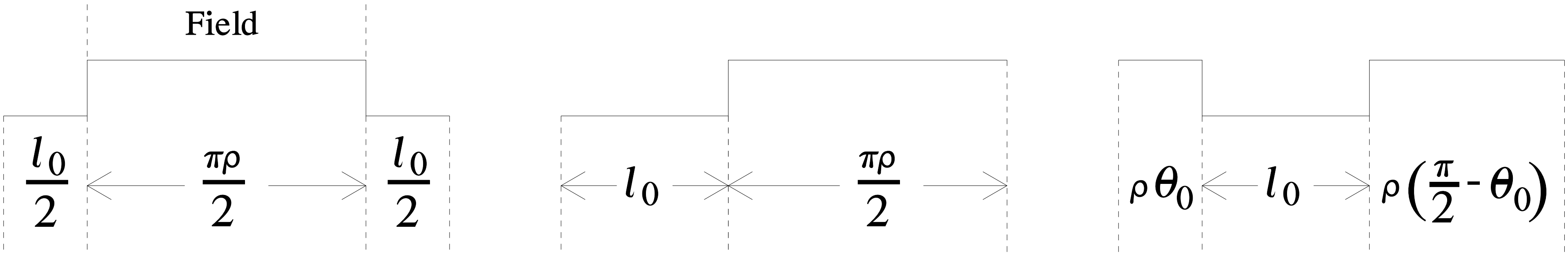
- $B_y$ larger for smaller gap between poles in an iron magnet.
- Tilt poles $\rightarrow \frac{\partial B_y}{\partial x} \neq 0$
- Polefaces opening outward (a&c) less horiz focusing. (Perhaps even net defocus-ing.)
- Polefaces closing outward (b&d) more horiz focusing.

# Example Weak-focusing Synchrotron




- Ring made of four 90° sector bends.
- Put drift spaces between magnets.
  - Beam injected at one drift space with pulsed injection kicker.
  - Another straight section may have an rf cavity for acceleration.
    - Bend field must increase as beam accelerates.
    - rf frequency must increase with velocity for constant cirumference.

Three possible periodic-cell configurations:




beam from right to left.
Ignore kicker and rf cavity for present.

- Periodic cell: 1/4 of ring:
- Book uses left periodic cell. Let's do the middle one.

$$\mathbf{M}_{\text{cell}} = \mathbf{M}_{\text{H,drift}}(l_0)\mathbf{M}_{\text{H,bend}}\left(\frac{\pi}{2}\right),$$

or

$$\mathbf{M}_{\text{cell}} = \begin{pmatrix} 1 & l_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\sqrt{1-n}\pi}{2} & \frac{\rho}{\sqrt{1-n}}\sin\frac{\sqrt{1-n}\pi}{2} \\ -\frac{\sqrt{1-n}}{\rho}\sin\frac{\sqrt{1-n}\pi}{2} & \cos\frac{\sqrt{1-n}\pi}{2} \end{pmatrix}.$$

$$\mathbf{M}_{\rm cell} = \begin{pmatrix} \cos\frac{\sqrt{1-n}\pi}{2} - \frac{l_0\sqrt{1-n}}{\rho}\sin\frac{\sqrt{1-n}\pi}{2} & \frac{\rho}{\sqrt{1-n}}\sin\frac{\sqrt{1-n}\pi}{2} + l_0\cos\frac{\sqrt{1-n}\pi}{2} \\ -\frac{\sqrt{1-n}}{\rho}\sin\frac{\sqrt{1-n}\pi}{2} & \cos\frac{\sqrt{1-n}\pi}{2} \end{pmatrix}.$$

For now skip rest of §2.6 in book. Probably easier to understand after we do Chapter 5.

# Momentum compaction factor

For a flat ring, the closed design orbit must have

$$\oint \frac{ds}{\rho(s)} = 2\pi.$$

In straight sections with no field $\rho = \infty$.

From the cyclotron radius, we have $\dfrac{1}{\rho} = \dfrac{qB_\perp}{p}$, so substituting, we find

$$p = \frac{q}{2\pi} \oint B_\perp ds.$$

The design circumference is $L = \oint ds$.

Momentum compaction is ratio: $\dfrac{\text{fractional change in circumference}}{\text{fractional change in momentum}}$, i. e.

$$\alpha_p = \frac{dL}{L} \Big/ \frac{dp}{p} = \frac{p}{L}\frac{dL}{dp}.$$

# Hubble Telescope example

What's the momentum compaction of the HST?

Nearly circular orbit at $r = R_\oplus + 559$km (from Wikipedia).

$$
\begin{aligned}
F = \frac{GMm}{r^2} &= ma = m\frac{v^2}{r} = \frac{p^2}{mr}, \\
p^2 r &= GMm^2 \\
2\ln(p) + \ln(r) &= \ln(GMm^2) \\
2\frac{dp}{p} + \frac{dr}{r} &= 0, \\
\frac{dL}{L} &= \frac{dr}{r} = -2\frac{dp}{p} \\
\alpha_p &= \frac{dL}{L} \bigg/ \frac{dp}{p} = -2.
\end{aligned}
$$

# Mom. comp. of our WF synchrotron

Circumference: $L = 2\pi\rho + 4l_0.$

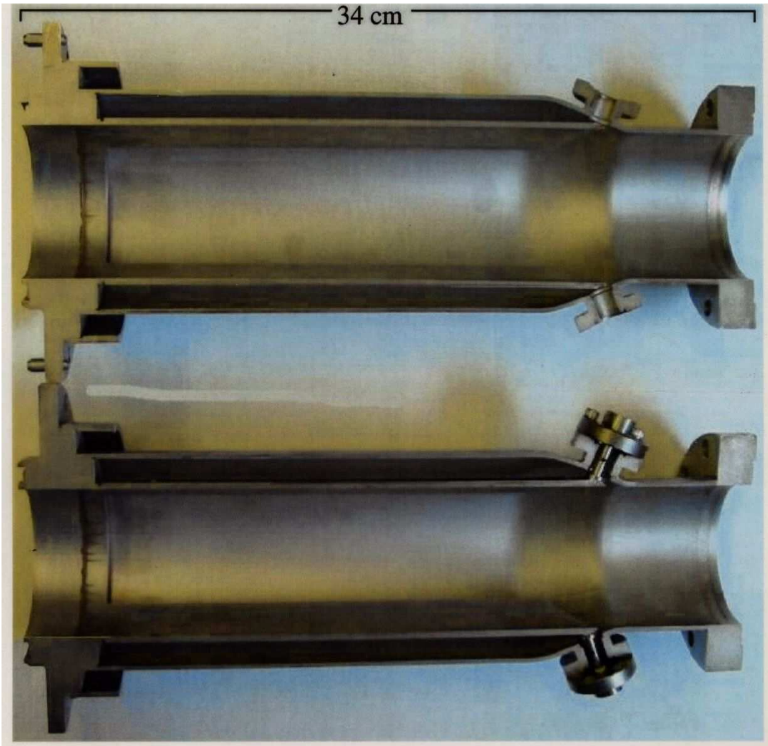
$$L + dL = 2\pi(\rho + d\rho) + 4l_0.$$
$$\frac{dL}{L} = \frac{2\pi d\rho}{2\pi\rho + 4l_0} = \frac{1}{1 + 2l_0/(\pi\rho)}\,\frac{d\rho}{\rho}$$

$$p = qB\rho$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dB}{B} = \left(1 + \frac{\rho}{B}\frac{\partial B}{\partial\rho}\right)\frac{d\rho}{\rho} = (1-n)\frac{d\rho}{\rho}.$$

$$\alpha_p = \frac{1}{\left(1 + \frac{2l_0}{\pi\rho}\right)(1-n)}.$$

# Beam position monitors




RHIC stripline beam monitor.

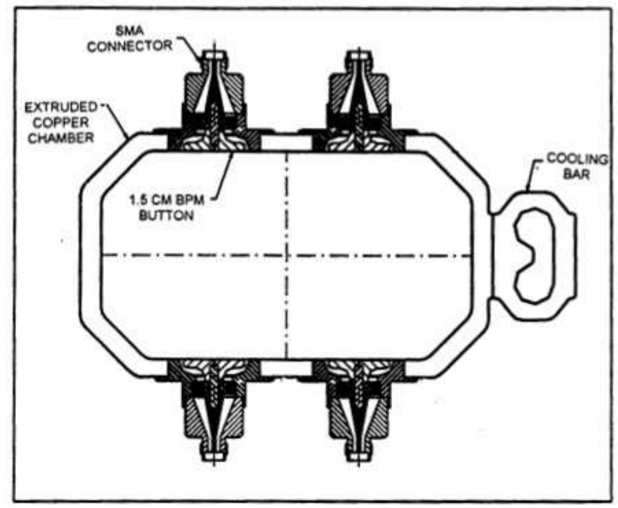
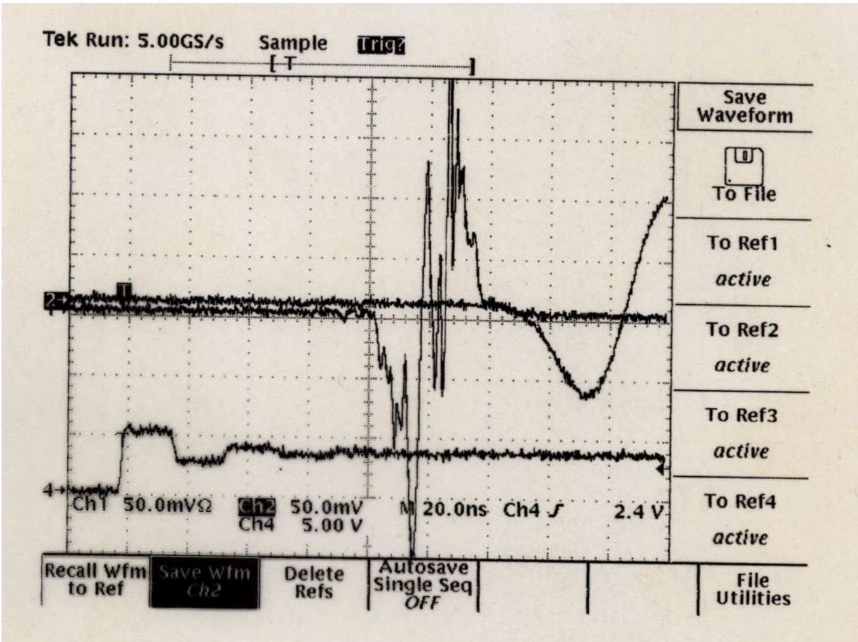
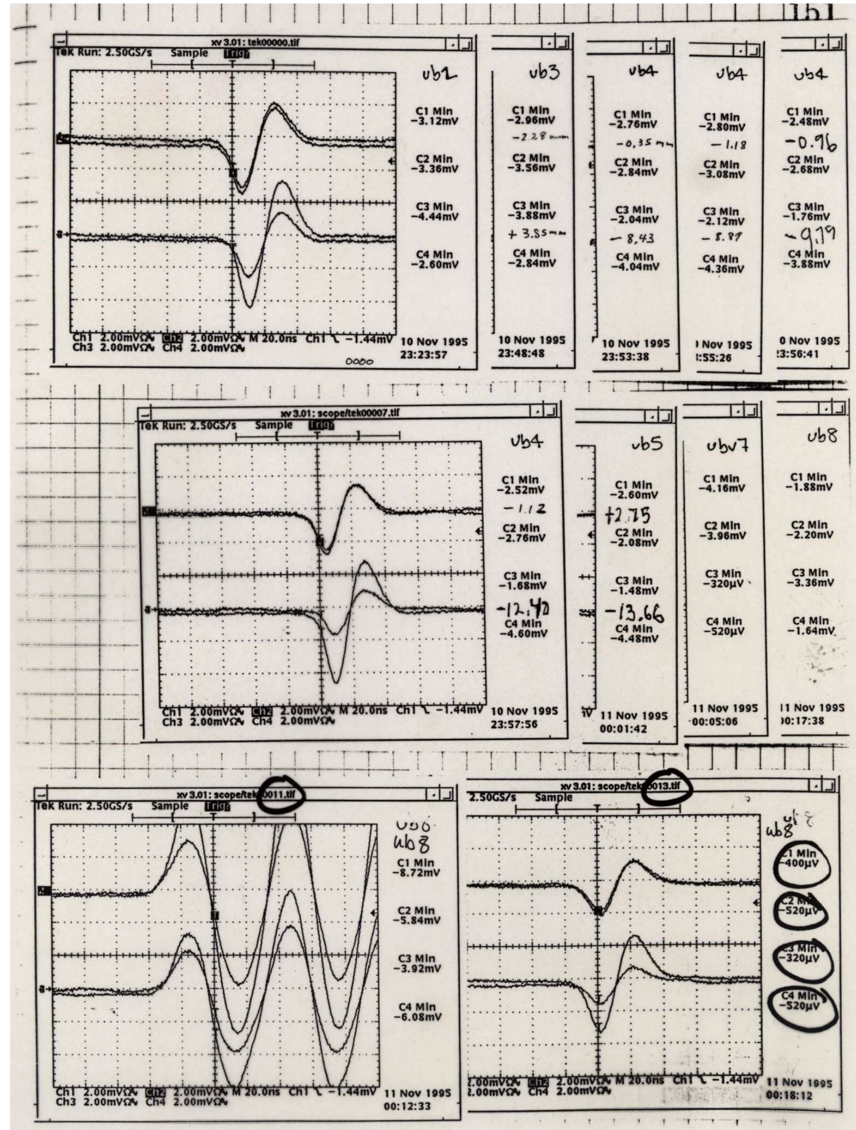



**Figure 9.** PEP-II high energy ring arc vacuum chamber cross section taken through the BPM buttons

PEP beam button monitor.
from S. R. Smith, SLAC-PUB-7244 (1966).

# BPM data from RHIC transfer line



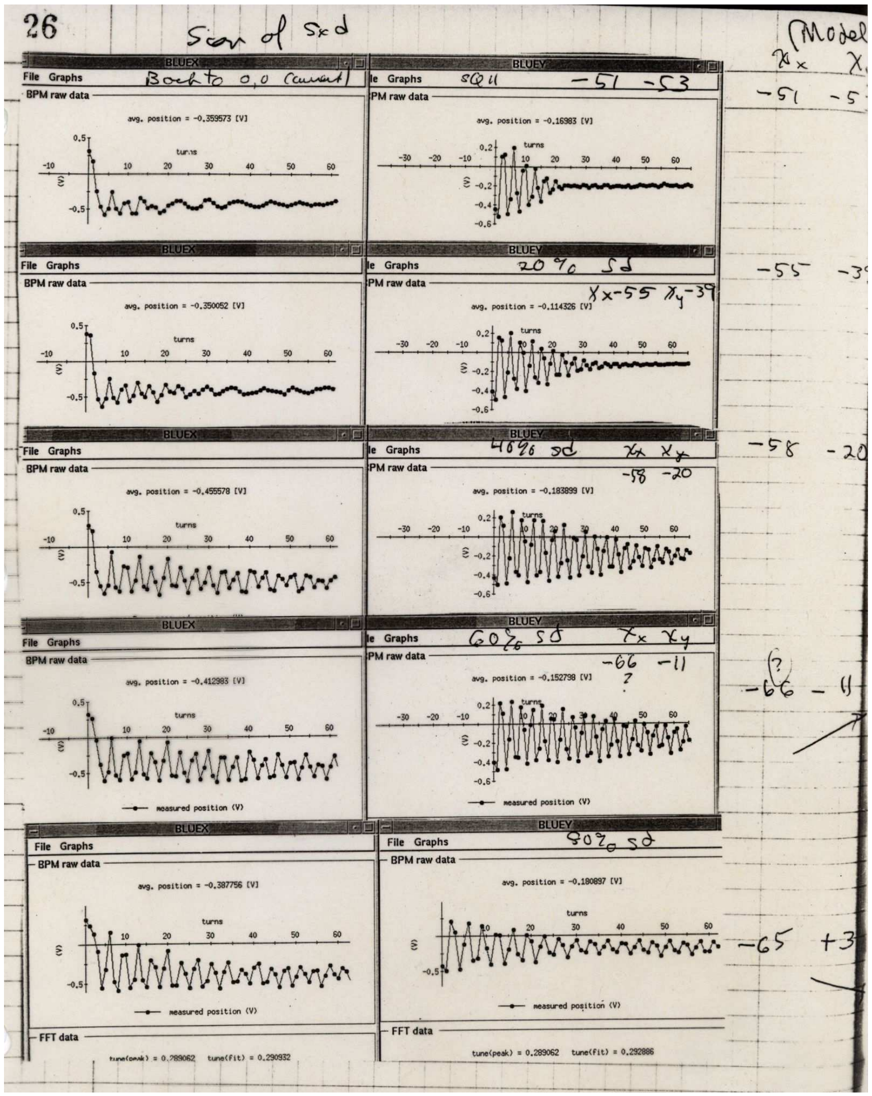



Left: Raw signal of single bunch passing a stripline. (Unfiltered)

Right: Electrode voltages measured with 20 MHz filters.

$$\text{Position} \propto \frac{V_1 - V_2}{V_1 + V_2}.$$

# BPM turn-by-turn data



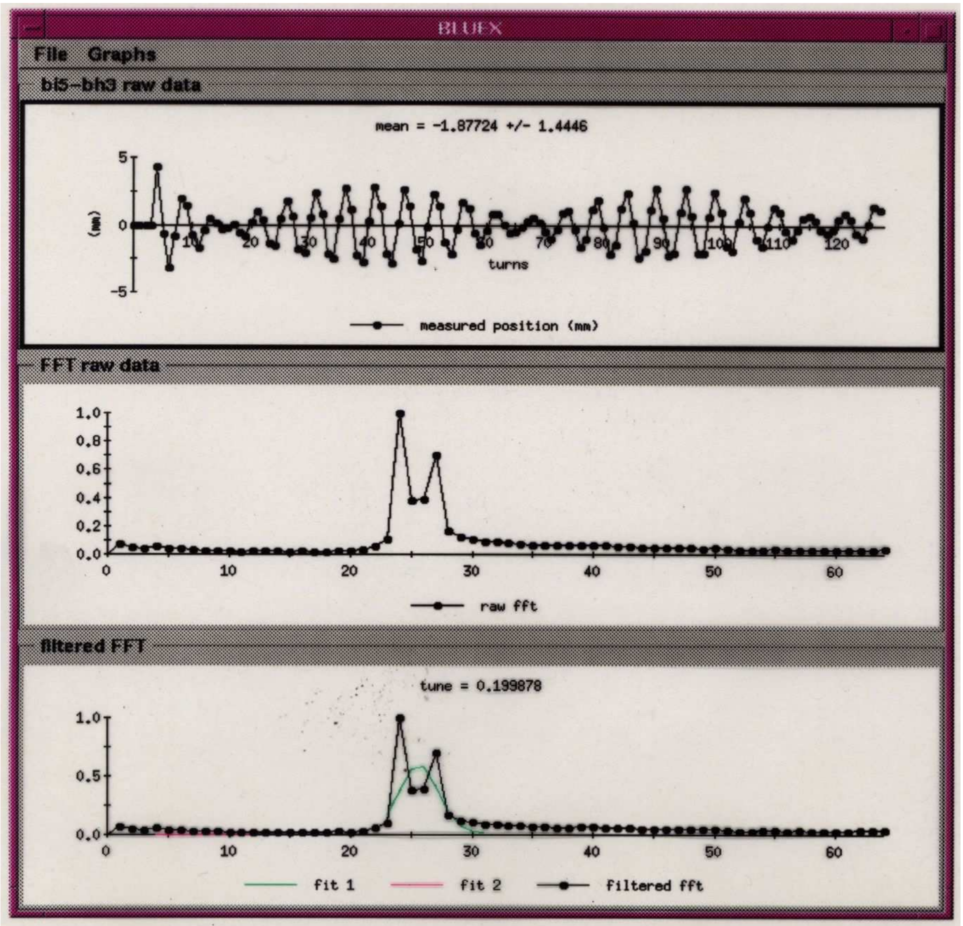

RHIC turn-by-turn meas.

- Left col. is horizontal.
- Right col. is vertical.
- Decoherence from chromaticity

$$\xi = \frac{1}{Q}\frac{dQ}{d\delta}.$$

- Chromaticity adj by sextupoles.
  - More about $\xi$ later in week.

# Tune measurement from turn-by-turn data




- Top: data from horiz. BPM.
- Middle: Raw FFT.
  - Tune from FFT (128 turns).
- Bottom: Fit to single peak.
  - Double peak caused by HV coupling.
  - Note beating in top graph.