USPAS Accelerator Physics 2013
Duke University

Lattice Extras: Linear Errors, Doglegs, Chicanes, Achromatic Conditions, Emittance Exchange

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Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi} - \text{periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \Psi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance} \qquad \mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \qquad \mathrm{Tr}\, M = 2\cos\mu$$

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using lattice parameters and $\Delta\Psi = \Psi(s_2) - \Psi(s_1)$

$$M_{s_1\to s_2} = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\Psi + \alpha(s_1)\sin\Delta\Psi] & \boxed{\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\Psi} \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\Psi+[1+\alpha(s_1)\alpha(s_2)]\sin\Delta\Psi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\Psi - \alpha(s_2)\sin\Delta\Psi] \end{pmatrix}$$

(C&M Eqn 5.52)

- This does not have a pretty form like the periodic matrix

However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

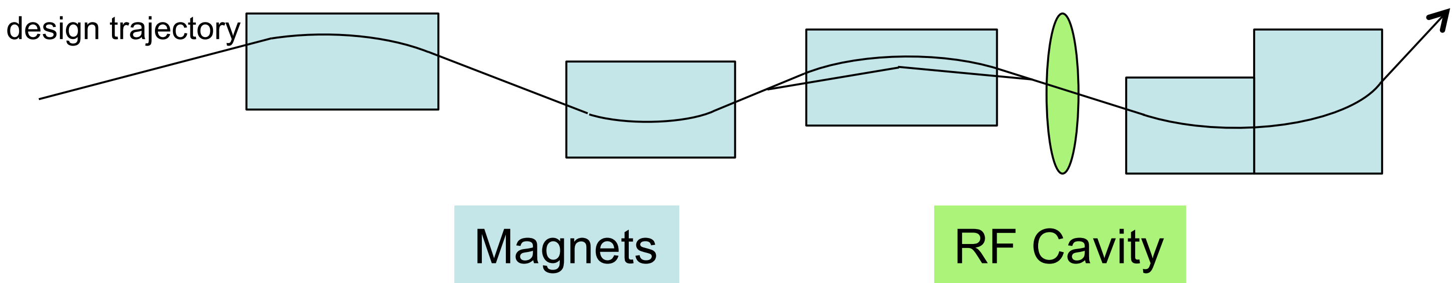
A common use of this matrix is the m_{12} term:

$$\boxed{\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\Psi)\; x'(s_1)}$$

Effect of angle kick on downstream position

General phase advance, NOT phase advance/cell μ

Design Orbit Perturbations



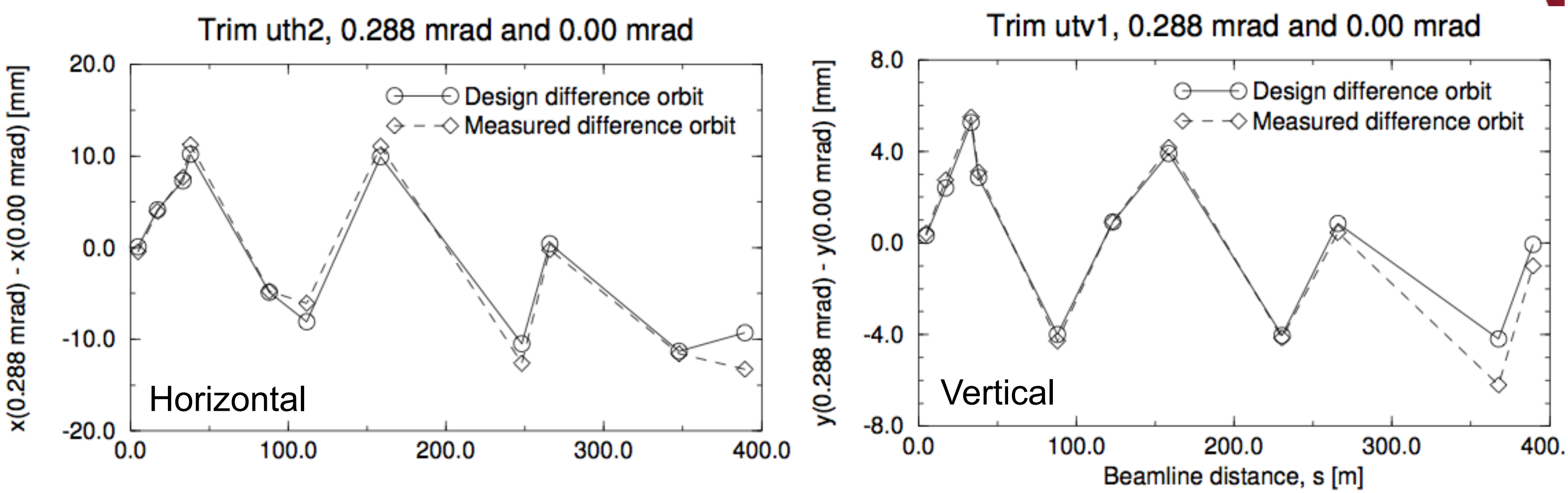
- Sometimes need a local change $\Delta x(s)$ to the design orbit
  - But we really only get changes in angle $\Delta x'$ from magnets
  - e.g. small dipole "corrector": $\Delta x' = B_{\mathrm{corrector}} L_{\mathrm{corrector}}/(B\rho)$
  - Changes to/corrections of design orbit from dipole correctors
  - Linear errors add up via linear superposition

$$\begin{pmatrix} \Delta x(s_2) \\ \Delta x'(s_2) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x'(s_1) \end{pmatrix}$$

$$\Delta x(s_2) = \Delta x'(s_1)\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\Psi$$

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\Psi - \alpha(s_2)\sin\Delta\Psi]$$

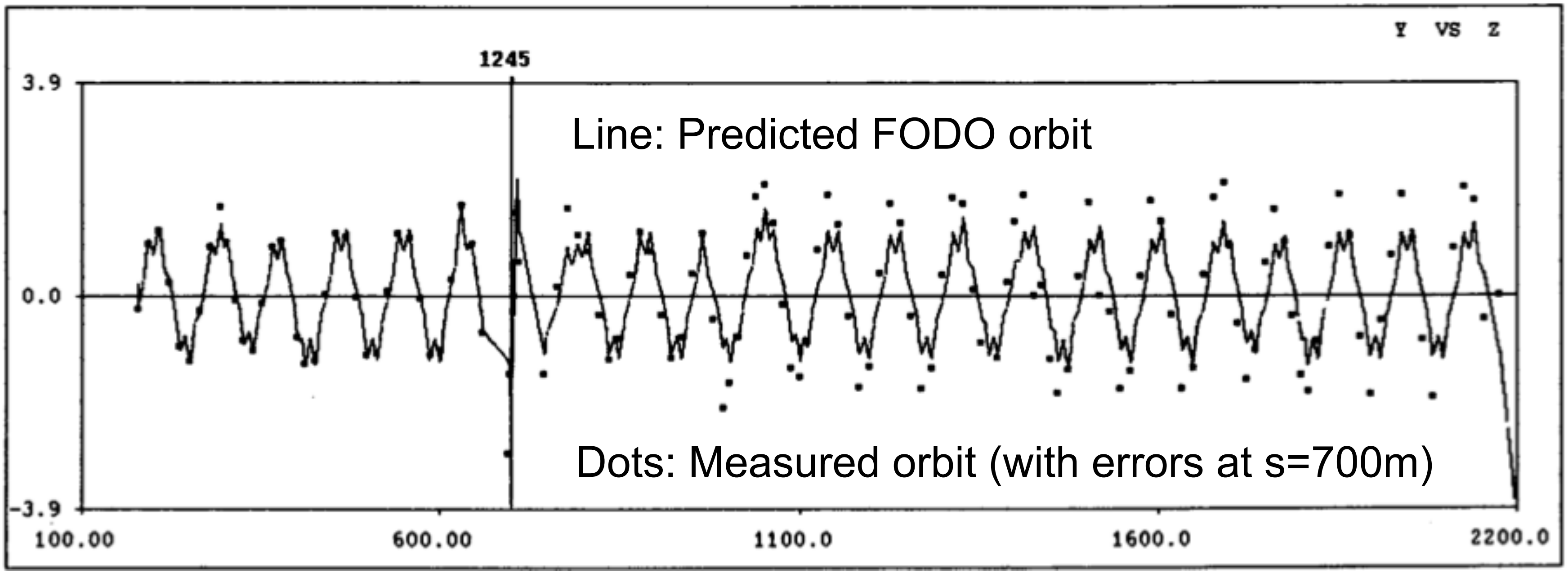
Checking Optics with Difference Orbits



- Steering changes through linear elements are linear
  - Compare calculated optical transport of Δx’ to measured
  - Can localize focusing errors with enough BPMs
  - Above picture is for single-pass AGS to RHIC transfer line
  - Strong nonlinearities introduce dependence on initial orbits

Physics of the AGS to RHIC Transfer Line Commissioning, T. Satogata, W.W. MacKay, et al, EPAC’96

Checking Optics with Difference Orbits



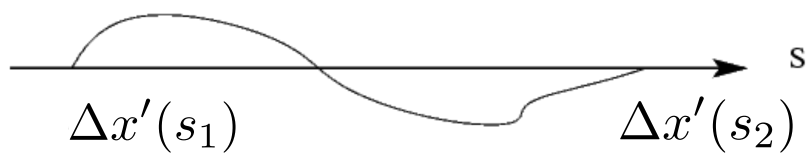
- PEP-II vertical orbit during commissioning
  - Projected vertical betatron oscillation in near-FODO lattice
  - Discrepancy starts at s~700m
  - Later found two quadrupole pairs had ~0.1% errors
  - Can clearly discriminate systematic optics errors from random BPM errors

Y. Cai, 1998, from M. Minty and F. Zimmermann, “Beam Techniques: Beam Control and Manipulation”

Control: Two-Bump

$$\Delta x(s_2) = \Delta x'(s_1)\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\Psi$$

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\Psi - \alpha(s_2)\sin\Delta\Psi]$$

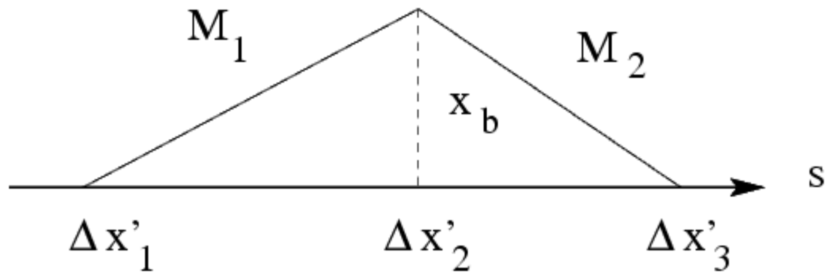


- A single orbit error changes all later positions and angles
  - Add another dipole corrector at a location where $\Delta\Psi = k\pi$ At this point the distortion from the original dipole corrector is all x' that we can cancel with the second dipole corrector.

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}} + \text{angle from } \mathrm{s}_2 \text{ dipole}$$

  - Called a **two-bump**: localized orbit distortion from two correctors
  - But requires $\Delta\Psi = k\pi$ between correctors

Control: Three-Bump (see computer lab)



- A general local orbit distortion from three dipole correctors
  - Constraint is that net orbit change from sum of all three kicks must be zero

$$\begin{pmatrix} C_2 & S_2 \\ C_2' & S_2' \end{pmatrix} \left[\begin{pmatrix} C_1 & S_1 \\ C_1' & S_1' \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x_1' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x_2' \end{pmatrix} \right] + \begin{pmatrix} 0 \\ \Delta x_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Delta x_1' = \frac{x_b}{S_1} \quad \Delta x_2' = -\frac{C_2 S_1 + S_2 S_1'}{S_1 S_2} x_b \quad \Delta x_3' = \frac{S_2}{S_1^2} x_b$$

  - Bump amplitude $x_b = S_1 \Delta x_1'$
  - Only **three-bump** requirement is that S_1, $S_2 \neq 0$

Steering Error in Synchrotron Ring

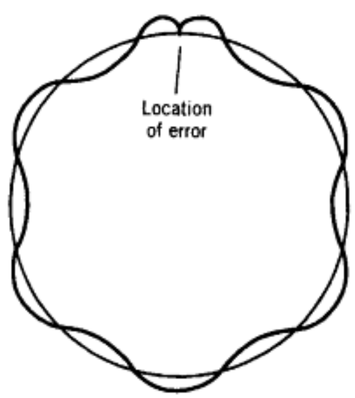
- Short steering error $\Delta x'$ in a ring with periodic matrix M
  - Solve for new periodic solution or design orbit (x_0, x'_0)

$$M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

  - Note that $(x_0=0, x'_0=0)$ is not the periodic solution any more!

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \Delta x'_0 \end{pmatrix}$$

$$\begin{aligned}(I-M)^{-1} = (I - e^{(2\pi Q)J})^{-1} &= \left(\left[e^{\pi Q J}\left(e^{-\pi Q J} - e^{\pi Q J}\right)\right]\right)^{-1} \\ &= -\left(2J\sin(\pi Q)\right)^{-1}\left(e^{\pi Q J}\right)^{-1} \\ &= \frac{1}{2\sin(\pi Q)}\left(J\cos(\pi Q) + I\sin(\pi Q)\right)\end{aligned}$$



New closed orbit

$$x_0 = \frac{\beta_0\, \Delta x'_0}{2}\tan(\pi Q) \qquad \Rightarrow \infty \text{ if } Q = k\pi$$

integer resonances

$$x'_0 = \frac{\Delta x'_0}{2}[1 - \alpha_0 \cot(\pi Q)]$$

Steering Error in Synchrotron Ring

- We can use the general propagation matrix to find the new closed orbit displacement at all locations around the synchrotron

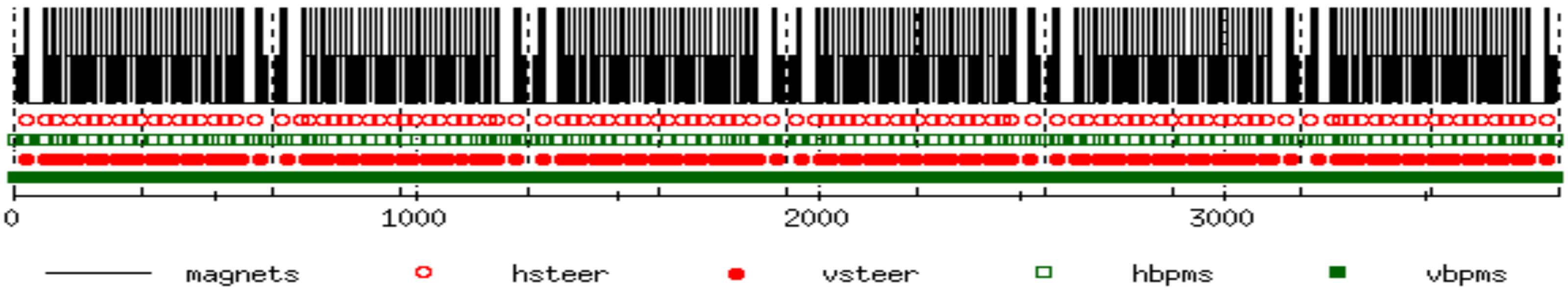
$$x(s) = \frac{\Delta x_0' \sqrt{\beta_0 \beta(s)}}{2\sin(\pi Q)} \cos[\Delta\Psi - \pi Q]$$

  - This displacement of the closed orbit changes its path length
  - If the revolution (RF) frequency is constant, then the beam energy changes, and there is an extra (small) term in the closed orbit displacement

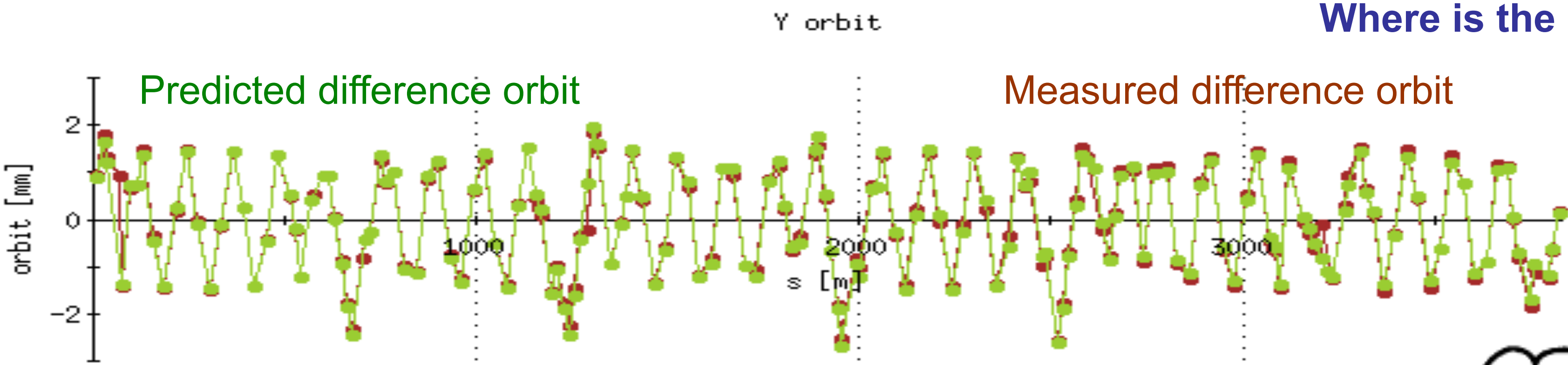
$$x(s) = \frac{\Delta x_0' \sqrt{\beta_0 \beta(s)}}{2\sin(\pi Q)} \cos[\Delta\Psi - \pi Q] + \Delta x_0' \frac{\eta_0 \eta(s)}{\alpha_p C}$$

where $\alpha_p \equiv \left(\frac{dL}{L} / \frac{dp}{p} \right)$ is the momentum compaction and C is the accelerator circumference.

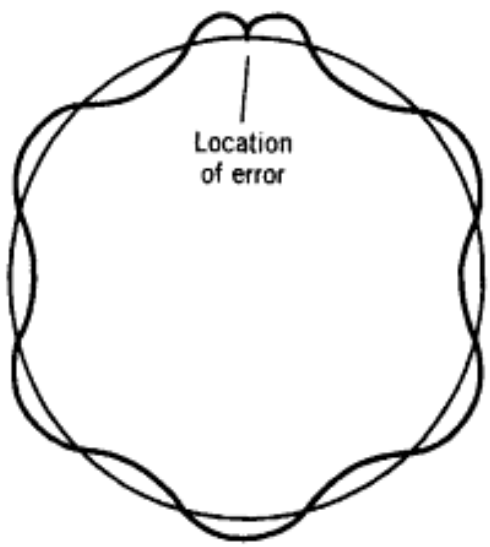
RHIC Vertical Difference Orbit



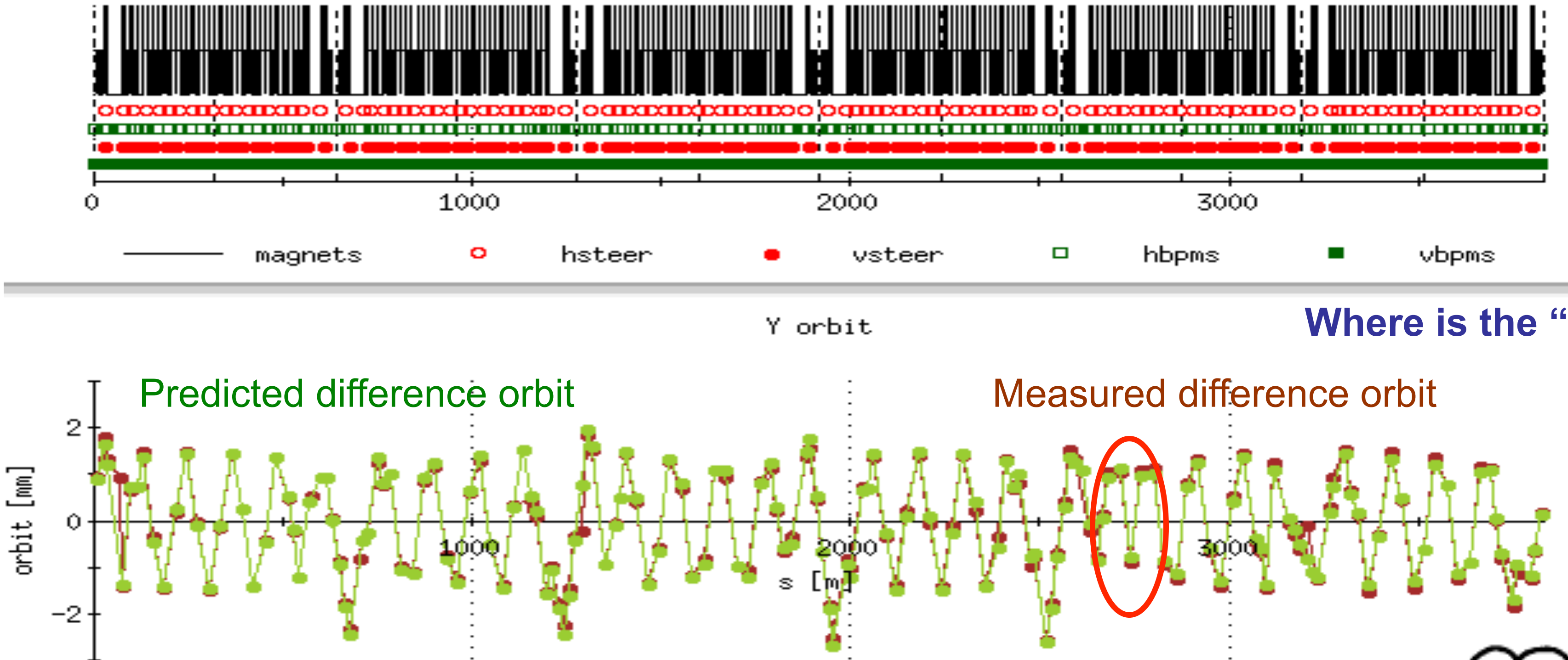
Where is the "cusp"?



- Ring difference orbits also measure optics
- Compare against modeled/design orbit
  - Integer part of tune readily visible: Q~30
  - Can easily find reversed, broken BPMs
  - Can also find (or eliminate) focusing errors



RHIC Vertical Difference Orbit



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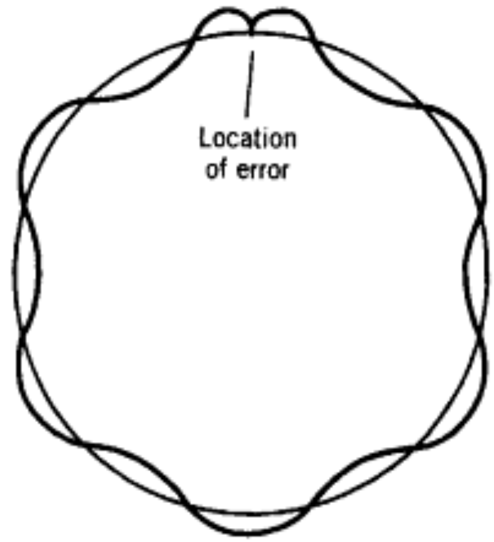
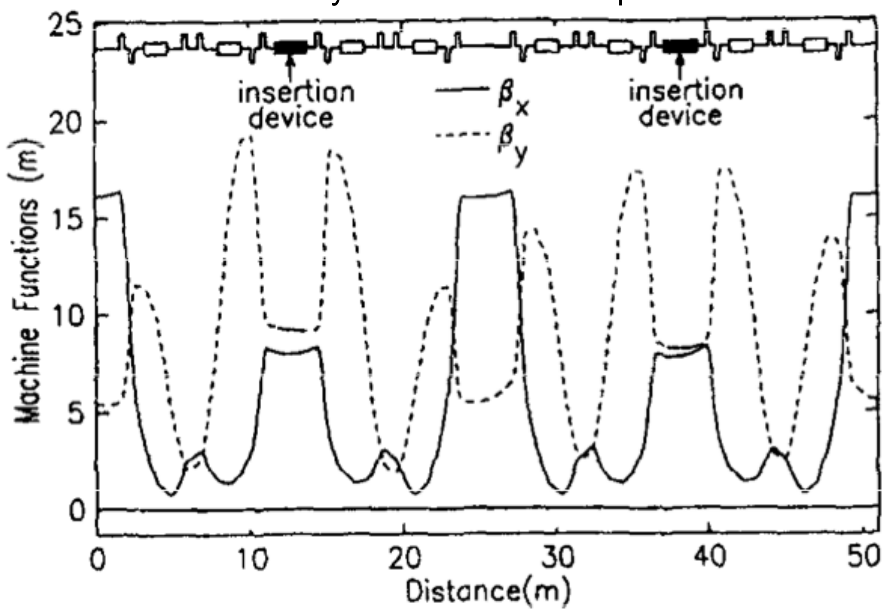
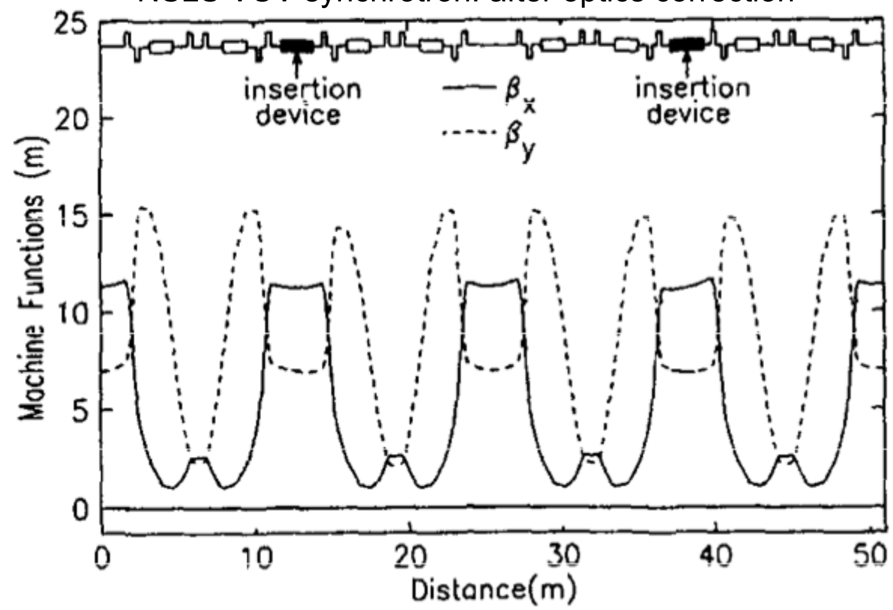


LOCO and SVD Statistical Methods

NSLS VUV synchrotron: before optics correction



NSLS VUV synchrotron: after optics correction



- Measuring many difference orbits in a ring gives a ton of information about optics, BPM and corrector errors
  - Singular value decomposition (SVD) optimization of error fits
  - Has become standard method for correction of synchrotron lattices, particularly synchrotron light sources, led to other methods
    - e.g. model independent analysis (MIA), independent component analysis (ICA)

J. Safranek, "Experimental Determination of Storage Ring Optics using Orbit Response Measurements", 1997

Focusing Error in Synchrotron Ring

- Short focusing error in a ring with periodic matrix M
  - Now solve for Tr M to find effects on tune Q

$$M_{\mathrm{new}} = M \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

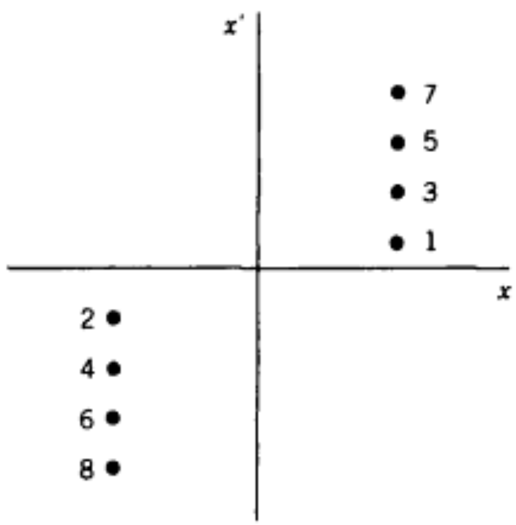
$$\frac{1}{2}\mathrm{Tr}\, M = \cos(2\pi Q_{\mathrm{new}}) = \cos(2\pi Q_0) - \frac{1}{2}\frac{\beta_0}{f}\sin(2\pi Q_0)$$

  - For small errors $Q_{\mathrm{new}} = Q_0 + \Delta Q$ we can expand to find

$$\Delta Q \approx \frac{1}{4\pi}\frac{\beta_0}{f}$$

  - Can be used for a simple measurement of β_0 at the quadrupole

  - Quadrupole errors also cause resonances when $Q = k/2$: **half-integer resonances**



Chromaticity Correction

Natural chromaticity $$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds$$

- How can we control chromaticity in our synchrotron ring?
  - We need a way to connect momentum offset δ to focusing
  - Dispersion (momentum-dependent position) and sextupoles (nonlinear focusing depending on position) come to rescue

$$x(s) = x_{\text{betatron}}(s) + \eta_x(s)\delta$$

$$\text{Sextupole B field} \quad B_y = b_2 x^2$$

$$B_y(\text{sext}) = b_2[x_{\text{betatron}}(s) + \eta_x(s)\delta]^2 \approx b_2 x^2_{\text{betatron}} + 2b_2 x_{\text{betatron}}(s)\eta_x(s)\delta$$

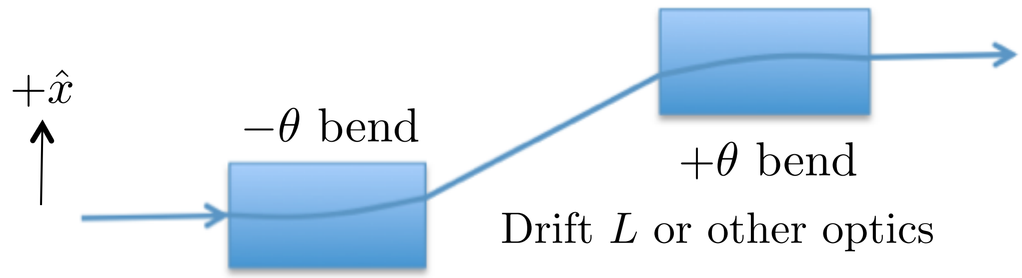
Nonlinear! like a quadrupole K(s)!

  - Total chromaticity from all sources is then

$$\boxed{\xi = -\frac{1}{4\pi Q}\oint [K(s) - b_2(s)\eta_x(s)]ds}$$

  - Strong focusing (large K) requires large sextupoles, **nonlinearity**!

Doglegs



- Displaces beam transversely without changing direction
- What is effect on 6D optics?

$$\mathbf{M}_{\text{dipole}}(\rho,\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} - \rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(C&M 3.102)

- Be careful about the coordinate system and signs!!
- If $\rho,\theta>0$, positive displacement points **out** from dipole curvature
- Be careful about order of matrix multiplication!

Reverse Bend Dipole Transport

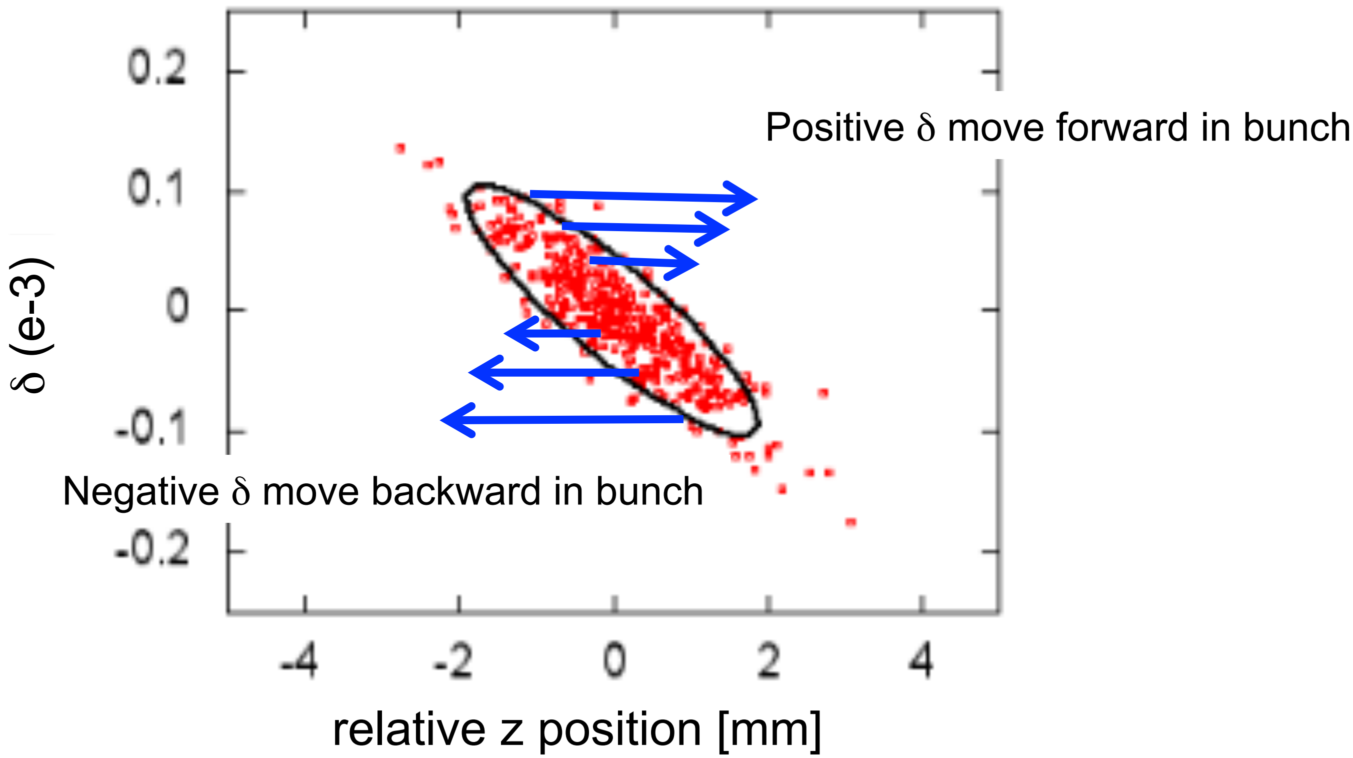
- What is the correct 6x6 transport matrix of a reverse bend dipole?
- It turns out to be achieved by reversing **both** ρ **and** θ
  - $\rho\theta$=L (which stays positive) so both must change sign

$$\mathbf{M}_{\mathrm{dipole}}(-\rho,-\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \boxed{-\rho(1-\cos\theta)} \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \boxed{-\sin\theta} \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{\sin\theta} & \boxed{\rho(1-\cos\theta)} & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} - \rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

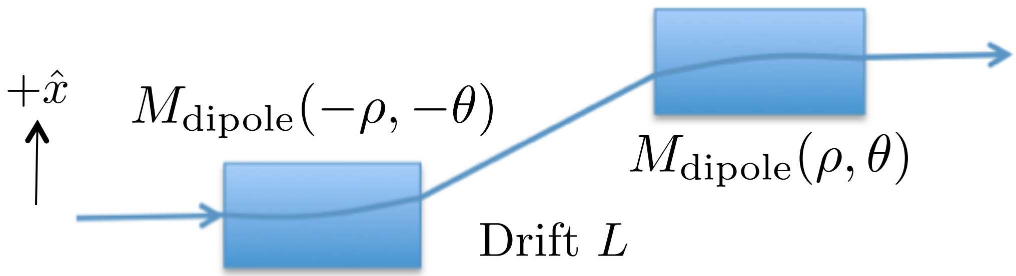
$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Aside: Longitudinal Phase Space Drift

- Wait, what was that M_{56} term with the relativistic effects?
  - Recall longitudinal coordinates are (z, δ)
  - This extra term is called "ballistic drift": not in all codes!
    - Important at low to modest energies and for bunch compression
    - Relativistic terms enter converting momentum p to velocity v



Weak Dogleg



$$\mathbf{M}_{\mathrm{dogleg}} = \mathbf{M}_{\mathrm{dipole}}(\rho, \theta)\mathbf{M}_{\mathrm{drift}}\mathbf{M}_{\mathrm{dipole}}(-\rho, -\theta)$$

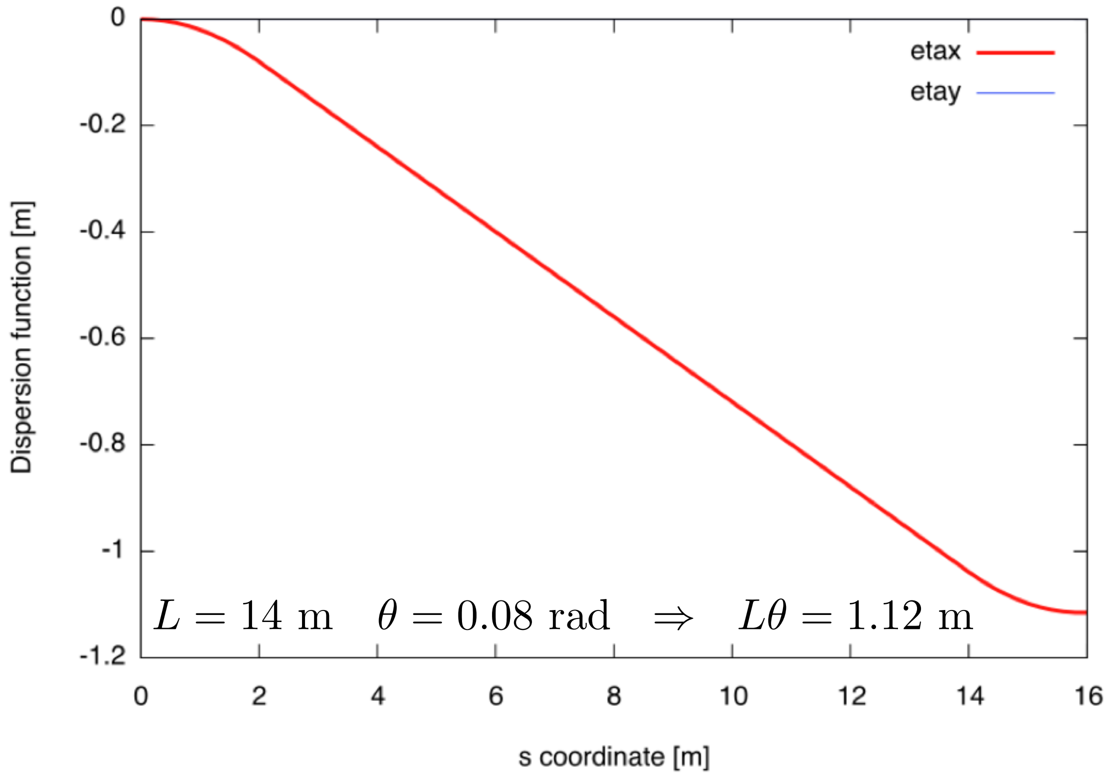
$$\mathbf{M}_{\mathrm{weak\ dogleg}} = \begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ 0 & 1 & -\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L + 2\rho\theta & \boxed{-L\theta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} (\eta, \eta')_{\mathrm{in}} = 0 \\ \Rightarrow \; (\eta, \eta')_{\mathrm{out}} = (-L\theta, 0) \end{matrix}$$

Strong dogleg can also be derived:

$$D = -L\cos\theta\sin\theta \qquad D' = \frac{L\sin^2\theta}{\rho}$$

Dogleg Dispersion



For weak dogleg

$$(\eta, \eta')_{\text{in}} = 0$$
$$\Rightarrow \quad (\eta, \eta')_{\text{out}} = (-L\theta, 0)$$

Does this make sense?

$$\Delta x'(\delta) = \frac{BL}{(B\rho)} = \frac{q}{p(1+\delta)}[BL] \approx \frac{q}{p}[BL](1-\delta) = (1-\delta)\Delta x'(\delta = 0)$$

A small momentum offset of $+\delta$ reduces the dipole kick by a factor of delta, and this is magnified to a transverse offset from design at the end of the dogleg by $-\delta L\theta$.

Achromatic Dogleg

- How can we make an achromatic dogleg?

$$(\eta, \eta')_{\mathrm{in}} = (0\ \mathrm{m}, 0) \quad \Rightarrow \quad (\eta, \eta')_{\mathrm{out}} = (0\ \mathrm{m}, 0)$$

- Use an I insertion (e.g. four consecutive $\pi/2$ insertions)

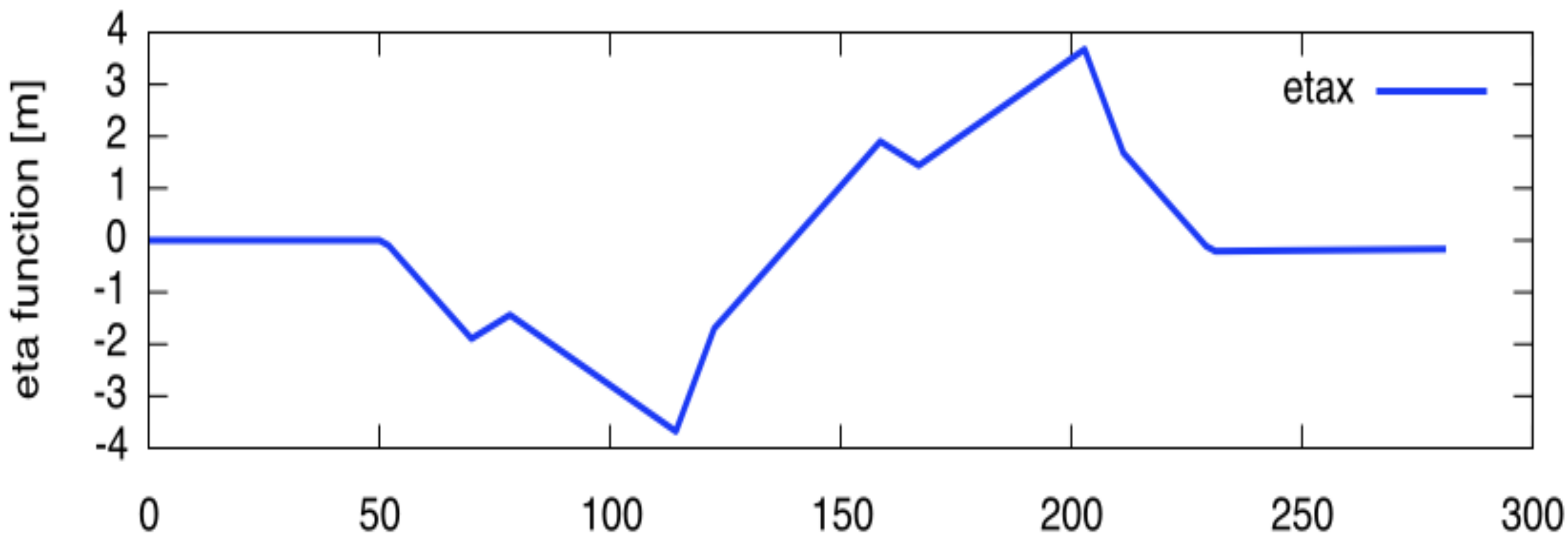
$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J} \qquad (\mathrm{Recall}\ \mathbf{J}^4 = \mathbf{I})$$

$$\mathbf{M}_{\mathrm{achromatic\ dogleg}} = \begin{pmatrix} \cos(2\theta) & \rho\sin(2\theta) & 0 \\ -\frac{\sin(2\theta)}{\rho} & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

achromatic!

- **Any** transport with net phase advance of $2n\pi$ will be achromatic ($n\pi$ if all dipoles bend in same direction)
  - common trick for matching dispersive bending arcs to non-dispersive straight sections.

Achromatic Dogleg



Achromatic Dogleg: Steffen CERN School Notes

Example of nondispersive translating system

Φ = sector magnet bend. angle

$\varphi = \ell\sqrt{k}$ = quadrupole magnet phase angle

d,λ = drift space lengths.

The system is nondispersive if the sinelike trajectory (with respect to the central symmetry point) goes through the mid-point of the bending magnets, i.e. if

$$\rho \tan\frac{\Phi}{2} + \lambda = \frac{1}{\sqrt{k}} \frac{d\sqrt{k}\cos\varphi + 2\sin\varphi}{d\sqrt{k}\sin\varphi - 2\cos\varphi} .$$

Focusing also in the other plane may be obtained by adding a third quadrupole of opposite polarity at the symmetry point.

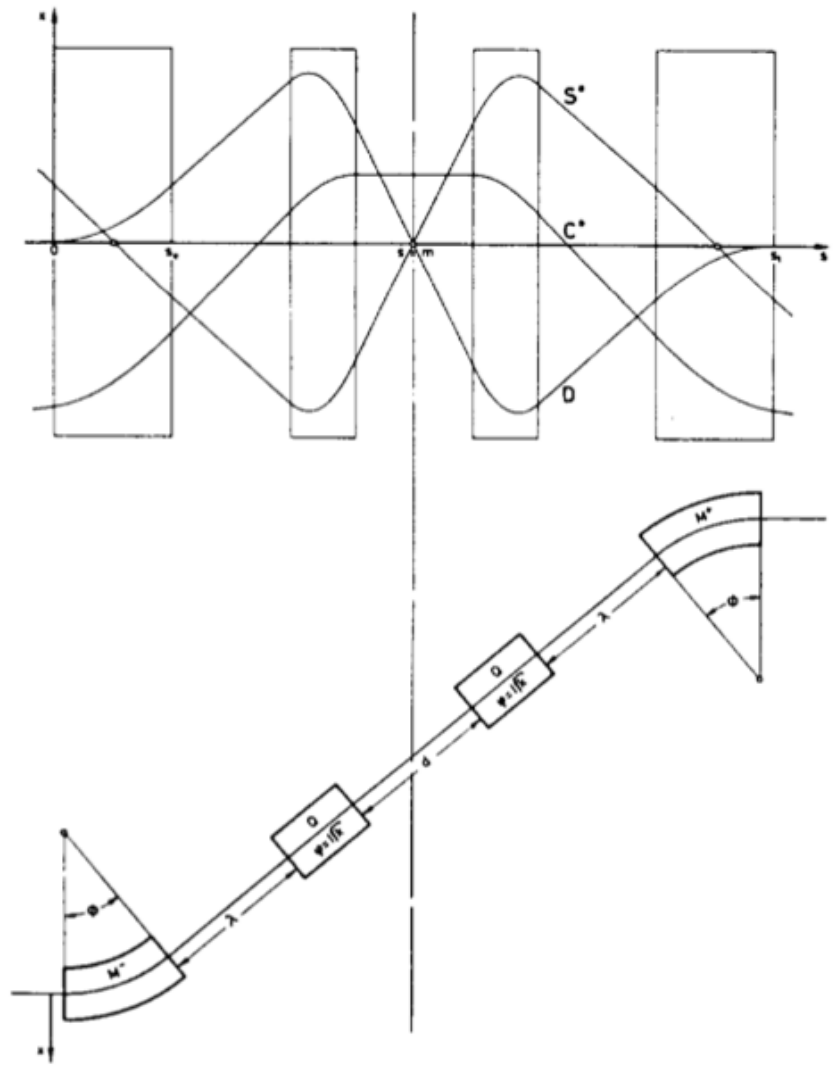


Fig. 15: Nondispersive translating system.

K. Steffen, CERN-85-19-V-1, 1985, p. 55

First-Order Achromat Theorem

- A lattice of n repetitive cells is achromatic (to first order, or in the linear approximation) iff $\mathrm{M}^n = I$ or each cell is achromatic
- Proof:

$$\text{Consider } \mathbf{R} \equiv \begin{pmatrix} \mathbf{M} & \bar{d} \\ 0 & 1 \end{pmatrix} \quad \text{where } \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_2 = \mathbf{R} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_1 \qquad \bar{d} = \begin{pmatrix} M_{16} \\ M_{26} \end{pmatrix}$$

$$\textbf{For } n \text{ cells}: \quad \mathbf{R}^n = \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^{n-1} + \mathbf{M}^{n-2} + \ldots + \mathbf{I})\bar{d} \\ 0 & 1 \end{pmatrix}$$

$$\text{but } \quad (\mathbf{M}^{n-1} + \mathbf{M}^{n-2} + \ldots + \mathbf{I}) = (\mathbf{M}^n - I)(\mathbf{M} - I)^{-1}$$

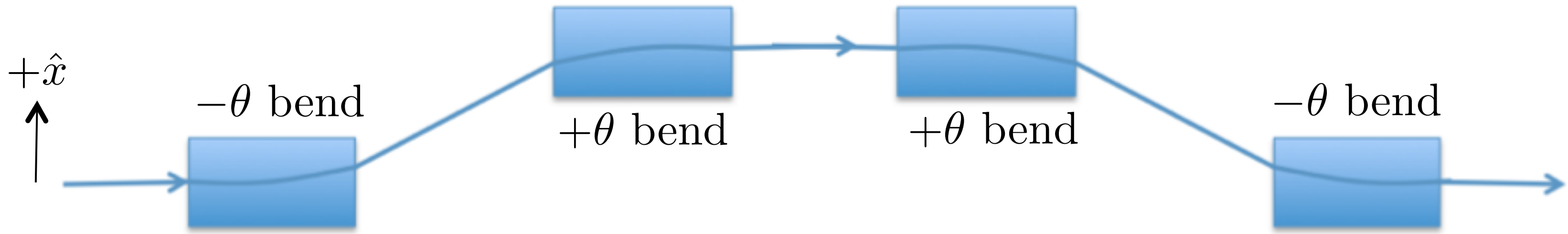
$$\text{So for } n \text{ cells}: \quad \mathbf{R}^n = \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^n - \mathbf{I})(\mathbf{M} - \mathbf{I})^{-1}\bar{d} \\ 0 & 1 \end{pmatrix}$$

- So the lattice is achromatic only if $\bar{d} = 0$ or $\mathbf{M}^n = \mathbf{I}$

$$\mathbf{M}^n = \mathbf{I}\cos\mu_{\text{tot}} + \mathbf{J}\sin\mu_{\text{tot}} \quad \Rightarrow \quad \mu_{\text{tot}} = 2\pi k$$

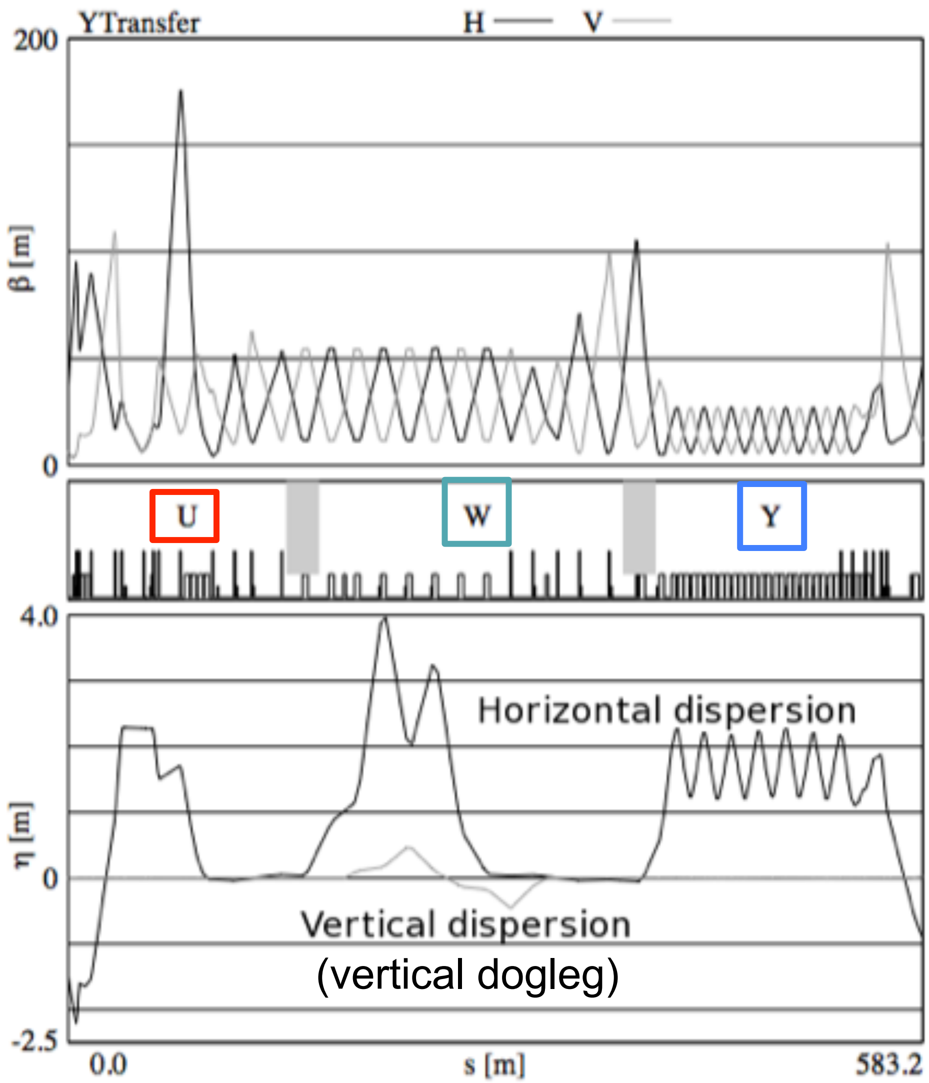
S.Y. Lee, “Accelerator Physics”

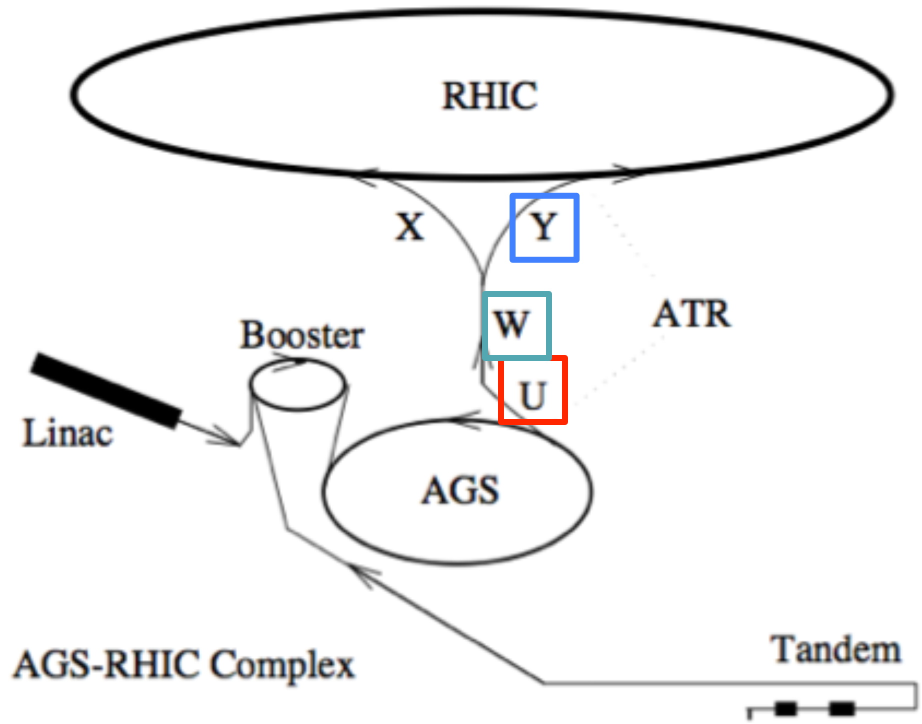
Chicane



- Divert beam around an obstruction
  - e.g. vertical bypass chicane in Fermilab Main Ring
  - e.g. horizontal injection chicane in CEBAF recirculating linac
  - Essentially a design orbit "4-bump" (4 dipoles)
- Usually need some focusing, optics between dipoles
- Usually design optics to be achromatic
  - Operationally null orbit motion at end of chicane vs changes in input beam energy
- Naively expect $M_{56}<0$ (bunch lengthening or decompression)
  - Higher energy particles ($+\delta$) have shorter path lengths
  - But can compress bunches with introduction of longitudinal correlation

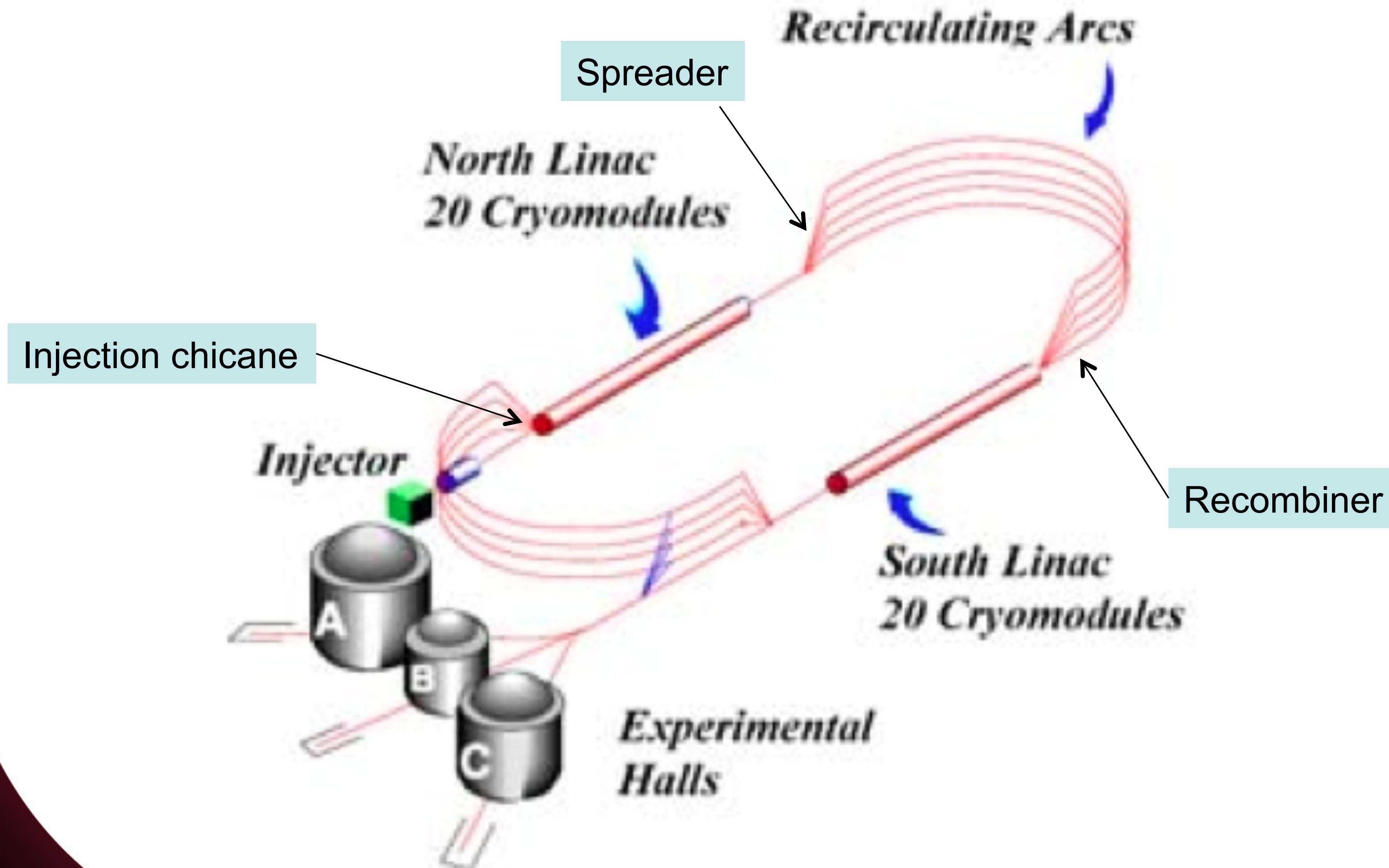
AGS to RHIC Transfer Line





The ATR vertical dogleg is not strictly a dogleg since the planes of the AGS and RHIC accelerators are not parallel

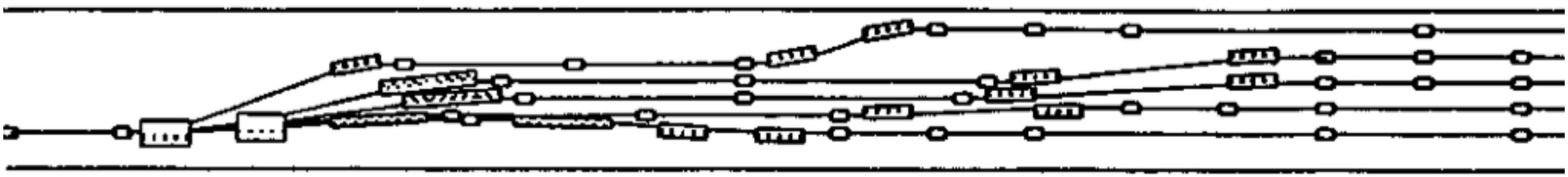
CEBAF



Recirculating Arcs
Spreader
North Linac
20 Cryomodules
Injection chicane
Injector
Recombiner
South Linac
20 Cryomodules
A
B
C
Experimental
Halls

CEBAF Spreaders/Recombiners

- Problem: Separate different energy beams for transport into arcs of CEBAF, and recombine before next linac
  - Achromats: arcs are FODO-like, linacs are dispersion-free
  - "I" insertion: 1 betatron wavelength between dipoles
  - Single dogleg: unacceptably high beta functions
  - Two consecutive "staircase" doglegs with same total phase advance was solution

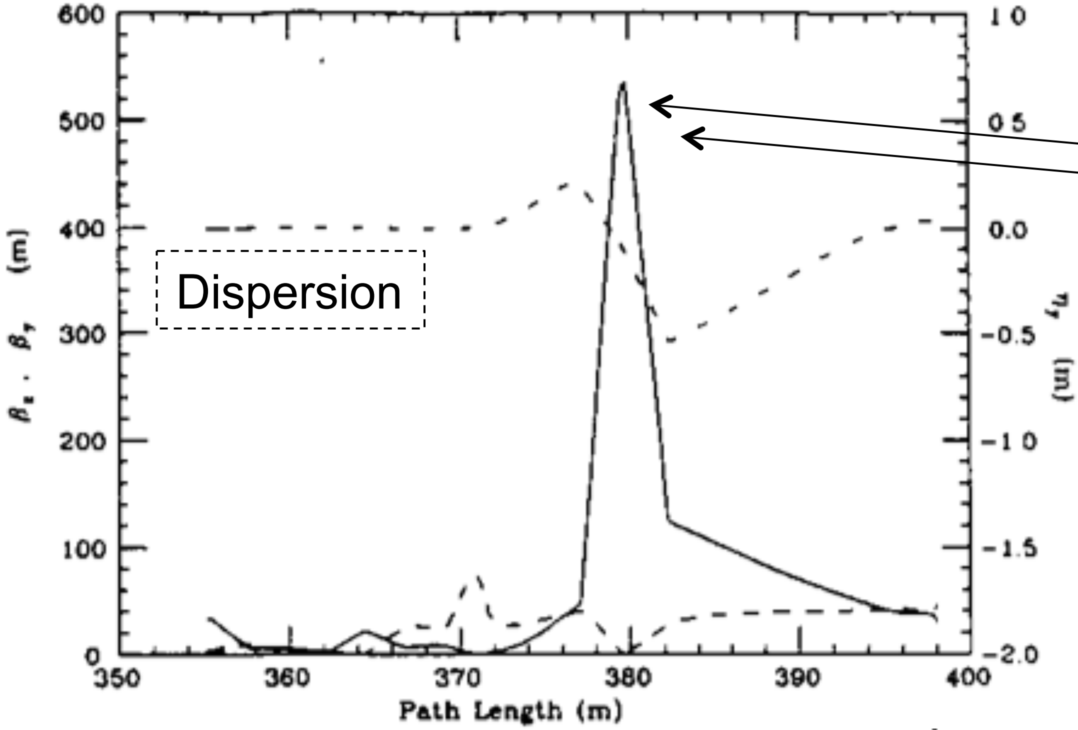


EAST ARC ELEVATION

  - Still quite a challenge in physical layout of real magnets!

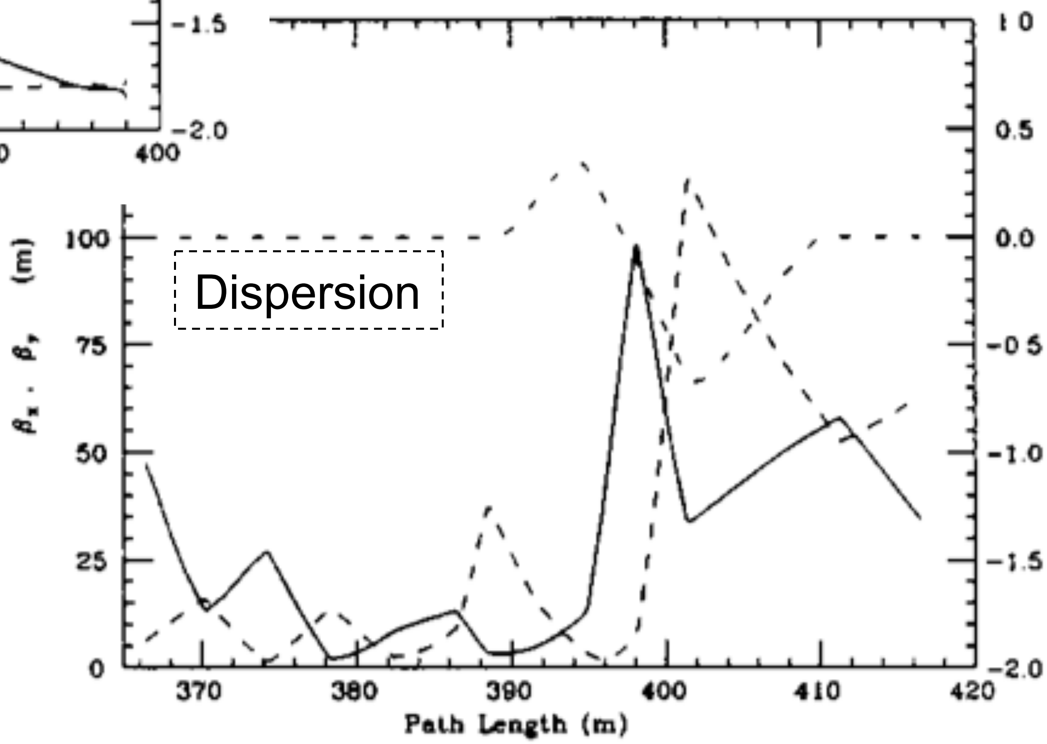
D. Douglas, R.C. York, J. Kewisch, "Optical Design of the CEBAF Beam Transport System", 1989

CEBAF Spreaders/Recombiners



“One step” recombiner
Unacceptably large vertical beta function/beam size (550 m)

“Staircase” two-step recombiner
Acceptable beta functions and beam sizes in both planes (100 m)



Mobius Insertion

- Fully coupled equal-emittance optics for e^+e^- CESR collisions
  - Symmetrically exchange horizontal/vertical motion in insertion
  - Horizontal/vertical motion are coupled
    - Only one transverse tune degree of freedom!

$$Q_{x,y}: \ \text{unrotated tunes} \qquad Q_{1,2} = \frac{Q_x + Q_y}{2} \pm \frac{1}{4} \qquad Q_1 - Q_2 = \frac{1}{2}$$

  - Match insertion to points where $\beta_x=\beta_y$ and $\alpha_x=\alpha_y$ with phase advances that differ by π between planes
    - Normal insertion: $\mathbf{M}_{\text{erect}} = \begin{pmatrix} \mathbf{T} & 0 \\ 0 & -\mathbf{T} \end{pmatrix}$
    - Rotated by 45 degrees around s axis: $\mathbf{M}_{\text{mobius}} = \begin{pmatrix} 0 & \mathbf{T} \\ \mathbf{T} & 0 \end{pmatrix}$
  - A purely transverse example of an **emittance exchanger**

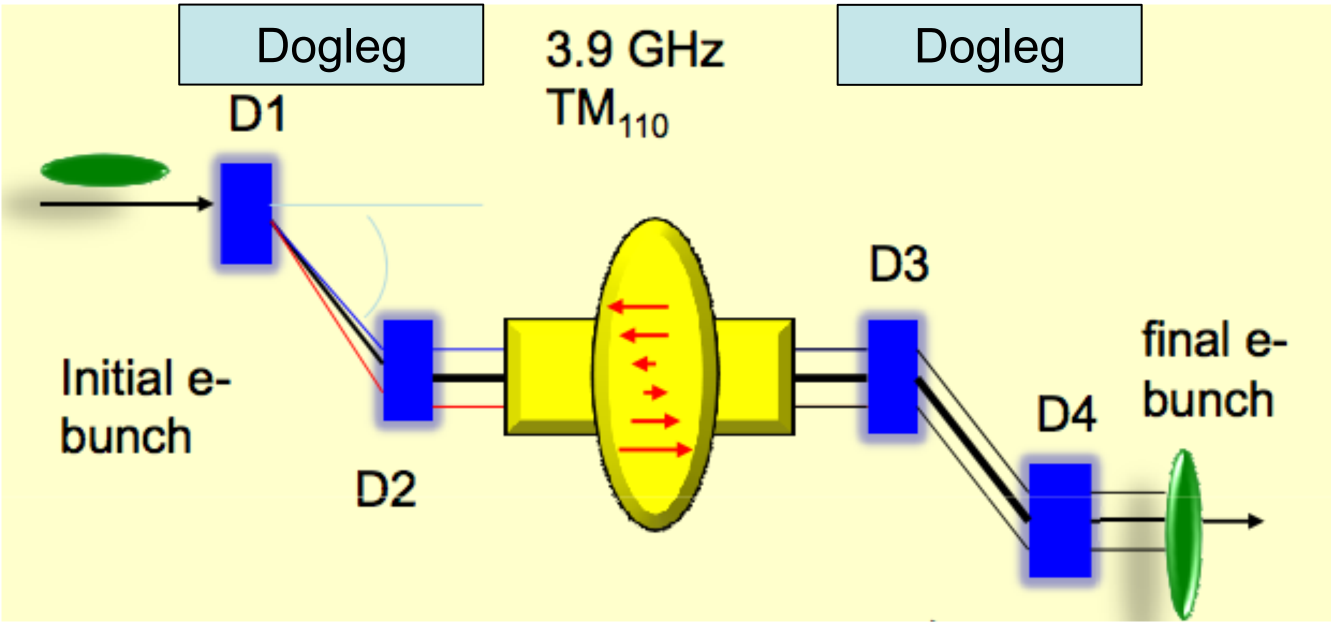
S. Henderson, R. Talman, et al., "Investigation of the Möbius Accelerator at CESR", Proc. of the 1999 Particle Accelerator Conference, New York, NY; R. Talman, "A Proposed Möbius Accelerator", Phys. Rev. Lett **74**, 1590-3 (1995).

Transverse/Longitudinal Emittance Exchange

- X-ray FELs demand ultra-low transverse emittance beam*
- State-of-the art photo-injectors can generate low 6-D emittance. Typically asymmetric emittances. Emittance exchange can swap transverse with the longitudinal emittance.
- Allows one to convert transverse modulations to longitudinal modulations : Beam shaping application
- Can also be used to suppress microbunching instability**

J.C.T. Thangaraj, Experimental Studies on an Emittance Exchange Beamline at the A0 Photoinjector, 2012

Fermilab A0 Emittance Exchanger



$\theta:$ Bending angle

$\eta:$ dogleg dispersion

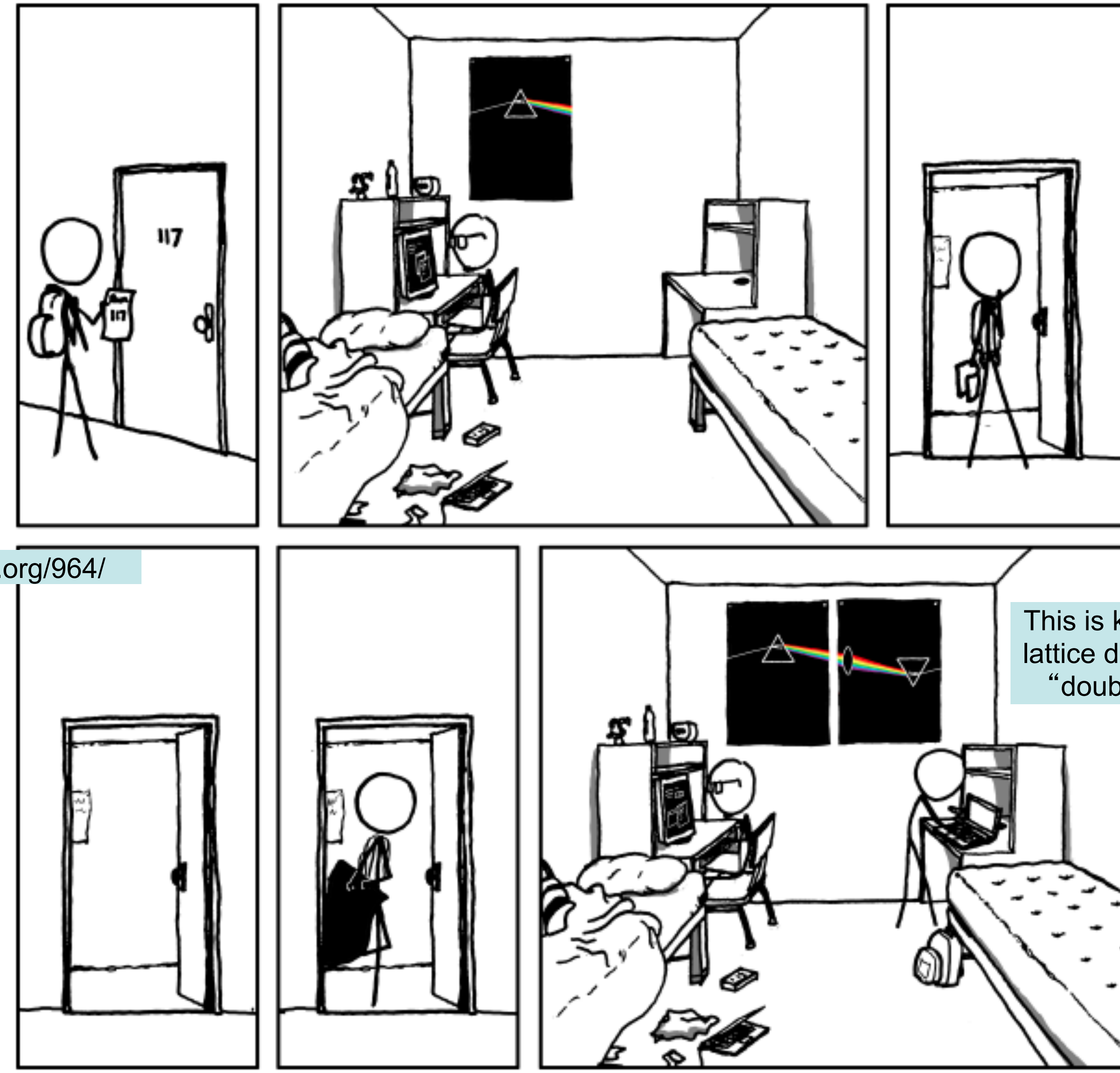
$L:$ dogleg length

$L_c:$ RF cell length

$$\mathbf{M} = \begin{pmatrix} 1 & \frac{L_c}{4} & -\frac{(4L+L_c)}{4\eta} & \eta - \frac{\theta(4L+L_c)}{4} \\ 0 & 1 & -\frac{1}{\eta} & -\theta \\ -\theta & \eta - \frac{\theta(4L+L_c)}{4} & 1 + \frac{\theta L_c}{4\eta} & \frac{\theta^2 L_c}{4} \\ -\frac{1}{\eta} & -\frac{4L+L_c}{4\eta} & \frac{\theta L_c}{4\eta^2} & 1 + \frac{\theta L_c}{4\eta} \end{pmatrix}$$

J.C.T. Thangaraj, Experimental Studies on an Emittance Exchange Beamline at the A0 Photoinjector, 2012

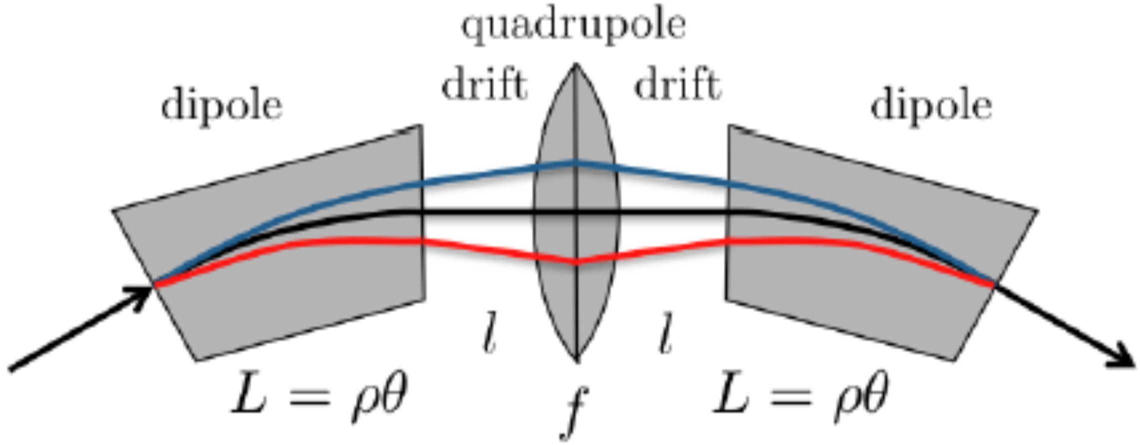
(xkcd interlude)



http://www.xkcd.org/964/

This is known in accelerator lattice design language as a "double bend achromat"

Double Bend Achromat (approximate)



- You calculated constraints for the double bend achromat in your homework last night

$$M_{\rm dipole} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho[1-\cos\theta] \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Keep lowest-order terms in θ, including θ^2 in upper right term since $\rho\theta = L$

$$M_{\rm dipole} = \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Double Bend Achromat (approximate)

$$\mathbf{M}_{\mathrm{DBA}} = \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{DBA}} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = 1 - \frac{(L+l)}{f}$$

$$S = \frac{(L+l)(2f-L-l)}{f}$$

$$\boxed{D = \theta\frac{(L+l)(4f-L-2l)}{2f}}$$

$$C' = -\frac{1}{f}$$

$$S' = 1 - \frac{(L+l)}{f} = C$$

$$D' = \theta\frac{(4f-L-2l)}{2f} = \frac{D}{L+l}$$

Double Bend Achromat (approximate)

- The periodic solutions for dispersion for the general M matrix were derived in class and the text

$$\eta(\text{periodic}) = \frac{[1-S']D + SD'}{2(1-\cos\mu)} = 0$$

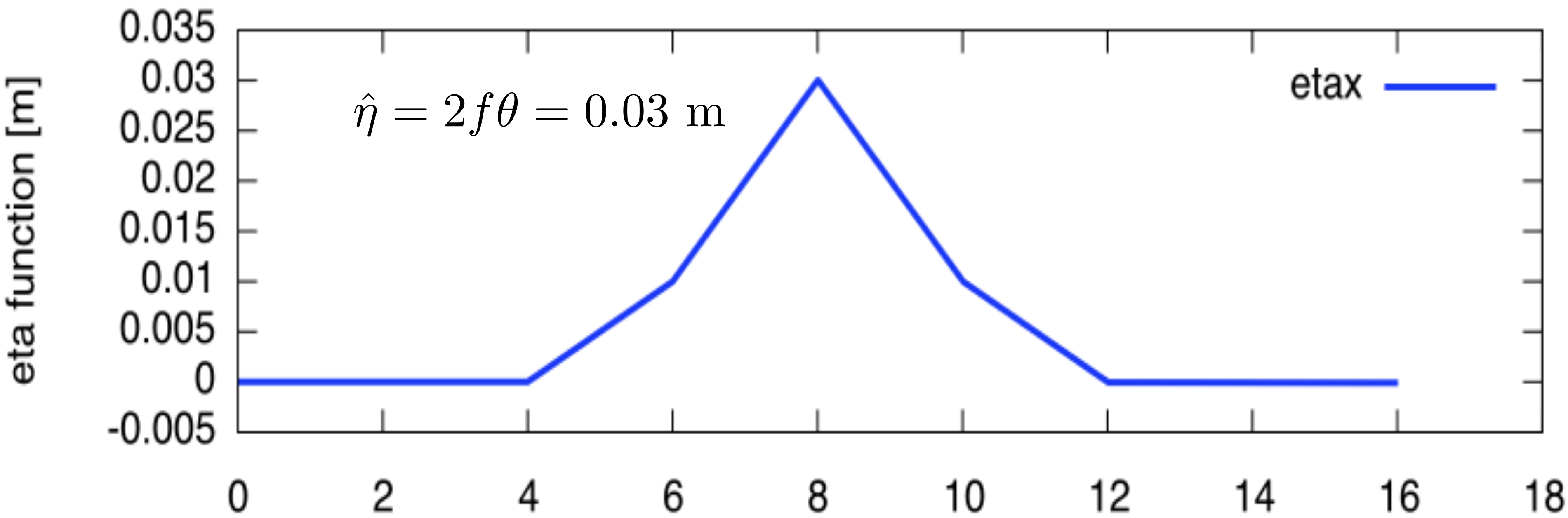
$$\eta'(\text{periodic}) = \frac{[1-C]D' + C'D}{2(1-\cos\mu)} = 0$$

- It turns out that the η' equation is satisfied automatically!
  - This is a consequence of the mirror symmetry of the system
- The η equation is satisfied if D=0:

$$D = \theta \frac{(L+l)(4f-L-2l)}{2f} = 0$$

$$\Rightarrow 4f - L - 2l = 0 \qquad \Rightarrow \qquad f = \frac{L+2l}{4} \qquad \hat{\eta} = \frac{(L+2l)\theta}{2} = 2f\theta$$

Double Bend Achromat

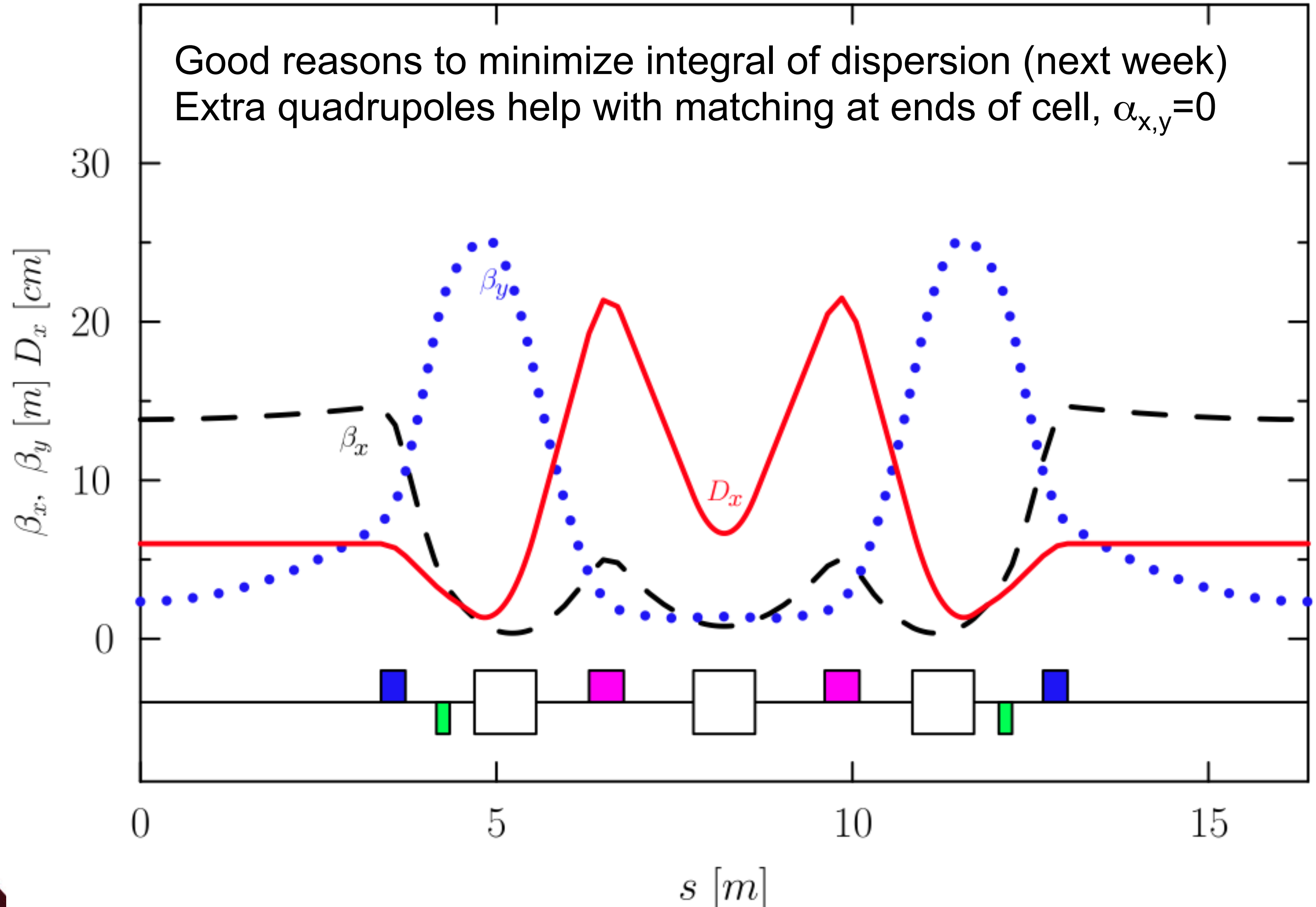


$$L = l = 2\ \mathrm{m} \qquad \theta = 0.01\ \mathrm{rad} \qquad f = \frac{L+2l}{4} = 1.5\ \mathrm{m} \qquad (KL_{\mathrm{quad}}) = 0.667\ \mathrm{m}^{-1}$$

$$\mathrm{Exact\ DBA:} \qquad f = \frac{l}{2} + \frac{\rho}{2}\tan(\theta/2) \qquad \hat{\eta} = \rho(1-\cos\theta) + l\sin\theta$$

- DBA is also known as a Chasman-Green lattice
  - Used in early third-generation light sources (e.g. NSLS at BNL)
  - More after we discuss synchrotron radiation, $\mathcal{H}$ functions

Triple Bend Achromat Cell (ALS at LBL)



L. Yang et al, Global Optimization of an Accelerator Lattice Using Multiobjective Genetic Algorithms, 2009