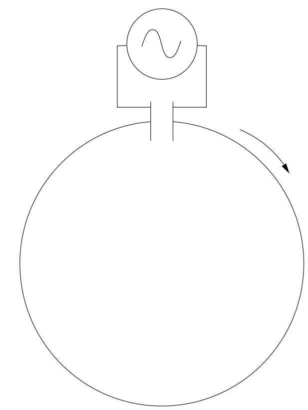
Transition Energy

Angular Revolution frequency:

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi\beta c}{L},$$



Ring with 1 rf cavity.

Differentiating $\ln(\omega)$ yields

$$\frac{d\omega}{\omega} = -\frac{d\tau}{\tau} = \frac{d\beta}{\beta} - \frac{dL}{L} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p}.$$

Define *phase slip factor*:

$$\eta_{\rm tr} = \frac{1}{\gamma^2} - \alpha_p = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\rm tr}^2}.$$

Note the sign flip (or transition) of $\frac{d\omega}{dp}$ at $\eta_{\rm tr} = 0$, i. e. when $\gamma = \gamma_{\rm tr} = \frac{1}{\sqrt{\alpha_p}}$.

Comment on convention

Note that a lot of people define η_{tr} with the opposite sign.

We follow the Ernest Courant's convention from

E. D. Courant, "Computer Studies of Phase-Lock Acceleration", International Conference on High Energy Accelerators" 201 (1961).

Basically, since it's called the phase-slip factor, we think it should be relative to $d\omega/\omega$ rather than $d\tau/\tau$.

(Perhaps some folks consider it better to fall on your bum than on your face.)

$$\frac{d\omega}{\omega} = -\frac{d\tau}{\tau} = \frac{d\beta}{\beta} - \frac{dL}{L} = \eta_{\rm tr}\,\frac{dp}{p}.$$

- Below transition energy, the change in frequency is dominated by the $\dfrac{d\beta}{\beta}$ term.
  - The particles sort of behave more nonrelativistically.

- As energy increases past transition, velocities approach speed of light, so that the $\dfrac{dL}{L}$ dominates.
  - The particles sort of behave more ultrarelativistically.

Voltage in the cavity as function of time:

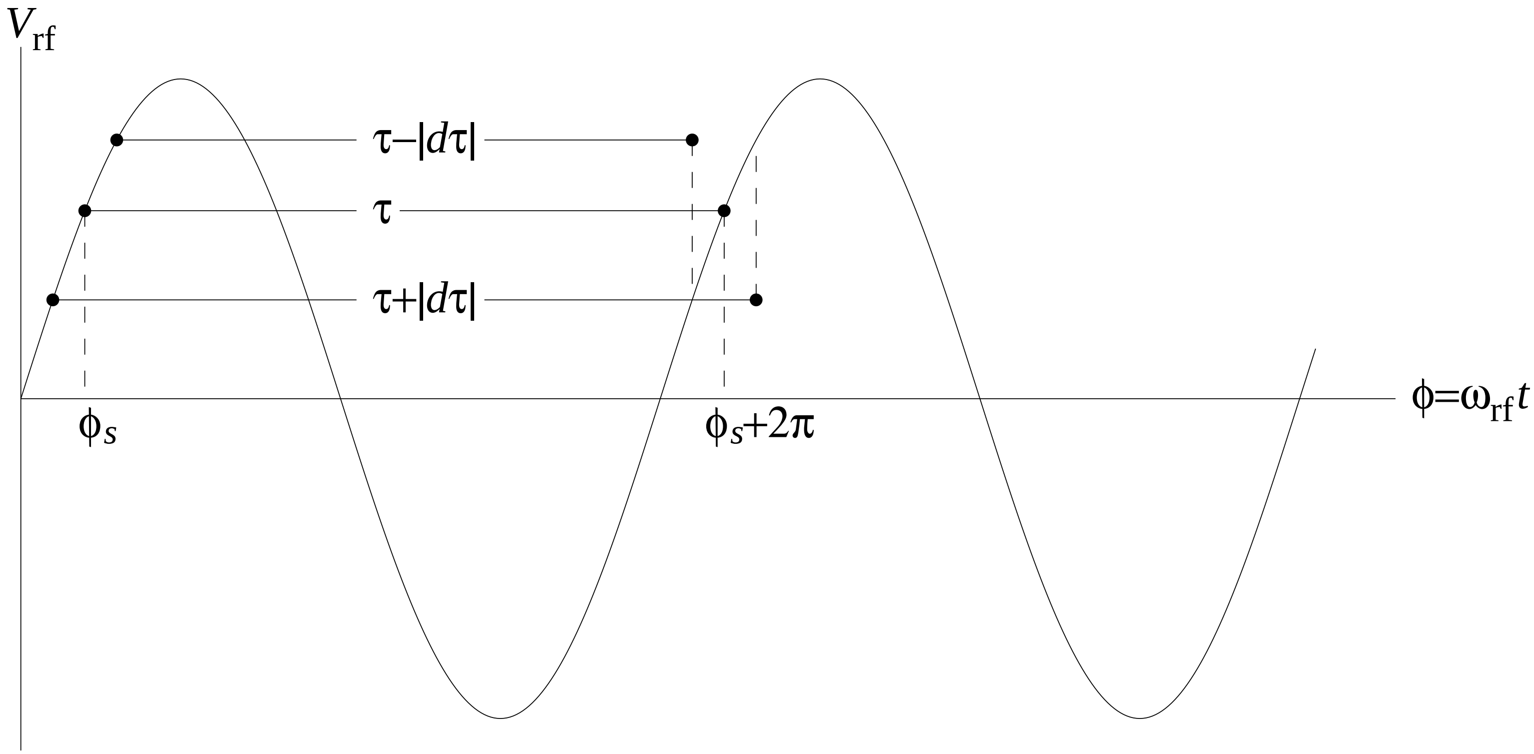
$$V_{\rm rf}(t) = V \sin(\omega_{\rm rf} t + \phi_s).$$

To understand stability, let us assume for the present that

$$\omega_{\rm rf} = \omega_{\rm rev} = \frac{2\pi\beta c}{L}.$$

(It makes the pictures easier.)

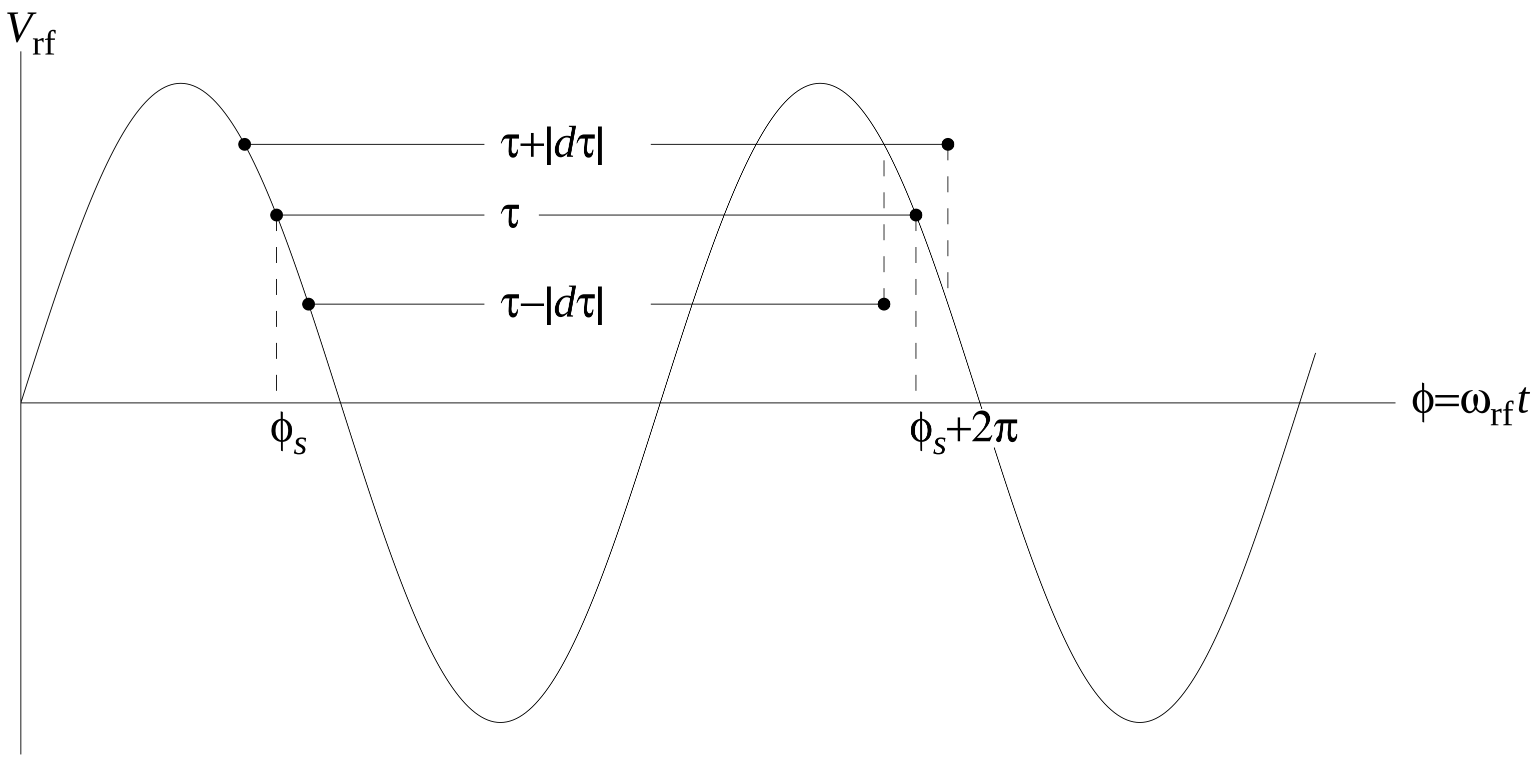
Phase staibility below transition



Below transition energy $\quad \gamma < \gamma_{\mathrm{tr}}: \quad \eta_{\mathrm{tr}} = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\mathrm{tr}}^2} > 0.$

- Increasing energy – takes less time per turn: $\quad \frac{d\tau}{\tau} = -\eta_{\mathrm{tr}}\frac{dp}{p}.$
- Note: plot shows energy gain for synchronous particle on each turn.

Phase stability above transition



Above transition energy $\gamma > \gamma_{\rm tr}$: $\eta_{\rm tr} = \dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_{\rm tr}^2} < 0.$

- Increasing energy – takes more time per turn: $\dfrac{d\tau}{\tau} = -\eta_{\rm tr}\dfrac{dp}{p}.$

Standing waves

A standing wave in a cavity can be considered as the superposition of traveling waves in opposite directions:

$$
\begin{aligned}
\frac{V}{2}\sin(kz+\omega_{\rm rf}t)-\frac{V}{2}\sin(kz-\omega_{\rm rf}t) = {} & \frac{V}{2}[\sin(kz)\cos(\omega_{\rm rf}t)+\cos(kz)\sin(\omega_{\rm rf}t)] \\
& -\frac{V}{2}[\sin(kz)\cos(\omega_{\rm rf}t)-\cos(kz)\sin(\omega_{\rm rf}t)] \\
= {} & V\sin(kz)\sin(\omega_{\rm rf}t).
\end{aligned}
$$

To quantify this a bit further, let's make a few simplifying assumptions:

1) There is only one accelerating gap of length g, located at $s = 0$.
2) The accelerating gap is much shorter than the distance traveled by the beam during one rf period, i. e., $g \ll \beta\lambda_{\rm rf}$.
3) The rf angular frequency is an integer multiple of the angular revolution frequency, ω_s, i. e., $\omega_{\rm rf} = h\omega_s$ for some integer, h, called the *harmonic number*.
4) The *synchronous particle* crosses the gap at time $t = 0$, when the rf phase is ϕ_s, and the voltage across the gap is $V \sin\phi_s$.
5) As energy increases the revolution frequency $\omega_s = \frac{2\pi\beta c}{L}$ increases, so we must increase the rf frequency as the energy is ramped.
   - This requires feedback on $\omega_{\rm rf}$ to keep L constant as B is ramped.
   - Exception: when $\beta \simeq 1$, such as high energy $e^{\pm}$ rings.

The energy gained by the synchronous particle per revolution is

$$\Delta U_s = qV \sin\phi_s,$$

and the effective electric field may be written as

$$\vec{E}(s,t) = \hat{s}\,E(s,t) = \hat{s}\,V\sin(\omega_{\rm rf}t + \phi_s)\sum_{n=-\infty}^{\infty}\delta(s - nL),$$

where L is the circumference of the synchronous particle's orbit.

Fourier series:
$$E(s,t) = \frac{V}{L}\sin(\omega_{\rm rf}t + \phi_s)\sum_{n=-\infty}^{\infty}\cos\left(\frac{2\pi ns}{L}\right)$$
$$= \frac{V}{L}\sum_{n=-\infty}^{\infty}\sin\left[\omega_s\left(ht - \frac{n}{v}s\right) + \phi_s\right],$$

where the synchronous particle's velocity is $v = \dfrac{L\omega_s}{2\pi}$.

The time that the synchronous particle passes a point s may be written

$$t_s = \frac{s}{v},$$

and the time for a generic particle

$$t = t_s + \delta t,$$

where the generic particle lags behind the synchronous particle by δt.

The longitudinal (energy/momentum) oscillations will typically much slower than the revolution period.

So we can average over one revolution period:

$$\langle E(\delta t)\rangle = \frac{V}{L}\sin(\omega_{\rm rf}\,\delta t + \phi_s)$$

for the effective field seen by the generic particle with lag δt.

A generic particle will then gain

$$\Delta U = qV \sin(\omega_{\rm rf}\delta t + \phi_s)$$

per turn which agrees with

$$\Delta U_s = qV \sin\phi_s$$

for $\delta t = 0$.

Define generic values (δ) relative to synchronoous values (subscript "s"):

$$\begin{aligned}
\text{total energy} \quad & U = U_s + \delta U. \\
\text{momentum} \quad & p = p_s + \delta p. \\
\text{angular frequency} \quad & \omega = \omega_s + \delta\omega. \\
\text{revolution period} \quad & \tau = \tau_s + \delta\tau, \qquad \text{with} \quad \mathrm{sign}(\delta\omega) = -\mathrm{sign}(\delta\tau). \\
\text{relative phase} \quad & \varphi = \delta\phi = \phi - \phi_s.
\end{aligned}$$

Again, the rf frequency is $\omega_{\mathrm{rf}} = h\omega_s$.

Energy gains per turn

$$\begin{aligned}
\delta U &= qV \sin\phi = V\sin(\phi_s + \varphi), \\
\delta U_s &= qV \sin\phi_s.
\end{aligned}$$

Construct a difference equation

Energy difference (generic − synchrounous) at beginning of $n^{\rm th}$ turn:

$$(\delta U)_n = U - U_s.$$

At beginning of the $n+1^{\rm th}$ turn:

$$(\delta U)_{n+1} = (U + \Delta U) - (U_s + \Delta U_s).$$

Relative change in energy per turn:

$$\Delta(\delta U) = \Delta U - \Delta U_s = qV(\sin\phi - \sin\phi_s).$$

Turn it into a differential equation (divide by τ_s):

$$\frac{d(\delta U)}{dt} \simeq \frac{\Delta(\delta U)}{\tau_s} = \frac{qV}{2\pi}\omega_s(\sin\phi - \sin\phi_s).$$

Define the energy variable:

$$W = -\frac{\delta U}{\omega_{\rm rf}} = -\frac{U - U_s}{\omega_{\rm rf}}.$$

$$\frac{dW}{dt} = \frac{qV}{2\pi}(\sin\phi_s - \sin\phi).$$

Want to change the canonical variables:

$$(\delta t, \quad -\delta U) \to (\omega_{\rm rf}\,\delta t, \quad W).$$

- Note that this preserves the phase-space areas.

$$\Delta\varphi \simeq \frac{d\varphi}{dt}\,\tau_s = \omega_{\mathrm{rf}}\,\delta t \tag{1}$$

After one revolution the difference in arrival times (gen − sync)

$$\Delta(\delta t) = \tau - \tau_s = \delta\tau = -\eta_{\mathrm{tr}}\tau\,\frac{dp}{p}, \tag{2}$$

since

$$\frac{d\tau}{\tau} = -\eta_{\mathrm{tr}}\,\frac{dp}{p}.$$

Combining (1) and (2) from previous page:

$$\frac{d\varphi}{dt} \simeq \frac{\Delta\varphi}{\tau_s} = \frac{\omega_{\mathrm{rf}}}{\tau_s}\,\Delta(\delta t) = -\frac{\omega_{\mathrm{rf}}}{\tau_s}\,\eta_{\mathrm{tr}}\,\frac{dp}{p}$$

$$U^2 = p^2c^2 + m^2c^4 \quad\Rightarrow\quad 2U\,\Delta U = 2pc^2\,\Delta p$$

$$\begin{aligned}\frac{\Delta p}{p} &= \frac{\Delta U}{p^2c^2} = \frac{\Delta U}{U}\,\frac{U^2}{p^2c^2} = \frac{1}{\beta^2}\,\frac{\Delta U}{U}\\ &= \frac{1}{\beta^2 U_s}\,(-\omega_{\mathrm{rf}}W).\end{aligned}$$

Longitudinal oscillation equation

$$\frac{d\varphi}{dt} = -\frac{\omega_{\mathrm{rf}}^2 \eta_{\mathrm{tr}}}{\beta^2 U_s} W.$$

$$\begin{aligned}\ddot{\varphi} = \frac{d^2\varphi}{dt^2} &= -\frac{\omega_{\mathrm{rf}}^2 \eta_{\mathrm{tr}}}{\beta^2 U_s} \frac{dW}{dt} \\ &= -\frac{\omega_{\mathrm{rf}}^2 \eta_{\mathrm{tr}}}{\beta^2 U_s} \frac{qV}{2\pi h} (\sin\phi_s - \sin\phi).\end{aligned}$$

Equation of longitudinal phase oscillation relative to synchronous particle:

$$\ddot{\varphi} + \frac{h\omega_s^2 \eta_{\mathrm{tr}} qV}{2\pi\beta^2 U_s} (\sin\phi_s - \sin\phi) = 0.$$

- However, if the energy steps from the cavity are large enough, then we should consider using difference equations rather than the approximation of the differential equations. (See further on.)

Small oscillations

For small amplitudes: $\sin\phi = \sin(\phi_s + \varphi) \simeq \varphi\cos\phi_s + \sin\phi_s.$

$$0 = \ddot{\varphi} + \frac{h\omega_s^2\eta_{\rm tr} qV}{2\pi\beta^2 U_s}(\sin\phi_s - \sin\phi) \simeq \ddot{\varphi} + \left(\frac{h\omega_s^2\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}\right)\varphi.$$

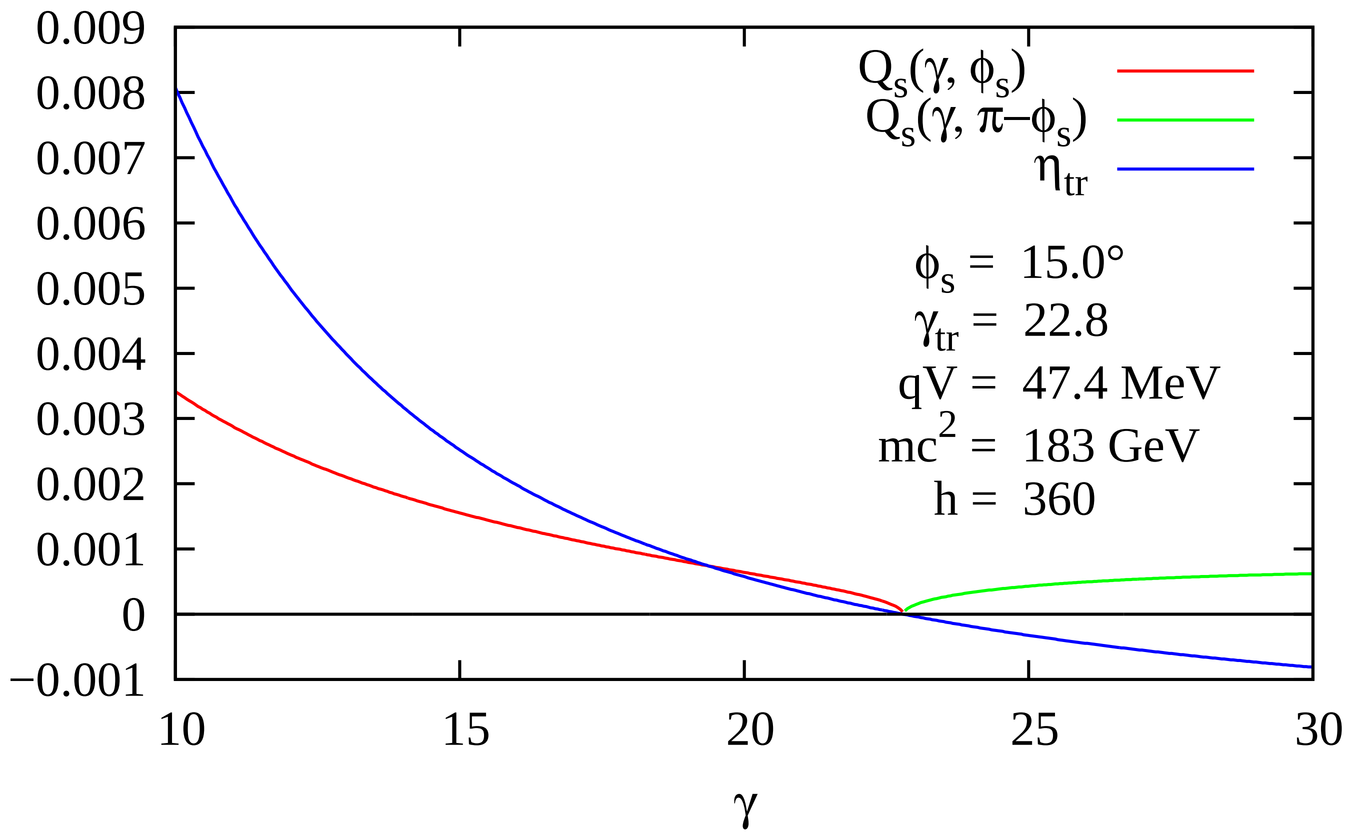
Define the angular *synchrotron oscillation* frequency

$$\Omega_s = \omega_s\sqrt{\frac{h\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$$

Synchrotron tune: $Q_s = \frac{\Omega_s}{\omega_s} = \sqrt{\frac{h\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$

- For oscillations motion about the synchronous phase $\eta_{\rm tr}\cos\phi_s$ must remain positive (or at least have the same sign as qV).
- Since $\eta_{\rm tr}$ flips sign when the beam accelerates through transition, the synchrounous phase must shift to maintain stability (e. g. $\phi_s \to \pi - \phi_s$).

Q_s and $\eta_{\rm tr}$ vs γ for RHIC with $^{197}{\rm Au}^{+79}$ beam; $V_{\rm rf} = 600$ kV.



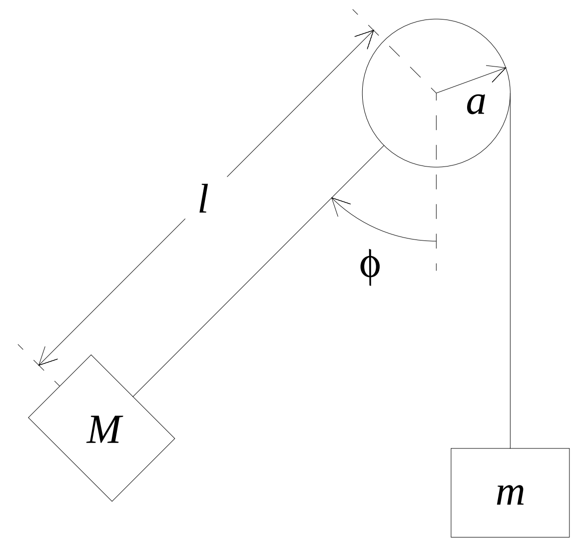
- Notice how the synchrotron frequency drops to zero at transition.
- Longitudinal phase-space becomes almost frozen around transition.
  - $f_{\rm rev} \simeq 78$ kHz. Cavity filling is a few microseconds.
  - Can shift ϕ_s in a few turns.

Large amplitude oscillations

$$\ddot{\varphi} + \frac{\Omega_s^2}{\cos\phi_s}[\sin(\varphi+\phi_s) - \sin\phi_s] = 0.$$

- Mechanical analog: the biased pendulum.
  - Weight M swings from pivoting cylinder.
  - String wrapped around cylinder holds m.

$$\ddot{\phi} + \frac{g}{l}\left(\sin\phi - \frac{ma}{Ml}\right) = 0.$$



$$\sin\phi_s = \frac{ma}{Ml},$$

$$\frac{\Omega_s^2}{\cos\phi_s} = \frac{g}{l}.$$

- Equilibrium for $\phi = \phi_s = \sin^{-1}\left(\frac{ma}{Ml}\right).$

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0. \qquad (\text{Of course } \dot{\varphi} = \dot{\phi}.)$$

Notice that $\quad \dfrac{d(\dot{\phi}^2)}{dt} = 2\ddot{\phi}\,\dfrac{d\phi}{dt}. \qquad$ So

$$d\left(\dot{\phi}^2\right) = \frac{2\Omega_s^2}{\cos\phi_s}(-\sin\phi\,d\phi) + 2\Omega_s^2\tan\phi_s\,d\phi,$$

which after integration becomes

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi - \phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,,$$

where ϕ_0 is the phase at $t = 0$.

- If $\phi_s = 0$, then it is a normal unbiased pendulum (with $m = 0$).
  - Stable fixed point at the bottom, i. e. $(0, 0)$.
  - Unstable fixed points at $(\phi, \dot{\phi}) = (\pm\pi, 0)$.

- If $|\phi_s| < \pi$, then we have
  - a stable fixed point at $(\phi_s, 0)$,
  - an unstable fixed point at $(\phi_1, 0) = (\pi - \phi_s, 0)$,
    - If ϕ is just less than $\pi - \phi_s$ the torque is restorative, i. e. $Mgl \sin\phi > mga$.
    - If ϕ is just a bit more than $\pi - \phi_s$, the net torque is away from ϕ_s, i. e. $mga > Mgl \sin\phi$.
    - Notice that the stable region shrinks to zero as ϕ_s increases to $\frac{\pi}{2}$. For stability we must have $|\phi_s| < \frac{\pi}{2}$.
  - A second unstable fixed point $(\phi_2, 0)$ may be obtained from

$$\frac{1}{\Omega_s}\dot{\phi}_2 = \pm\sqrt{\frac{2(\cos\phi_2 - \cos\phi_1)}{\cos\phi_s} + 2(\phi_2 - \phi_1)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_1^2}\,.$$

Squaring gives and setting $\dot\phi_1 = \dot\phi_2 = 0$,

$$0 = \frac{2(\cos\phi_2 - \cos\phi_1)}{\cos\phi_s} + 2(\phi_2 - \phi_1)\tan\phi_s,$$

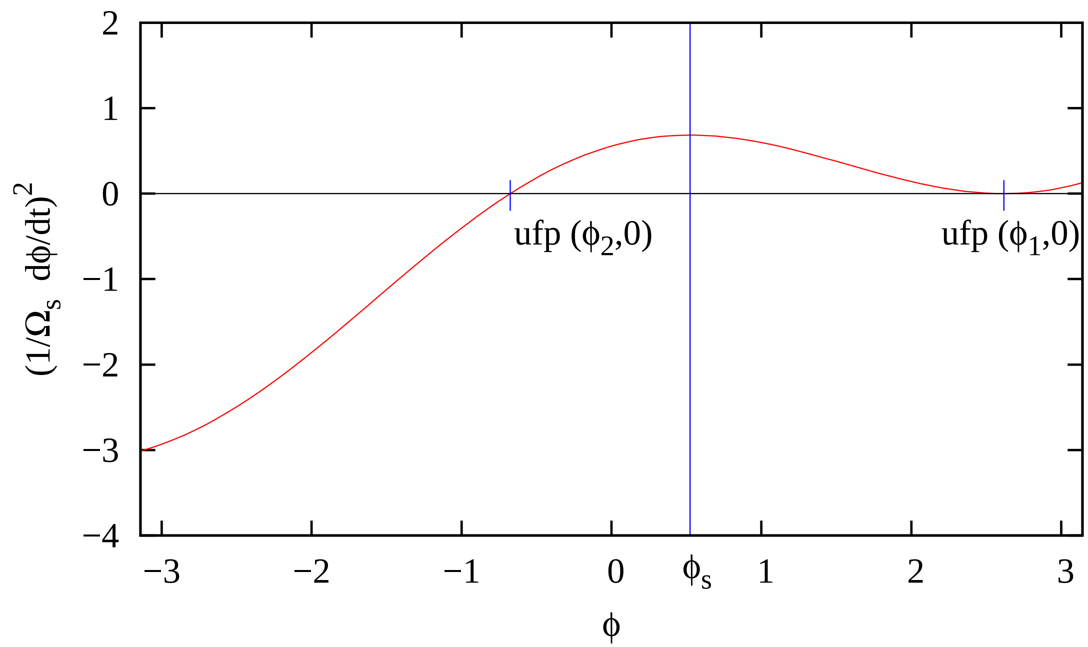
and with $\phi_1 = \pi - \phi_s$, we find the transcendental equation:

$$\left(\frac{1}{\Omega_s}\,\dot\phi_2\right)^2 = 0 = \cos\phi_2 + \cos\phi_s + (\phi_2 + \phi_s - \pi)\sin\phi_s,$$

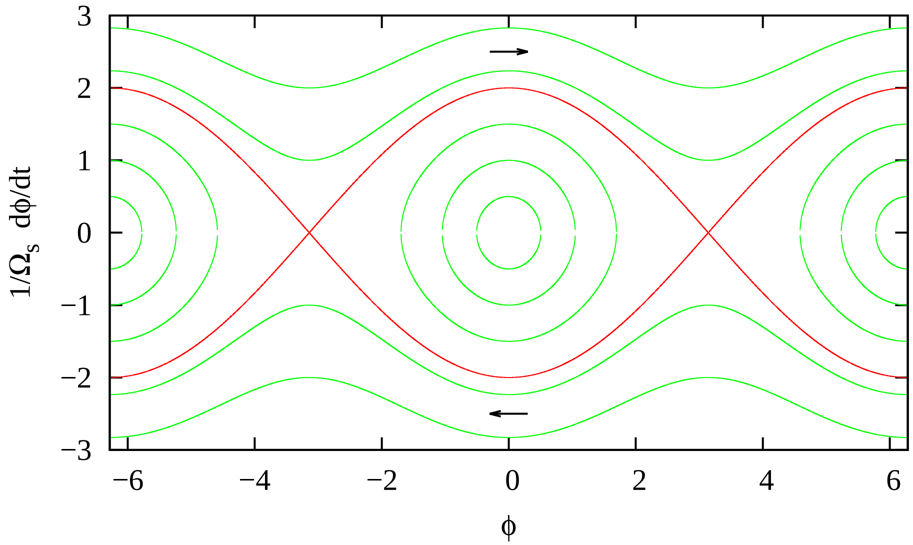
It can be solved numerically.

In this example:

$$\begin{aligned}\phi_s &= \ \ 30^\circ,\\ \phi_1 &= \ \ \phi_s,\\ \phi_2 &\simeq -36.7^\circ.\end{aligned}$$

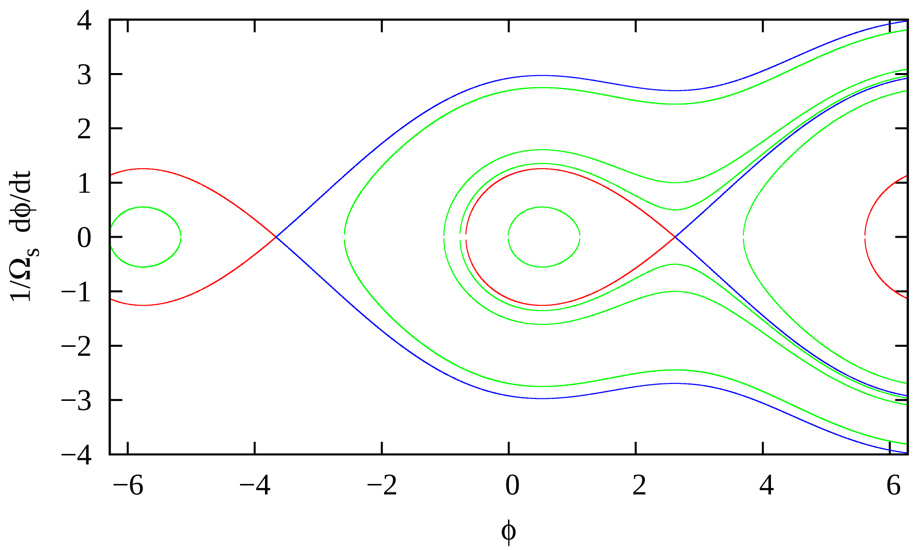


Separatrices and buckets



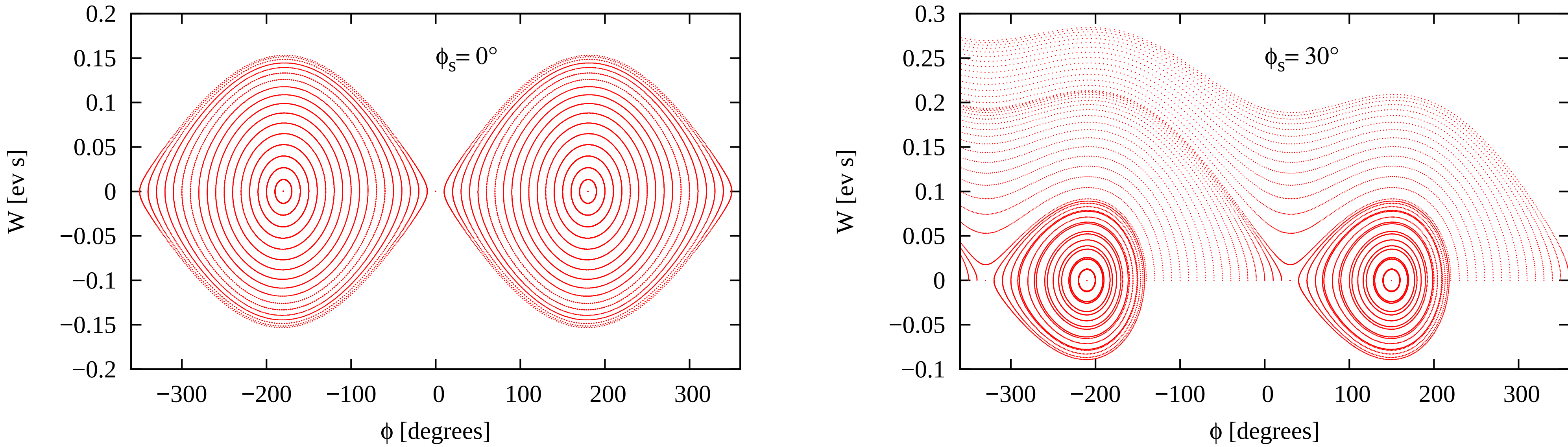
Stationary buckets

- $\phi_s = 0$.
- Separatrix in red.
- Elliptical flow inside separatrix.
- Particles outside not contianed.



Accelerating buckets

- $\phi_s = 30°$.
- Separatrix in red.
- Elliptical flow inside separatrix.
- Particles outside not contained.



$$\frac{dW}{dt} = \frac{qV}{2\pi}[\sin\phi_s - \sin(\varphi + \phi_s)],$$
$$\frac{d\varphi}{dt} = -\frac{\omega_{\rm rf}^2\eta_{\rm tr}}{\beta^2 U_s}\, W.$$

Integrated as difference equations:

$$W_{j+1} = W_j + \frac{qV}{2\pi}[\sin\phi_s - \sin(\varphi_j + \phi_s)],$$

$$\varphi_{j+1} = \varphi_j - \frac{\omega_{\rm rf}^2\eta_{\rm tr}}{\beta^2 U_s}\, W_{j+1}.$$

Wrong way to integrate

When we write simulation codes to integrate

$$\frac{d\phi}{dt} = \alpha W, \qquad \text{and} \qquad \frac{dW}{dt} = -\beta\phi,$$

where α and β are constants. Making a 2$^{\text{nd}}$ order differential equation:

$$\frac{d^2\phi}{dt^2} + \alpha\beta\,\phi = 0,$$

we know that the solution is simple harmonic motion.

For numerical integration we might try the difference equations:

$$\begin{aligned} \phi_{n+1} &= \phi_n + \alpha W_n\,\Delta t, \\ W_{n+1} &= W_n - \beta\phi_n\,\Delta t. \end{aligned}$$

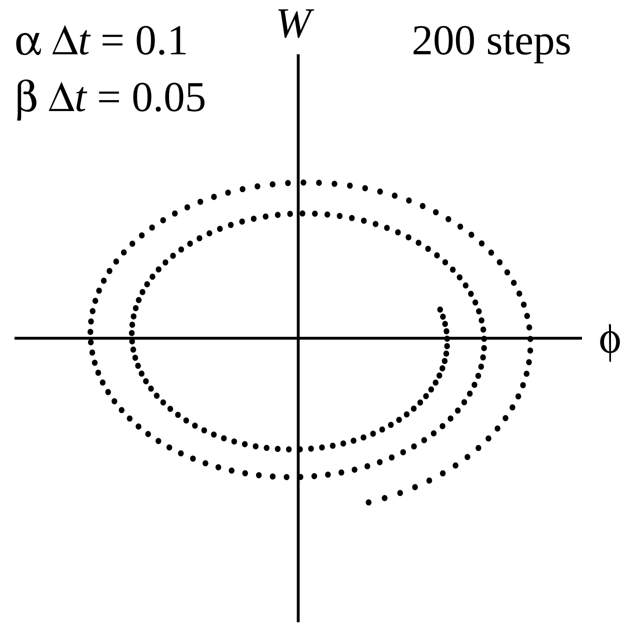
What's wrong with this?

It becomes obvious when we write the difference equations in matrix form:

$$\begin{pmatrix} \phi_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha\Delta t \\ -\beta\Delta t & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ W_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \phi_n \\ W_n \end{pmatrix}$$

We find

$$|\mathbf{M}| = 1 + \alpha\beta(\Delta t)^2 \neq 1.$$



- Instead of an ellipse in the (ϕ, W)-plane, we get a spiral.
  - If $\alpha\beta > 0$, it spirals outward.
  - if $\alpha\beta < 0$, it spirals inward to a point.

- In the limit of $\Delta t \to 0$, we should get the correct answer.

Better way: Leapfrog integration

- Stagger the integration steps:

$$\begin{array}{ccccccccc} \phi_{\frac{1}{2}} & \longrightarrow & & \phi_{1+\frac{1}{2}} & \longrightarrow & & \phi_{2+\frac{1}{2}} & & \\ & W_1 & \longrightarrow & & W_2 & \longrightarrow & & W_3 \end{array}$$

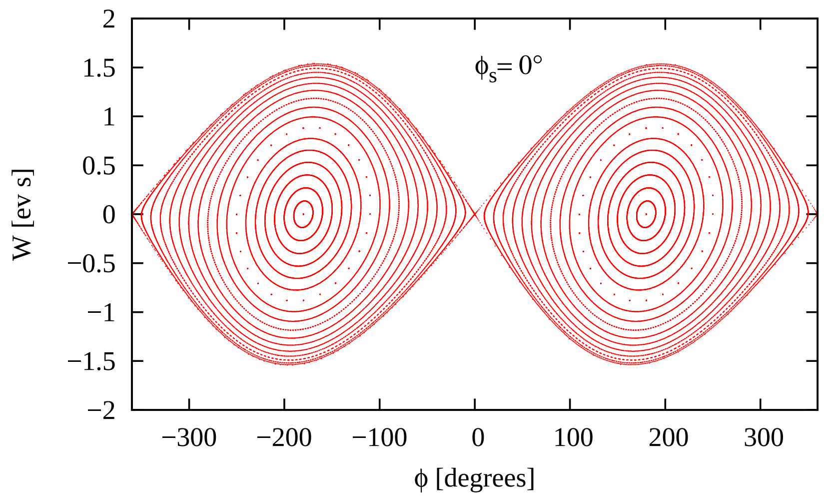
This is actually more like what we expect for a ring with a single cavity:

1. Go around the ring from downstream of cavity to upstream.
2. Then go through the cavity.

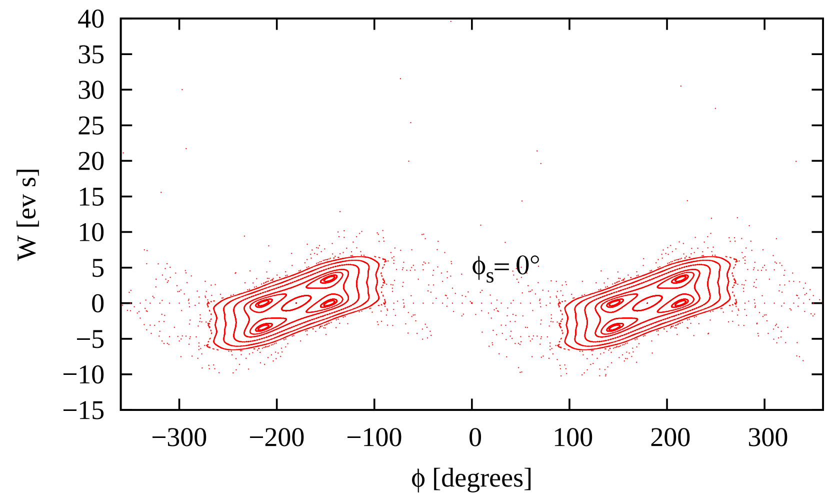
$$\begin{aligned} \begin{pmatrix} \phi_{n+\frac{1}{2}} \\ W_{n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\beta\Delta t & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha\Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha\Delta t \\ -\beta\Delta t & 1-\alpha\beta\,\Delta t^2 \end{pmatrix} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix}, \end{aligned}$$

Now $|\mathbf{M}| = 1$.

Cranking up the rf voltage for the simulation for $\phi_s = 0$ (back on p. 24), we get the distorted buckets:



100 times the voltage.



3000 times the voltage.

Hamiltonian formalism

Simple method to concoct a Hamiltonian: Work backwards from equations of motion.

$$\frac{dW}{dt} = -\frac{\partial H}{\partial \varphi} = \frac{qV}{2\pi h}[\sin\phi_s - \sin(\phi_s + \varphi)],$$
$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial W} = \frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W.$$

An obvious solution to this pair of equations is

$$H = \frac{1}{2}\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W^2 - \frac{qV}{2\pi h}[\varphi \sin\phi_s + \cos(\varphi + \phi_s)].$$

For small amplitudes this becomes

$$H \simeq \frac{1}{2}\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W^2 + \frac{qV\cos\phi_s}{4\pi h}\varphi^2 + \text{constant},$$

which is just the Hamiltonian for a harmonic oscillator.

A much more complicated but rigorous method is given in CM§ 7.6.

- It includes terms for synchro-betatron coupling via the dispersion functions η_x and η_x':

$$
\begin{aligned}
H_2 = {} & -p_s + \frac{p_s K}{2} x_\beta^2 + \frac{p_\beta^2}{2p_s} + \frac{\omega_{\mathrm{rf}}^2}{2\beta^3 U_s c}\left(\frac{1}{\gamma^2} - \frac{\eta_x}{\rho}\right) W^2 \\
& - \frac{qV}{\omega_{\mathrm{rf}}} \sum_{n=-\infty}^{\infty} \delta(s - nL) \cos\left[\phi_s + \varphi + \frac{2\pi h s}{L} - \frac{\omega_{\mathrm{rf}} U_s}{(p_s c)^2}(\eta_x p_\beta - p_s \eta_x' x_\beta)\right] \\
& - \frac{qV}{L\omega_{\mathrm{rf}}} \sin\phi_s \left[\varphi - \frac{\omega_{\mathrm{rf}} U_s}{(p_s c)^2}(\eta_x p_\beta - p_s \eta_x' x_\beta)\right].
\end{aligned}
$$

- The details of the derivation are there, but it is more time consuming than we want to go into for this course.

- When things get complicated, folks frequently use tracking codes to study particular accelerators.

Adiabatic invariant

In the adiabatic approximation, the Poincaré-Cartan invariant gives:

$$I_L = \oint p dq = \oint W d\phi = \oint W \frac{d\phi}{dt} dt,$$

where the integral $\oint$ is over one cycle of the synchrotron oscillations.

Recalling that $\quad \dfrac{d\varphi}{dt} \simeq \dfrac{\omega_{\mathrm{rf}}^2 \eta_{\mathrm{tr}}}{\beta^2 U_s} W, \quad$ this becomes

$$I_L = \frac{h^2 \eta_{\mathrm{tr}} \omega_s^2}{\beta^2 \gamma m c^2} \oint W^2 dt.$$

Invariant for small oscillations

Small amplitude oscillations:

$$\varphi(t) = \varphi_m \sin(\Omega_s t + \psi_0),$$
$$W(t) = W_m \cos(\Omega_s t + \psi_0),$$

with

$$W_m = \frac{\Omega_s \beta^2 U_s}{\omega_{\rm rf}^2 \eta_{\rm tr}} \varphi_m, \qquad \text{since} \qquad W = \frac{\beta^2 U_s}{\omega_{\rm rf}^2 \eta_{\rm tr}} \dot{\varphi}.$$

The invariant may now be written as

$$I_L = \frac{h^2 \omega_s^2 \eta_{\rm tr}}{\beta^2 U_s} \oint \frac{\beta^2 U_s}{h^2 \omega_s^2 \eta_{\rm tr}} \varphi_m W_m \cos^2(\Omega_s t + \psi_0)\, \Omega_s\, dt = \pi \varphi_m W_m.$$

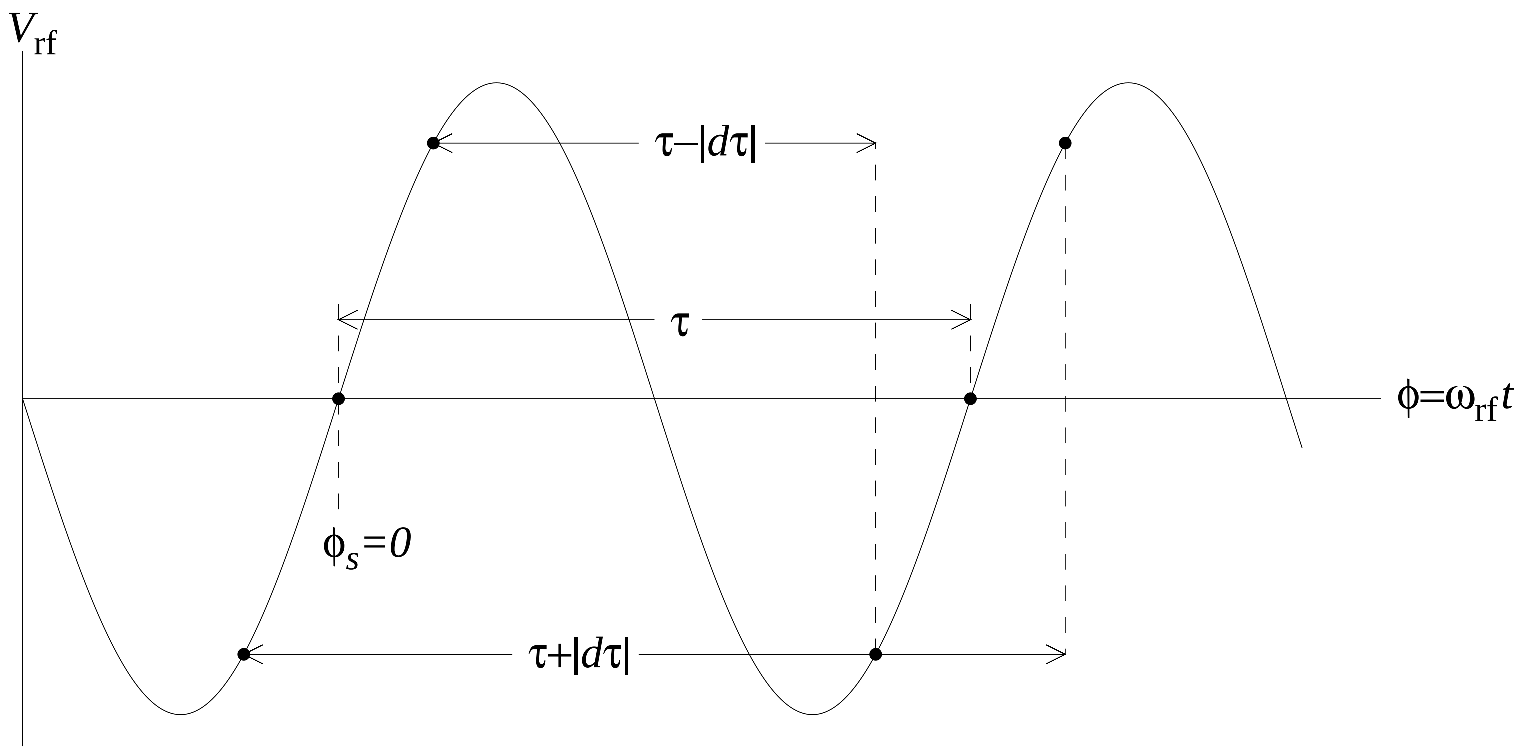
We may also write this as

$$I_L = \frac{\pi \omega_{\rm rf}^2 \eta_{\rm tr}}{\Omega_s \beta^2 U_s} W_m^2.$$

Squaring I_L gives

$$W_m^4 = \frac{qV \cos\phi_s \beta^2 U_s}{2\pi^3 h \omega_{\rm rf}^2 \eta_{\rm tr}} I_L^2.$$

Stationary bucket



Phase stability below transition with no acceleration: $\phi_s = 0.$

Recalling

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi - \phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,,$$

and with $\phi_s = 0$ for an unaccelerated synchronous particle, we obtain

$$\frac{1}{\Omega_s}\,\dot{\phi} = \pm\sqrt{2(\cos\phi - \cos\phi_m)},$$

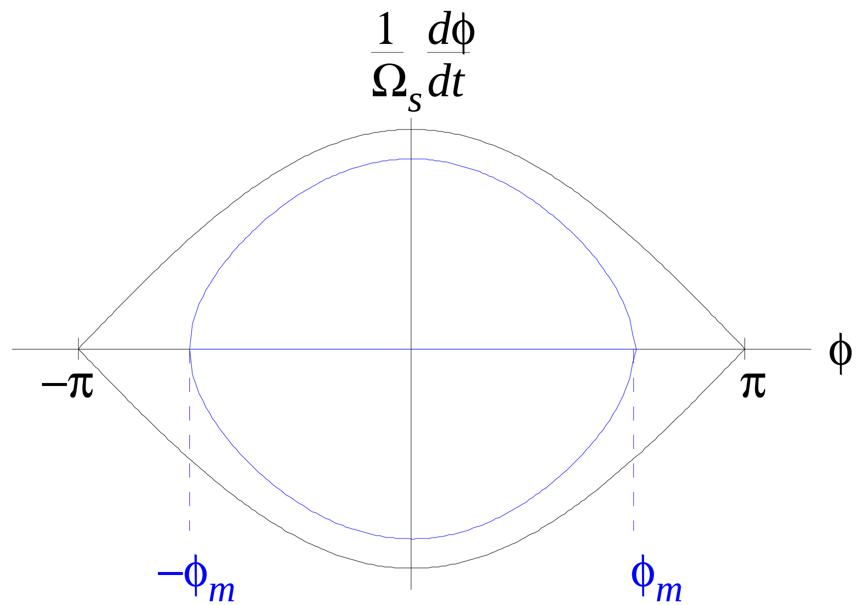
where I have taken $\dot{\phi}_0 = 0$ at $t = 0$.

For $\phi_m = \pi$,

$$\frac{1}{\Omega_s}\,\dot{\phi} = \pm\sqrt{2(\cos\phi + 1)} = 2\cos\frac{\phi}{2}.$$

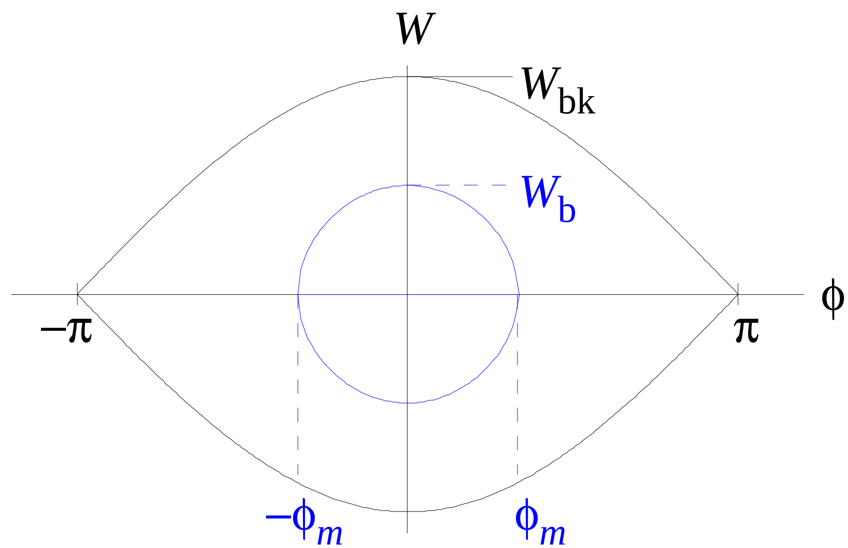
Small amplitudes:

$$\Omega_s = \omega_s\sqrt{\frac{h|\eta_{\rm tr}|}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$$



Reintroduce the canonical variable W:

$$\frac{1}{\Omega_s}\frac{d\phi}{dt} = \frac{2\pi c}{L}\sqrt{\frac{2\pi h^3 \eta_{\rm tr}}{U_s q V \cos\phi_s}}W.$$



Equation of separatrix:

$$W = \pm\frac{L}{\pi c}\sqrt{\frac{qVU_s}{2\pi h^3|\eta_{\rm tr}|}}\cos\frac{\phi}{2}.$$

Area of stationary bucket:

$$A_{\rm bk} = 2\int_{-\pi}^{\pi} W\,d\phi = \frac{8L}{\pi c}\sqrt{\frac{qVU_s}{2\pi h^3|\eta_{\rm tr}|}}.$$

$$W_{\rm b} = \frac{A_{\rm bk}}{8}\sin\frac{\phi_m}{2}.$$

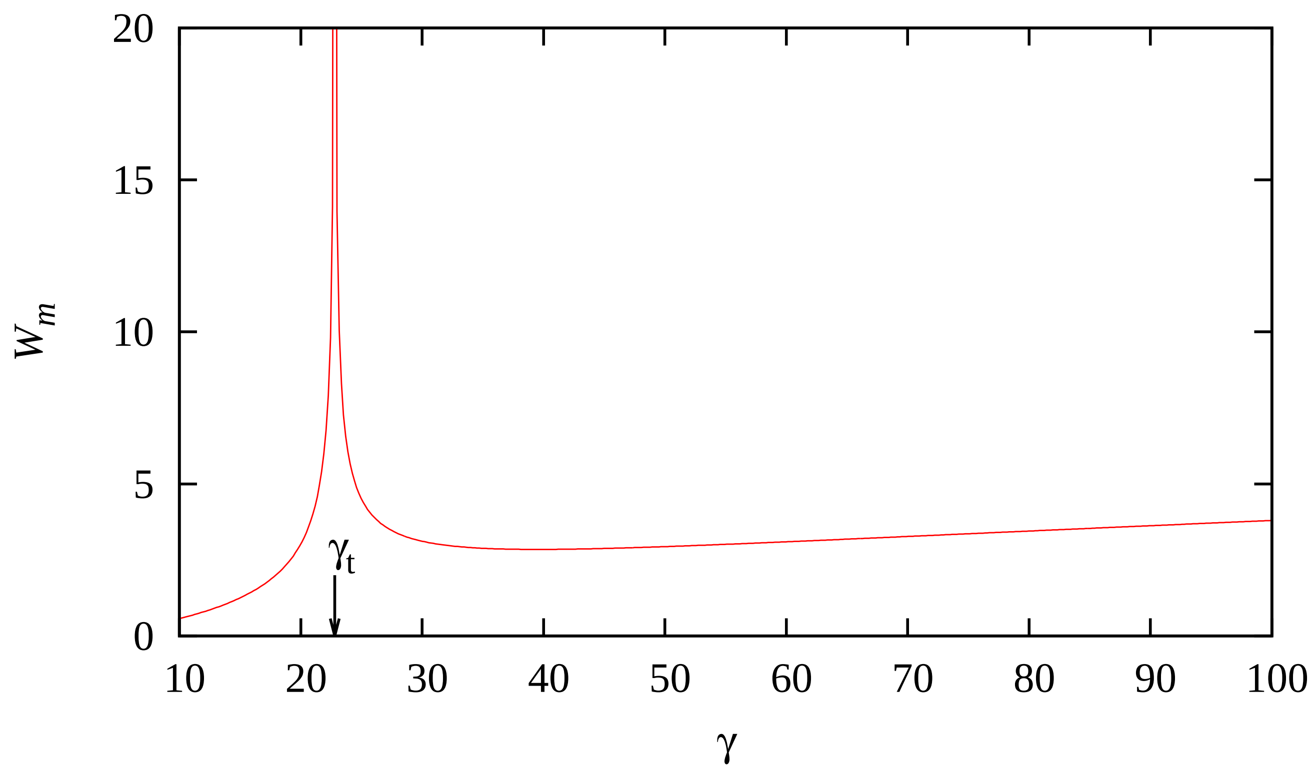
$$\frac{\Delta p}{p} = \frac{1}{\beta^2 U_s}(-\omega_{\rm rf}W). \quad \text{(See p. 15.)}$$

$$|W_{\rm b}| = \frac{\beta^2 U_s}{\omega_{\rm rf}}\frac{\delta p}{p},$$

Phase oscillation equation becomes:

$$W = \pm\frac{A_{\rm bk}}{8}\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\phi_m}{2}}.$$

Momentum spread gets big at transition



On the other hand the bunch length gets short at transition, since

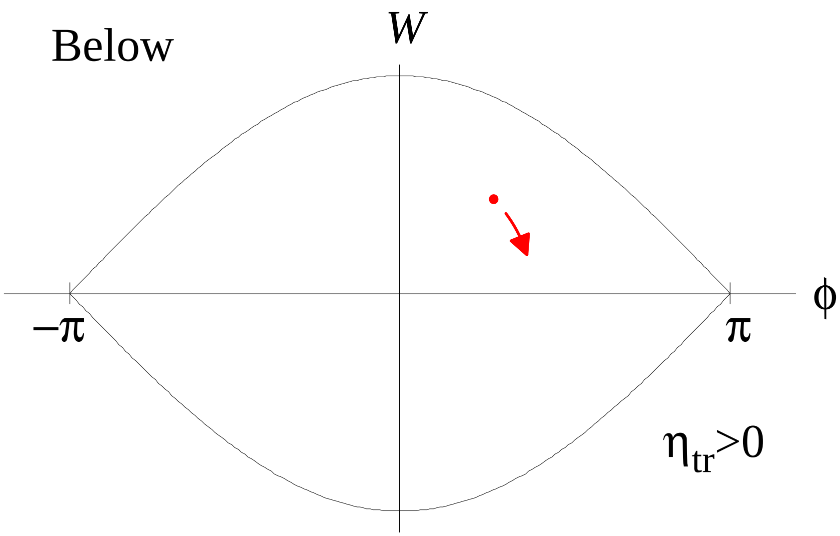
$$I_L = \pi \varphi_m W_m.$$

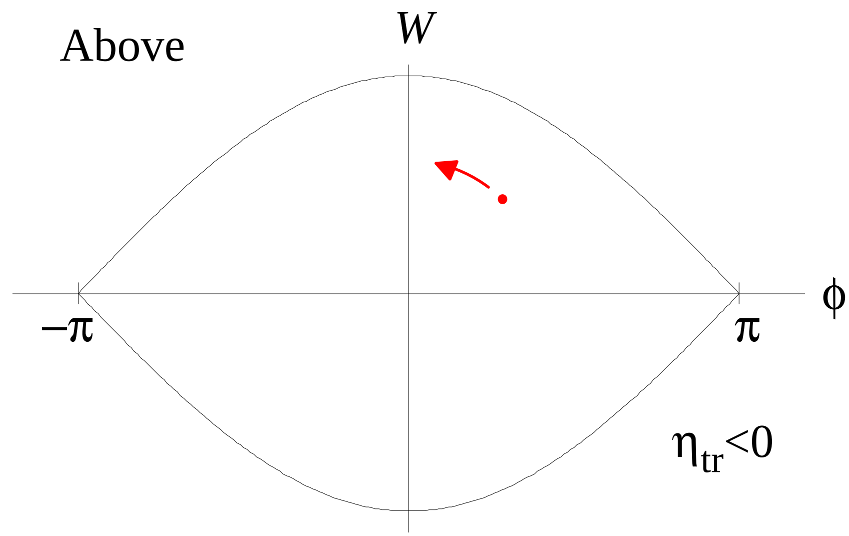
Note: Book is a wee bit inconsistent: § 7.7 uses W_m whereas § 7.8 uses W_b for the bunch height. (I'll have to fix that for the next printing.)

Direction of phase space rotation

Since $\dot{\phi} = \dfrac{\omega_{\rm rf}^2}{\beta^2 U_s}\,\eta_{\rm tr}\,W,$ we find that:

1. Below transition with $\eta_{\rm tr} > 0$, ϕ increases for $W > 0$, so a stable particle will move in a clockwise direction about the stable fixed point $(\phi_s, 0)$.
2. Above transition with $\eta_{\rm tr} < 0$, ϕ decreases for $W > 0$, so a stable particle will move in a counterclockwise direction about the stable fixed point $(\phi_s, 0)$.





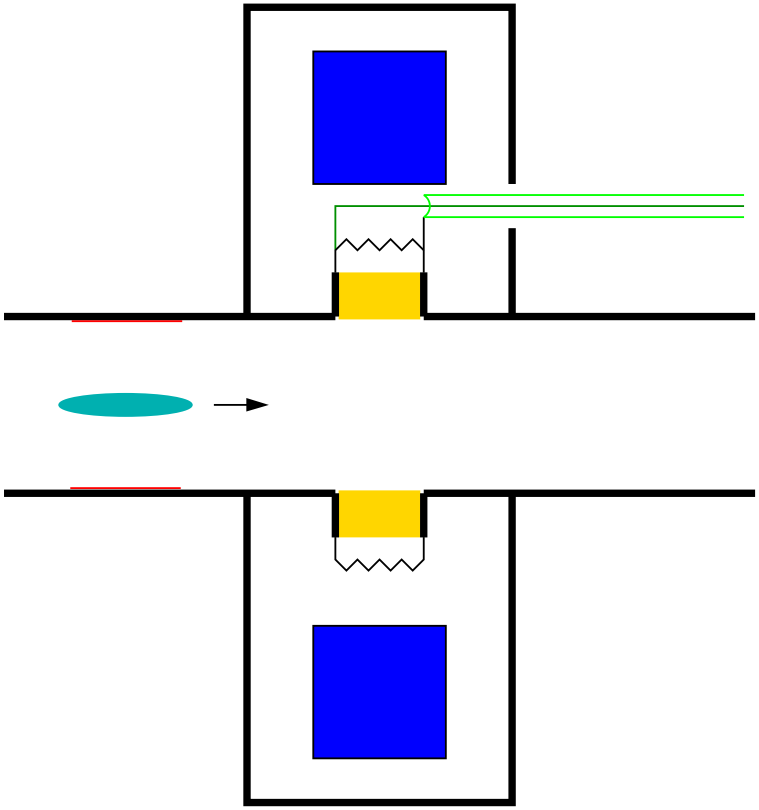
Note to Waldo: Do some demos with `split2`.

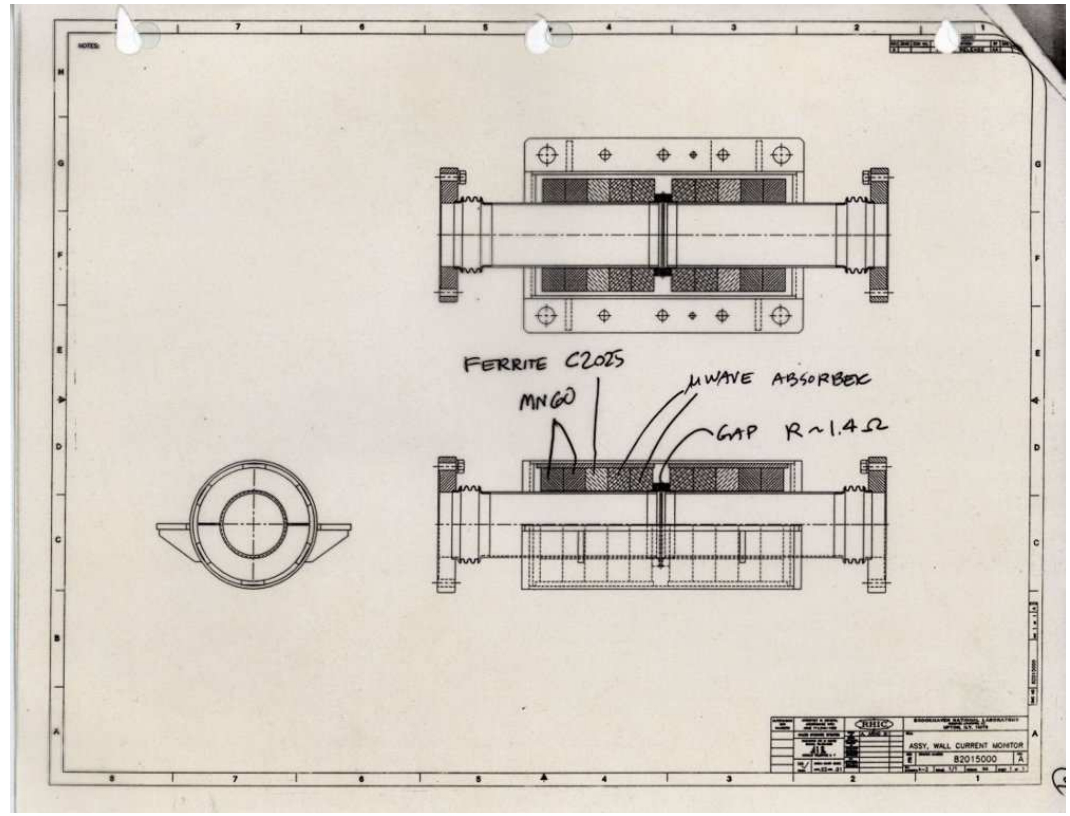
Measurement of the beam current

- Wall current monitors can measure the bunched current.
  - Weber's design gives flat response up to 6 GHz.
- DC current transformers (DCCT): measure unbunched cirulating current.
  - Better than a part in 10^4 accuracy.

- R. C. Weber, "Longitudinal Emittance: An Introduction to the Concept and Survey of Meaesurement Techniques Including Design of a Wall Current Monitor", in *Accelerator Instrumentation*, AIP Conf. Proc. 212, 85 (1989).

Wall current monitors (WCM)





- Beam (light blue) moving right with image charges (red) on beam pipe walls.
- Ceramic gap (gold) in pipe maintains vacuum integrity.
- Voltage measured across resisters. Ferrite (blue) with outer can enhance signal.

DC current transformers (DCCT)

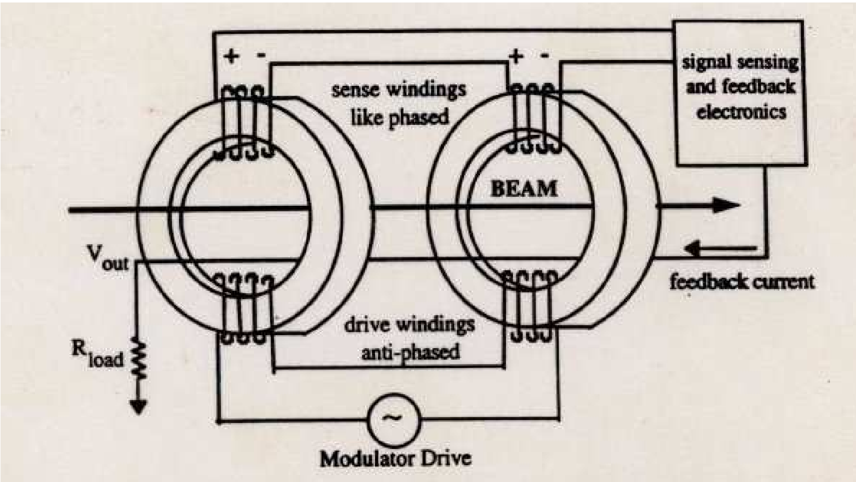


Figure 6. Magnetic modulator section of DC Transformer with flux nulling feedback.

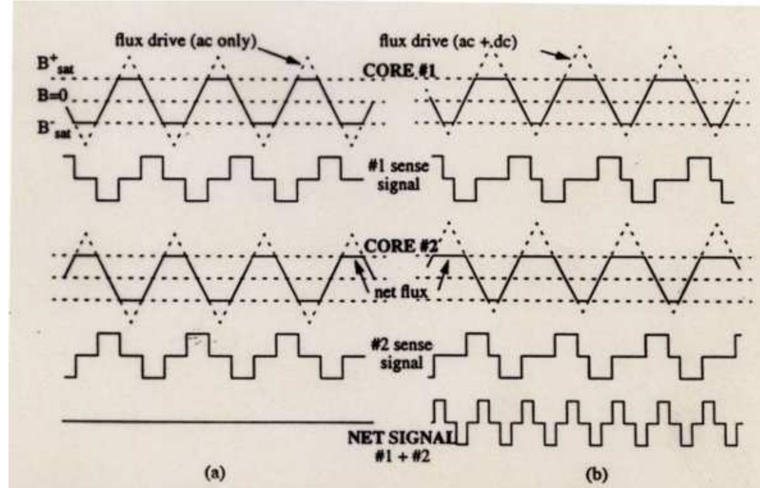
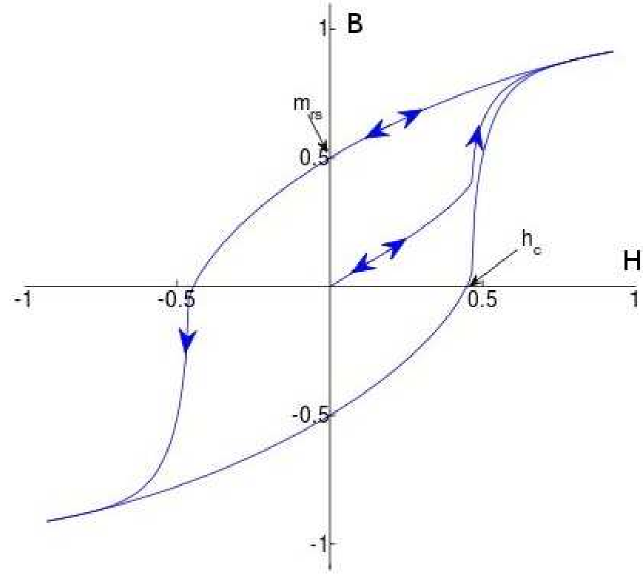


Figure 7. Magnetic modulator signals, (a) with no dc present and (b) with exaggerated dc presence.

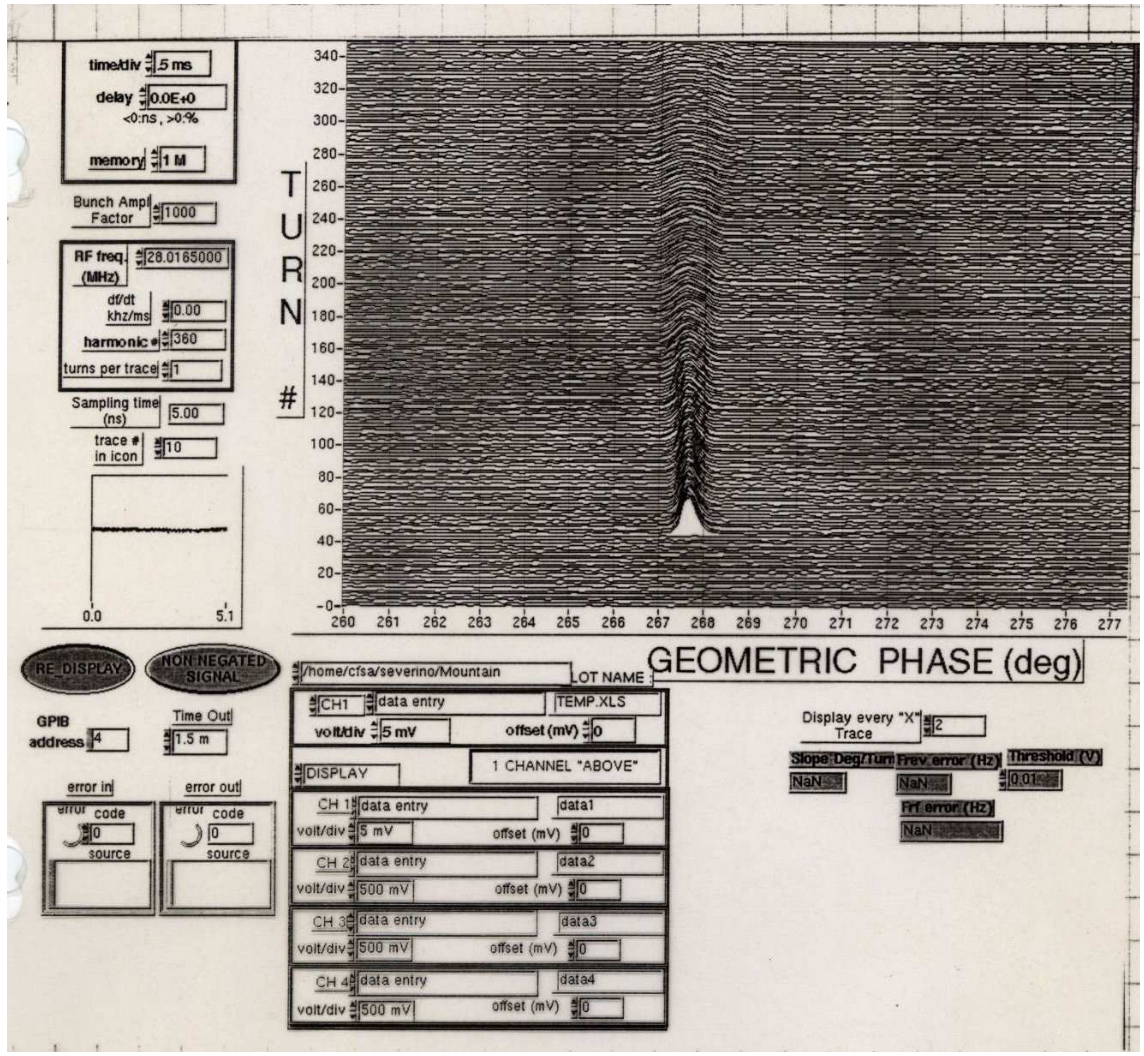


- Measures unbunched beam.
- Used for dc power supplies.

(Figs 6& 7 from R. C. Weber.)

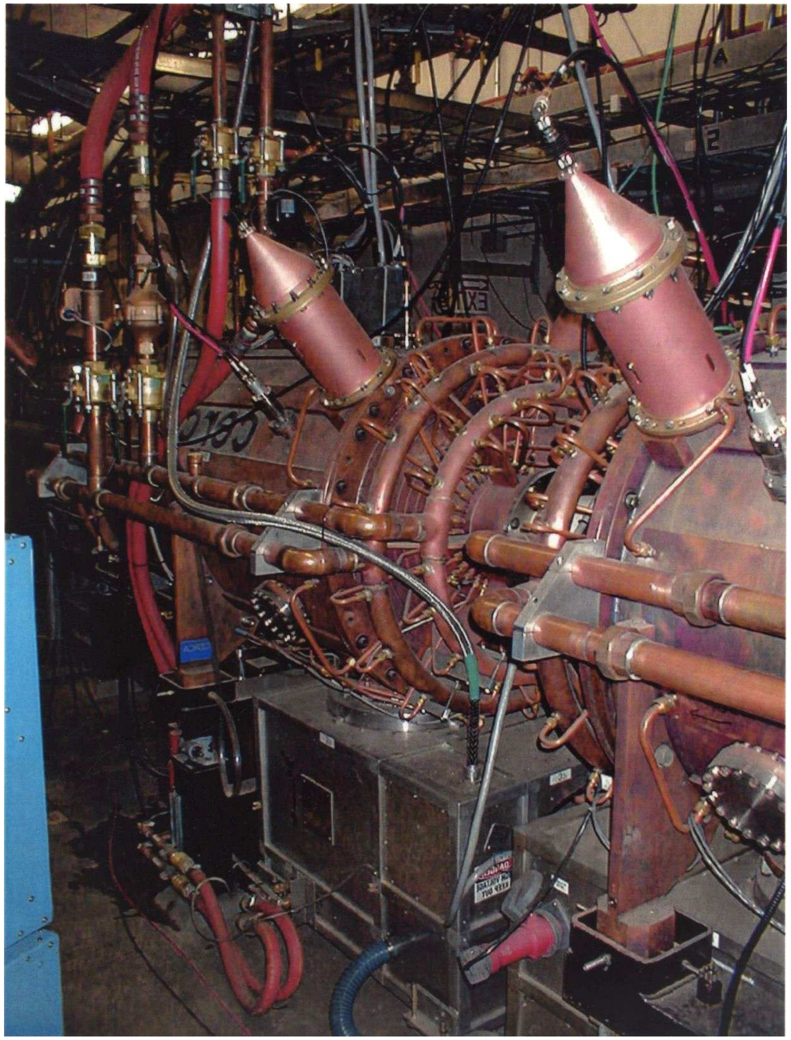
RHIC injection with no rf

$$f_{\rm rf} = 28.01653 \text{ MHz}.$$
$$h = 360.$$
$$f_s = \frac{f_{\rm rf}}{h} = 77.824 \text{ kHz}.$$
$$L = 3833.845 \text{ m}.$$
$$c = 299792458 \text{ m/s}.$$
$$\beta = \frac{f_s L}{c} = 0.99523.$$
$$\gamma = 10.255.$$

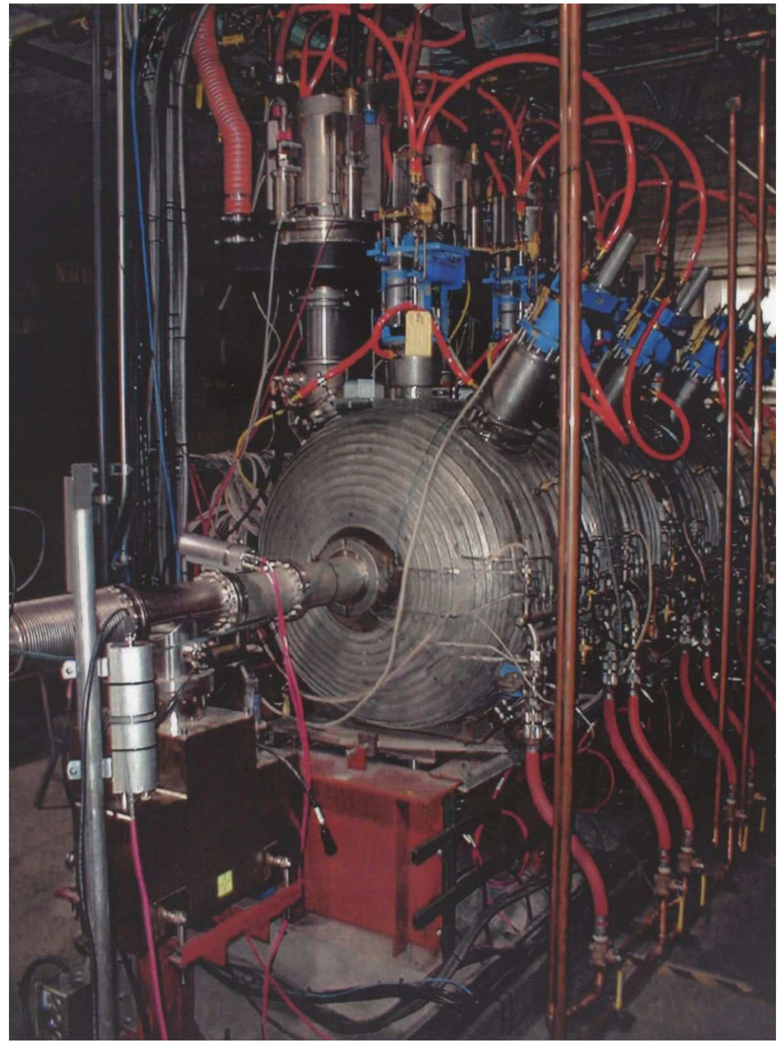


- Ah. Must be gold ions.

RHIC rf cavities



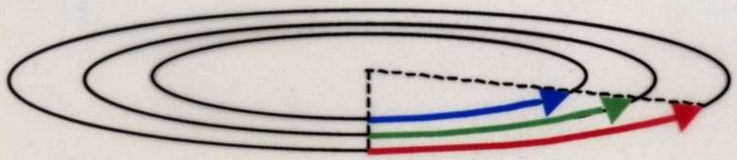
- 28 MHz accelerating ($h = 360$).



- 200 MHz storage ($h = 2520$).

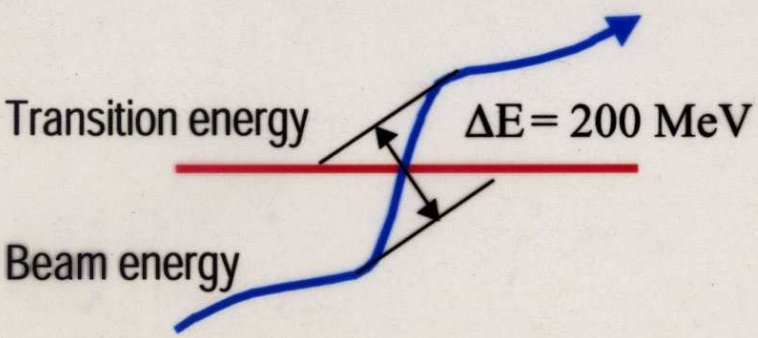
RHIC commissioning: Transition Xing

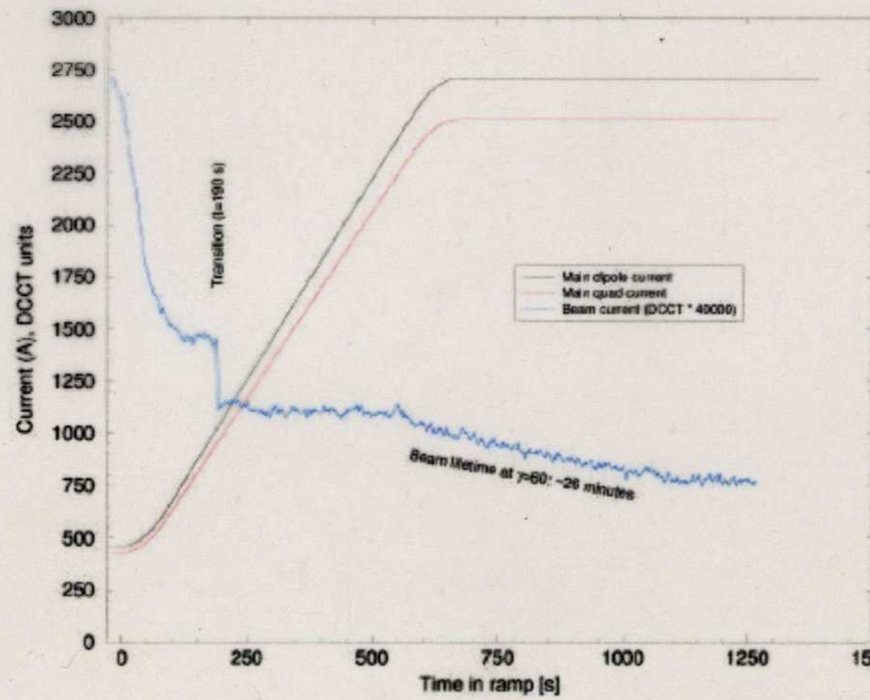
RHIC is first superconducting, slow ramping accelerator to cross transition energy:



Slow and fast particles remain in step.
⇨ increased particle interaction (space charge)
⇨ short, unstable bunches

Cross unstable transition energy with radial energy jump:





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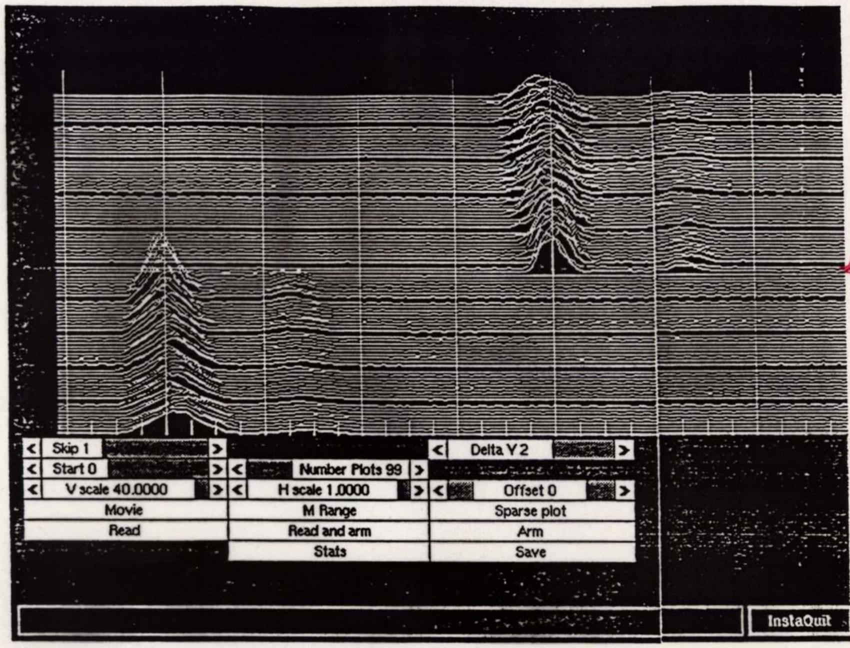
AGS transition crossing

LECROY 7200 ⇓ SDS

Notice bunch length decreases up to transition

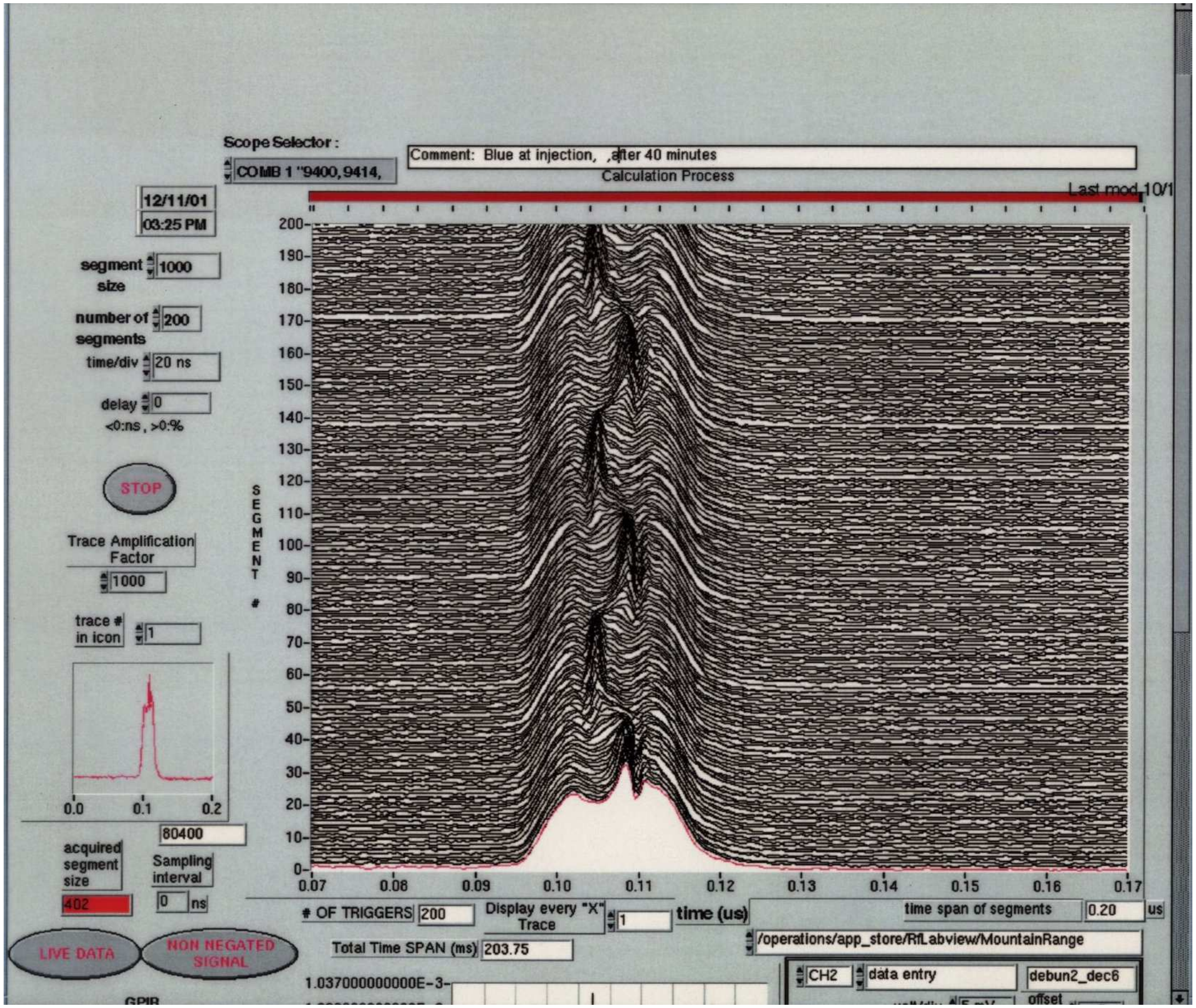


phase jump @ transition

* Background subtraction
* Evaluation of intensity and r.m.s. bunch length
* Evaluation of bunch "core" intensity and width
* Extraction of beam emittance (longitudinal)
* Evaluation of phase jump, rf voltage, peak intensity, skewness, kurtosis, etc.

Real soliton

Is it a soliton?

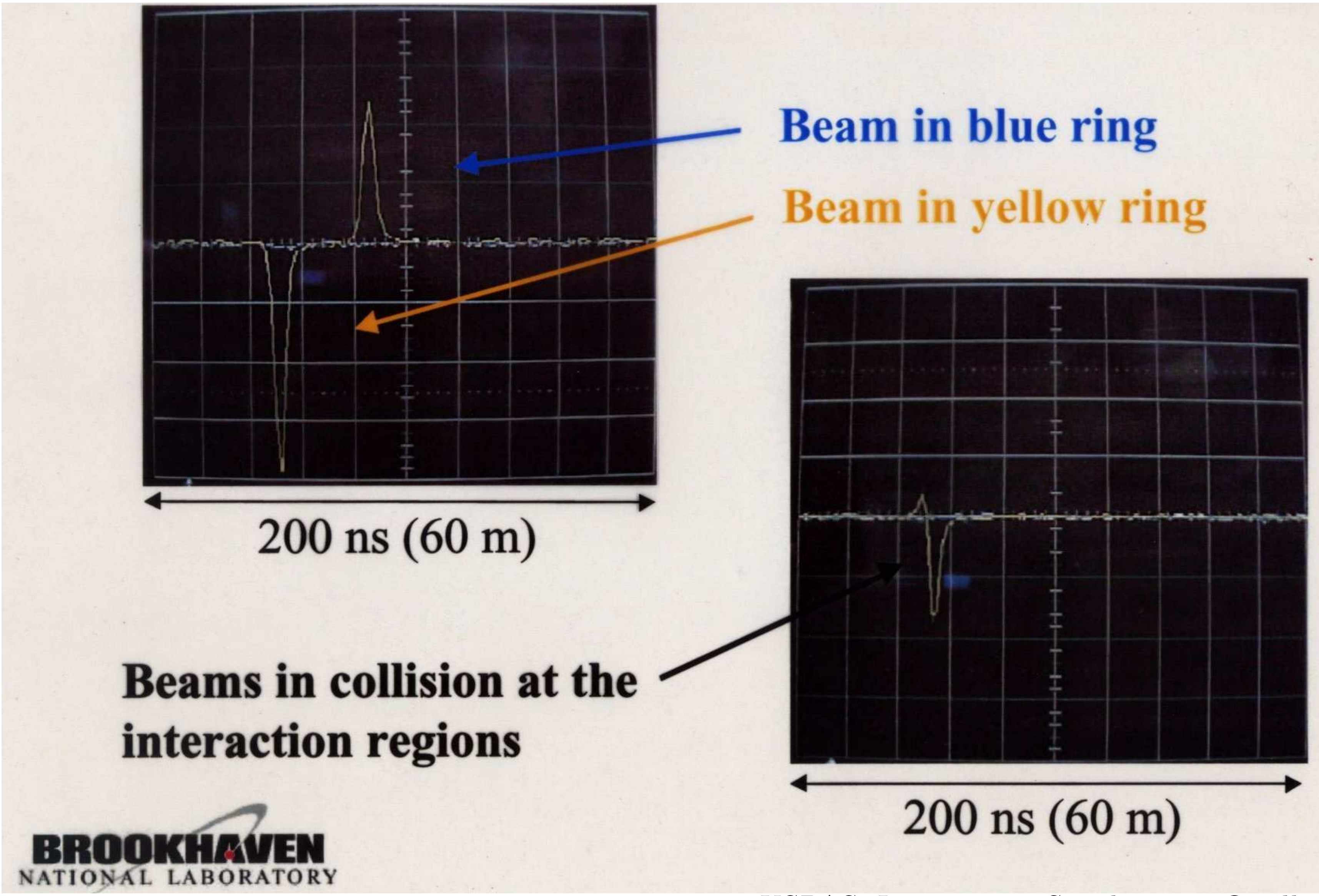


RHIC injection.

- Protons.
- after 40 minutes!

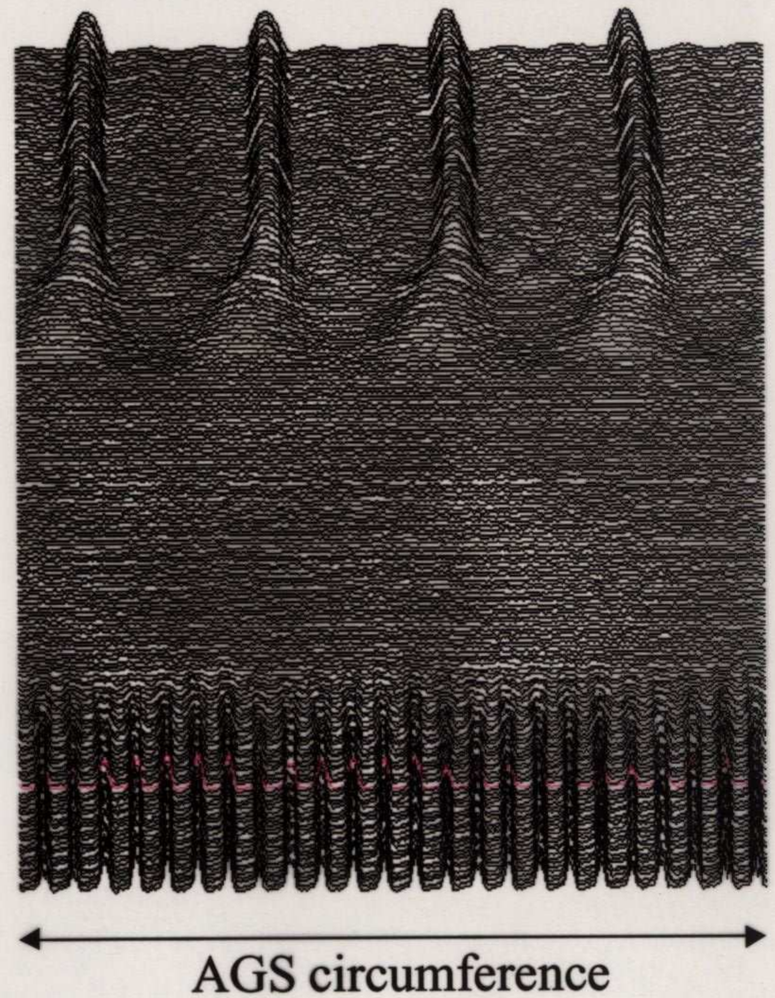
High intensity bunch.
Effect of space charge.

RHIC colliding beams



Debunching and rebunching in AGS

- 4×6 bunches injected from Booster
- Debunch / rebunch into 4 bunches at AGS injection
- Final longitudinal emittance: 0.3 eVs/nuc./bunch
- Achieved $4{\times}10^9$ Au ions in 4 bunches at AGS extraction on 8/4/00



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Booster and AGS merging