USPAS Accelerator Physics 2013
Duke University

Chapter 9: RF Linear Accelerators and RF Cavities

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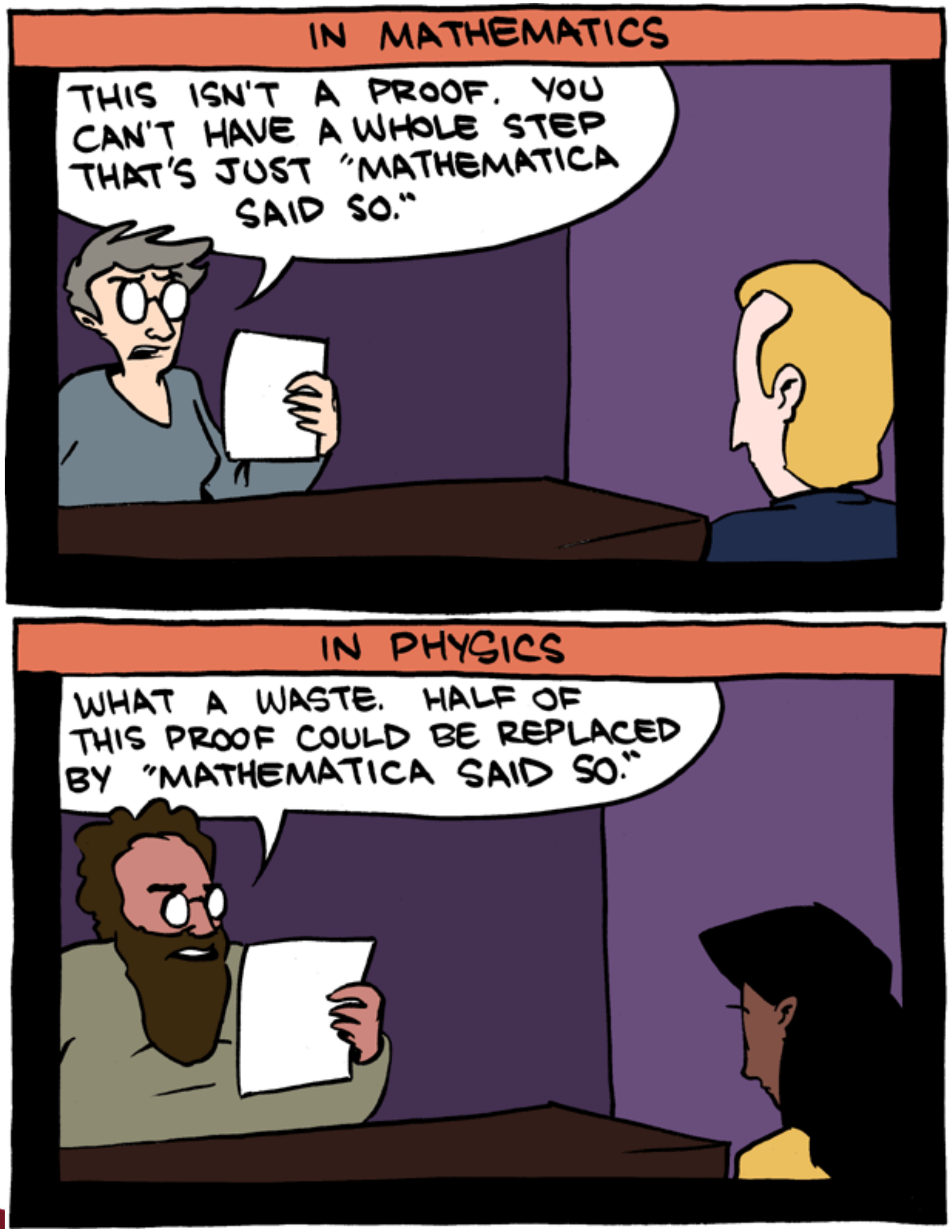

IN MATHEMATICS
THIS ISN'T A PROOF. YOU CAN'T HAVE A WHOLE STEP THAT'S JUST "MATHEMATICA SAID SO."
IN PHYSICS
WHAT A WASTE. HALF OF THIS PROOF COULD BE REPLACED BY "MATHEMATICA SAID SO."

RF Concepts and Design

- Much of RF is really a review of graduate-level E&M
  - See, e.g., J.D. Jackson, “Classical Electrodynamics”
  - The beginning of this lecture is hopefully review
    - But it’s still important so we’ll go through it
    - Includes some comments about electromagnetic polarization
  - We’ll get to interesting applications later in the lecture
    - (or tomorrow)
  - Particularly important are cylindrical waveguides and cylindrical RF cavities
    - Will find transverse boundary conditions are typically roots of Bessel functions
    - TM (transverse magnetic) and TE (transverse electric) modes
    - RF concepts (shunt impedance, quality factor, resistive losses)

9.1: Maxwell's Equations

Electric charge density

$$\vec{\nabla} \cdot \vec{D} = \rho$$ Electric charge generates electric displacement

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law

$$\vec{\nabla} \cdot \vec{B} = 0$$ No magnetic charges

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ Magnetic field generated by real or displacement current density

Electric current density

Maxwell's Equations are linear in the source terms ρ and $\vec{J}$. In general will generate linear (partial) differential equations to solve. Superposition valid!

Constitutive Relations and Ohm's Law

$$\boxed{\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \, \frac{\mathrm{C}}{\mathrm{N} \cdot \mathrm{m}^2}$$

$\vec{D}$: Electric displacement field

$\vec{E}$: Electric field

ε : Permittivity

$$\boxed{\varepsilon_0 \mu_0 = \frac{1}{c^2}}$$

$$\boxed{\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}}$$

$$\mu_0 = 4\pi \times 10^{-7} \, \frac{\mathrm{N} \cdot \mathrm{s}^2}{\mathrm{C}^2}$$

$\vec{B}$: Magnetic field

$\vec{H}$: Magnetizing field

ε : Permeability

$$\boxed{\vec{J} = \sigma \vec{E}}$$

σ : conductivity

Boundary Conditions

- Boundary conditions on fields at the surface between two media depend on the surface charge and current densities:
  - $E_{||}$ and $B_{\perp}$ are continuous
  - $D_{\perp}$ changes by the surface charge density ρ_{s} (scalar)
  - $H_{||}$ changes by the surface current density $\vec{J}_{\mathrm{s}}$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_{\mathrm{s}}$$

- C&M Chapter 9: no dielectric or magnetic materials

$$\mu = \mu_0 \qquad \epsilon = \epsilon_0$$

Wave Equations and Symmetry

- Taking the curl of each curl equation and using the identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

then gives us two **identical** wave equations

$$\boxed{\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}}$$

These **linear wave equations** reflect the deep symmetry between electric and magnetic fields. Harmonic solutions:

$$\vec{\Psi}(\vec{r}, t) = \hat{\Psi}(\vec{r}) \mathrm{e}^{-i\omega t} \qquad \text{for} \quad \vec{\Psi} = \vec{E}, \vec{H}$$

$$\Rightarrow \quad \nabla^2 \hat{\Psi} = \Gamma^2 \hat{\Psi} \qquad \text{where} \quad \Gamma^2 = (-\mu\epsilon\omega^2 + i\mu\sigma\omega)$$

Conductivity and Skin Depth

- For high conductivity, $\sigma \gg \omega\epsilon$, charges move freely enough to keep electric field lines perpendicular to surface (RF oscillations are adiabatic vs movement of charges)
  - Copper: $\sigma \approx 6 \times 10^7\ \Omega^{-1}\,\mathrm{m}^{-1}$
  - Conductivity condition holds for very high frequencies (10^{15} Hz)
- For this condition

$$\Gamma \approx \sqrt{\frac{\mu\omega\sigma}{2}}(1+i)$$

  - Inside the conductor, the fields drop off exponentially

$$\hat{\Psi}(z) = \Psi(0)\ \mathrm{e}^{-z/\delta}$$

$$\text{skin depth} \quad \boxed{\delta \equiv \sqrt{\frac{2}{\mu\omega\sigma}}}$$

e.g. for Copper

Frequency	Skin depth (μm)
60 Hz	8470
10 kHz	660
100 kHz	210
1 MHz	66
10 MHz	21
100 MHz	6.6

Surface Resistance and Power Losses

- There is still non-zero power loss for finite resistivity
  - Surface resistance: resistance to current flow per unit area

$$\text{surface resistance} \qquad R_s \equiv \sqrt{\frac{\mu\omega}{2\sigma}} = \frac{1}{\sigma\delta}$$

$$\text{surface power loss} \qquad \langle P_{\text{loss}} \rangle = \frac{R_s}{2} \int_{\text{S}} |H_{||}|^2 \, dS$$

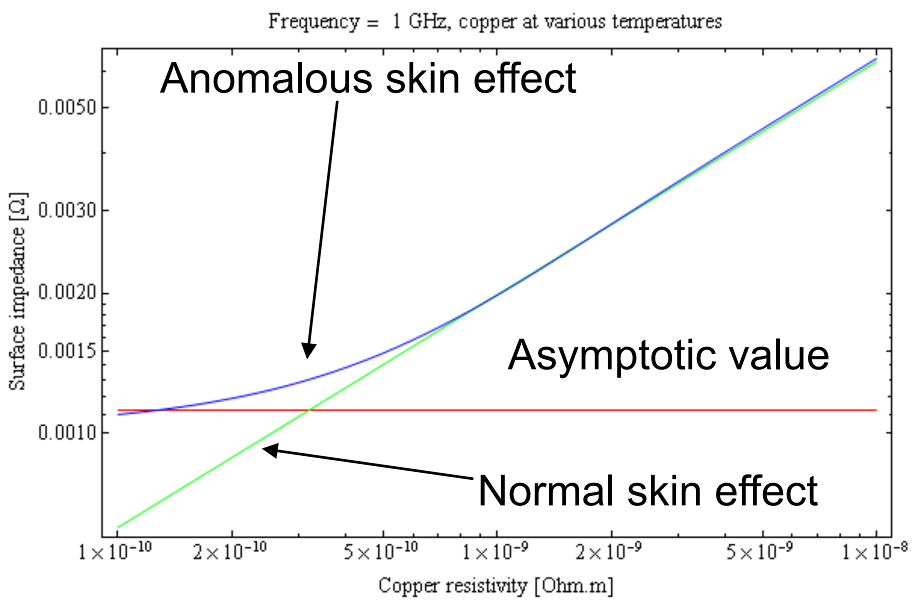
  - This isn't just applicable to RF cavities, but to transmission lines, power lines, waveguides, etc – anywhere that electromagnetic fields are interacting with a resistive media
  - Deriving the power loss is part of your homework!
  - We'll talk more about transmission lines and waveguides after one brief clarification…

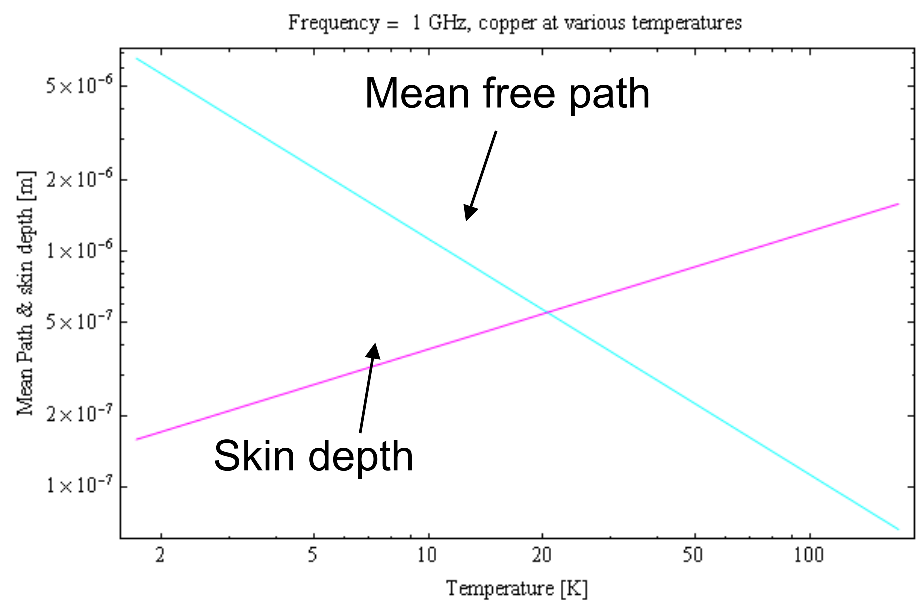
Anomalous Skin Effect

- Conductivity really depends on the frequency ω and mean time between electron interactions τ (or inverse temperature)

$$\sigma(\omega) = \frac{\sigma_0}{(1 + i\omega\tau)}$$

  - Classical limit is $\omega\tau \to 0$, adiabatic field wrt electron interactions
  - For non-classical limit, $\vec{J} = \sigma\vec{E}$ no longer applies since electrons see changing fields between single interactions





Reuter and Sondheimer, Proc. Roy. Soc. A195 (1948), from Calatroni SRF'11 Electrodynamics Tutorial

Plane Wave Properties

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x} - i\omega t}$$

Generally $F(z,t) = \int_{-\infty}^{\infty} A(\omega) \mathrm{e}^{i(\omega t - k(\omega) z)} d\omega$

$$k = \frac{2\pi}{\lambda_{\mathrm{rf}}} \quad \text{(wave number)}$$

$$v_{\mathrm{ph}} = \frac{\omega}{k} \quad \text{(phase velocity)}$$

$$v_{\mathrm{g}} = \frac{d\omega}{dk} \quad \text{(group velocity)}$$

The source-free divergence Maxwell equations imply

$$\vec{k} \cdot \vec{E}_0 = \vec{k} \cdot \vec{B}_0 = 0$$

so the fields are both transverse to the direction $\vec{k}$

(This will be the $\hat{z}$ direction in a lot of what's to come)

Faraday's law also implies that both fields are spatially transverse to each other

$$\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega} = \frac{n\hat{n} \times \vec{E}_0}{c} \qquad \hat{n} \equiv \frac{\vec{k}}{k} \quad \text{index of refraction}$$

Standing Waves

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

Note that Maxwell's equations are linear, so any linear combination of magnetic/electric fields is also a solution.

Thus a **standing wave** solution is also acceptable, where there are two plane waves moving in opposite directions:

$$\vec{E}(\vec{x},t) = \vec{E}_0\, \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{-i\vec{k}\cdot\vec{x}-i\omega t}\right)$$

$$\vec{B}(\vec{x},t) = \vec{B}_0\, \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{-i\vec{k}\cdot\vec{x}-i\omega t}\right)$$

Polarization

$$\vec{E}(\vec{x},t) = \vec{E}_0\,\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

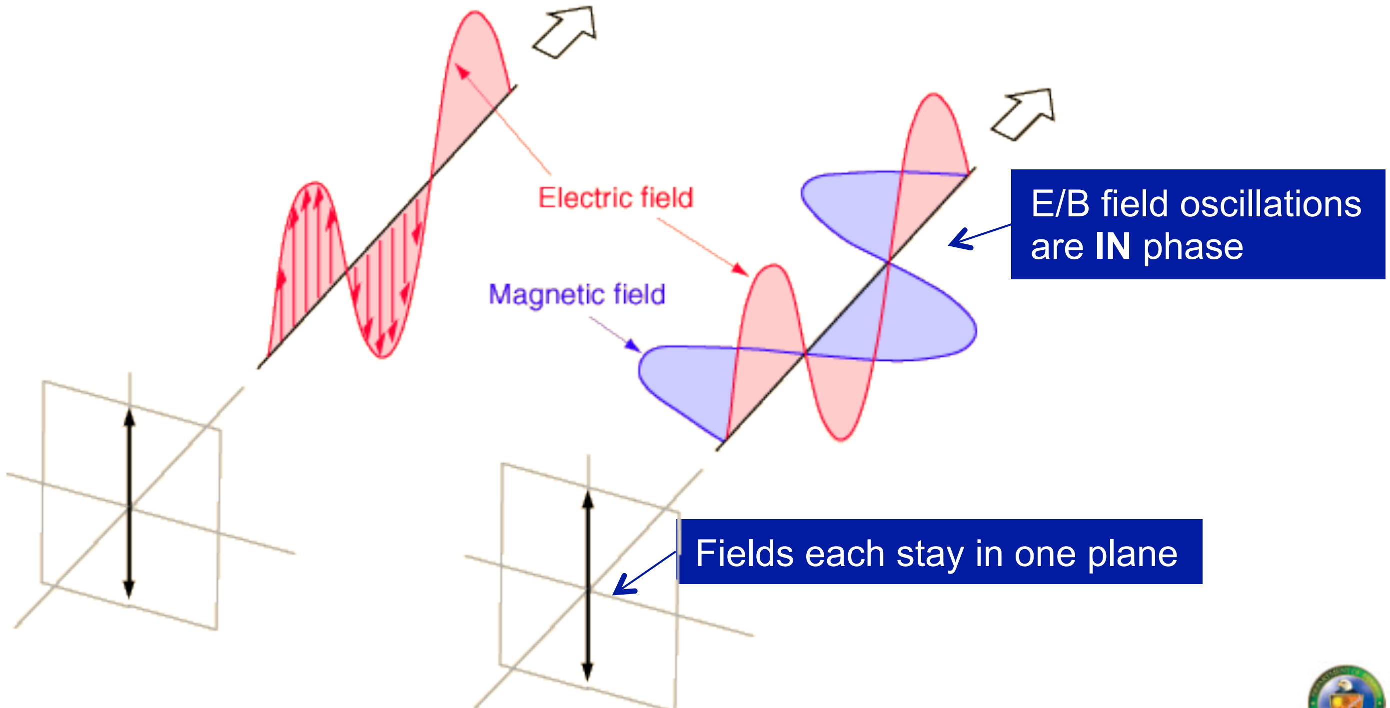
$$\vec{B}(\vec{x},t) = \vec{B}_0\,\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

- As long as the $\vec{E}$ and $\vec{B}$ fields are transverse, they can still have different transverse components.
  - So our description of these fields is also incomplete until we specify the transverse components at all locations in space
  - This is equivalent to an uncertainty in phase of rotation of $\vec{E}$ and $\vec{B}$ around the wave vector $\vec{k}$.
  - The identification of this transverse field coordinate basis defines the **polarization** of the field.

Linear Polarization

$$\vec{E}(\vec{x},t) = \vec{E}_0 \, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}_0 \, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t}$$

- So, for example, if $\vec{E}$ and $\vec{B}$ transverse directions are constant and do not change through the plane wave, the wave is said to be **linearly polarized**.

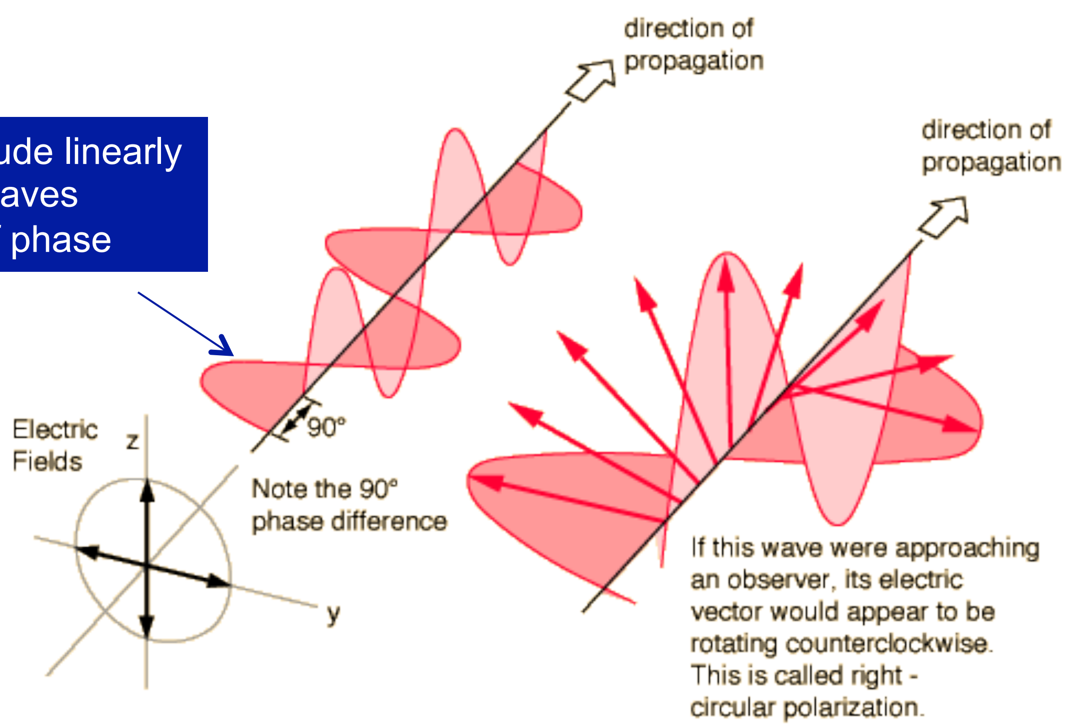


Circular Polarization

$$\vec{E}(\vec{x},t) = \vec{E}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t+\pi/2} \right)$$
$$\vec{B}(\vec{x},t) = \vec{B}_0 \left(\mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t+\pi/2} \right)$$

- If $\vec{E}$ and $\vec{B}$ transverse directions vary with time, they can appear as two plane waves traveling out of phase. This phase difference is 90 degrees for **circular polarization**.

Two equal-amplitude linearly polarized plane waves 90 degrees out of phase



9.2: Cylindrical Waveguides

- Consider a cylindrical waveguide, radius a, in z direction

$$\vec{E} = \vec{E}(r, \theta) \mathrm{e}^{i(\omega t - k_g z)}$$

  - k_g can be imaginary for attenuation down the guide
  - We'll find constraints on this cutoff wave number
  - Maxwell in cylindrical coordinates (math happens) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} = -k_c^2 E_z$$

$$\frac{\partial^2 E_z}{\partial z^2} = -k_g^2 E_z$$

$$\text{(and the same for } H_z\text{)}$$

$$k \equiv \left(\frac{\omega}{c}\right) \quad \text{(free space wave number)}$$

$$k_c^2 \equiv k^2 - k_g^2$$

$$E_r = -\frac{1}{k_c^2}\left[ik_g\frac{\partial E_z}{\partial r} + \frac{i\omega\mu}{r}\frac{\partial H_z}{\partial \theta}\right],$$

$$E_\theta = \frac{1}{k_c^2}\left[-\frac{ik_g}{r}\frac{\partial E_z}{\partial \theta} + i\omega\mu\frac{\partial H_z}{\partial r}\right],$$

$$H_r = \frac{1}{k_c^2}\left[\frac{i\omega\epsilon}{r}\frac{\partial E_z}{\partial \theta} - ik_g\frac{\partial H_z}{\partial r}\right],$$

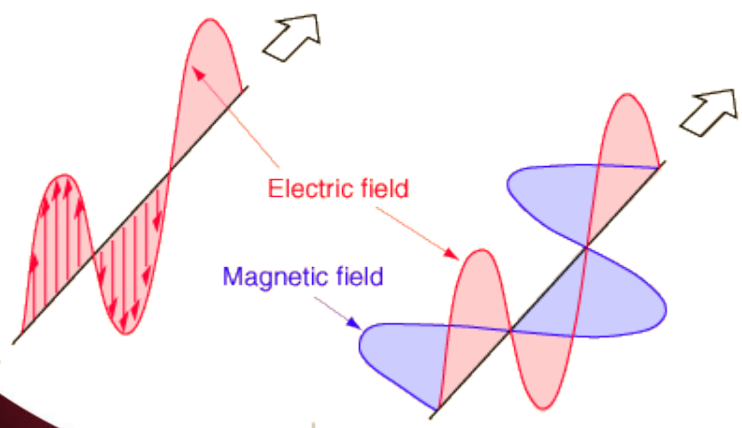
$$H_\theta = -\frac{1}{k_c^2}\left[i\omega\epsilon\frac{\partial E_z}{\partial r} + \frac{ik_g}{r}\frac{\partial H_z}{\partial \theta}\right].$$

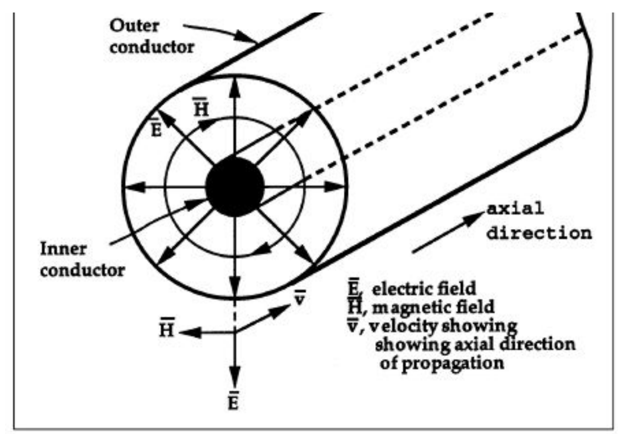
Transverse Electromagnetic (TEM) Modes

- Various subsets of solutions are interesting
  - For example, the $E_z, H_z = 0$ nontrivial solutions require

$$k_c^2 = k^2 - k_g^2 = 0$$

  - The wave number of the guide matches that of free space
  - Wave propagation is similar to that of free space
  - This is a TEM (transverse electromagnetic) mode
    - Requires multiple separate conductors for separate potentials or free space
    - Sometimes similar to polarization pictures we had before





Coaxial cable

Transverse Magnetic (TM) Modes

- Maxwell's equations are linear so superposition applies
  - We can break all fields down to TE and TM modes
  - **TM**: Transverse magnetic (H_z=0, E_z≠0)
  - TE: Transverse electric (E_z=0, H_z≠0)
  - TM provides particle energy gain or loss in z direction!

$$\text{Separate variables } E_z(r,\theta) = R(r)\Theta(\theta)$$

$$\text{Boundary conditions } E_z(r=a) = E_\theta(r=a) = 0$$

  - Maxwell's equations in cylindrical coordinates give

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k_c^2 - \frac{n^2}{r^2}\right)R = 0$$

Bessel equation

$$\frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0$$

SHO equation
=> n integer, real

TM Mode Solutions

- The radial equation has solutions of Bessel functions of first $J_n(k_c r)$ and second $N_n(k_c r)$ kind
  - Toss N_n since they diverge at r=0
  - Boundary conditions at r=a require $J_n(k_c a)=0$
  - This gives a constraint on k_c

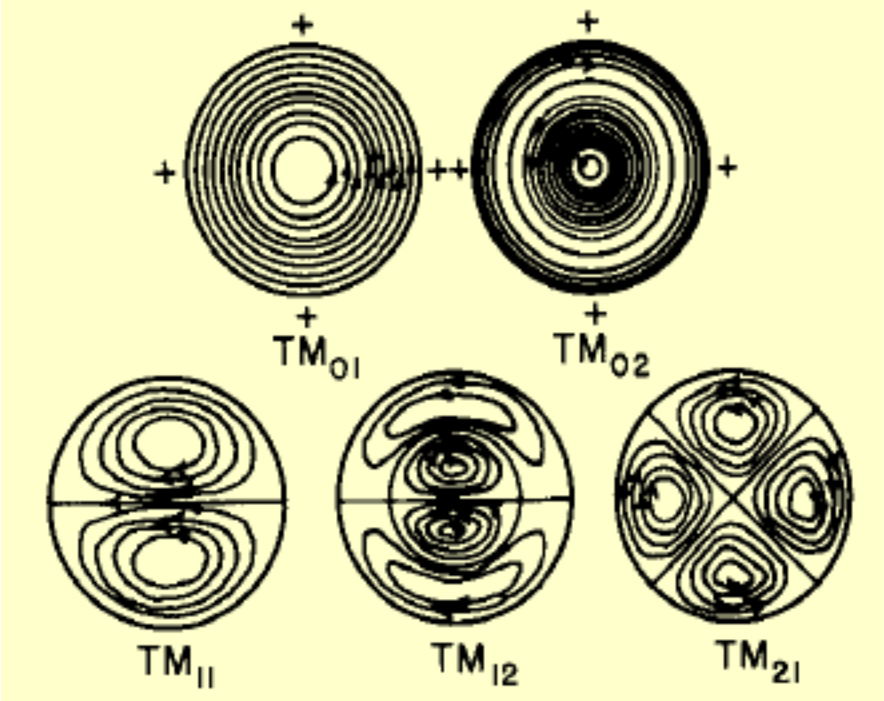
$$k_c = X_{nj}/a \quad \text{where} \quad X_{nj} \text{ is the j}^{\text{th}} \text{ nonzero root of } J_n$$

$$\Rightarrow E_z(r,\theta,t) = (C_1 \cos n\theta + C_2 \sin n\theta) J_n\left(\frac{X_{nj}}{a} r\right) \mathrm{e}^{\mathrm{i}(\omega \mathrm{t} - \mathrm{k_g z})}$$

  - This mode of this field is commonly known as the TM_{nj} mode
  - The first index corresponds to theta periodicity, while the second corresponds to number of radial Bessel nodes
  - TM_{01} is the usual fundamental accelerating mode

TM Mode Visualizations

$$E_z(r,\theta,t) = (C_1 \cos n\theta + C_2 \sin n\theta) J_n\left(\frac{X_{nj}}{a} r\right) \mathrm{e}^{\mathrm{i}(\omega \mathrm{t} - \mathrm{k_g z})}$$



- Java app visualizations are also linked to the class website
  - For example, http://www.falstad.com/emwave2 (TM visualization)
    - Scroll down to Circular Modes 1 or 2 in first pulldown

Cutoff Frequency

- It's easy to see that there is a minimum frequency for our waveguide that obeys the boundary conditions
  - Modes at lower frequencies will suffer resistance
  - Dispersion relation in terms of radial boundary condition zero

$$k^2 - k_g^2 = \left(\frac{\omega}{c}\right)^2 - k_g^2 = k_c^2 = \left(\frac{X_{nj}}{a}\right)^2$$

  - A propagating wave must have real group velocity $k_g^2 > 0$
  - This gives the expected lower bound on the frequency

$$\frac{\omega}{c} \geq \frac{X_{nj}}{a} \quad \Rightarrow \quad \boxed{\text{cutoff frequency } \omega_c = \frac{cX_{nj}}{a}}$$

  - Wavelength of lowest cutoff frequency for j=1 is

$$\boxed{\lambda_c = \frac{2\pi}{k_c} \approx 2.61a}$$

Dispersion or Brillouin Diagram

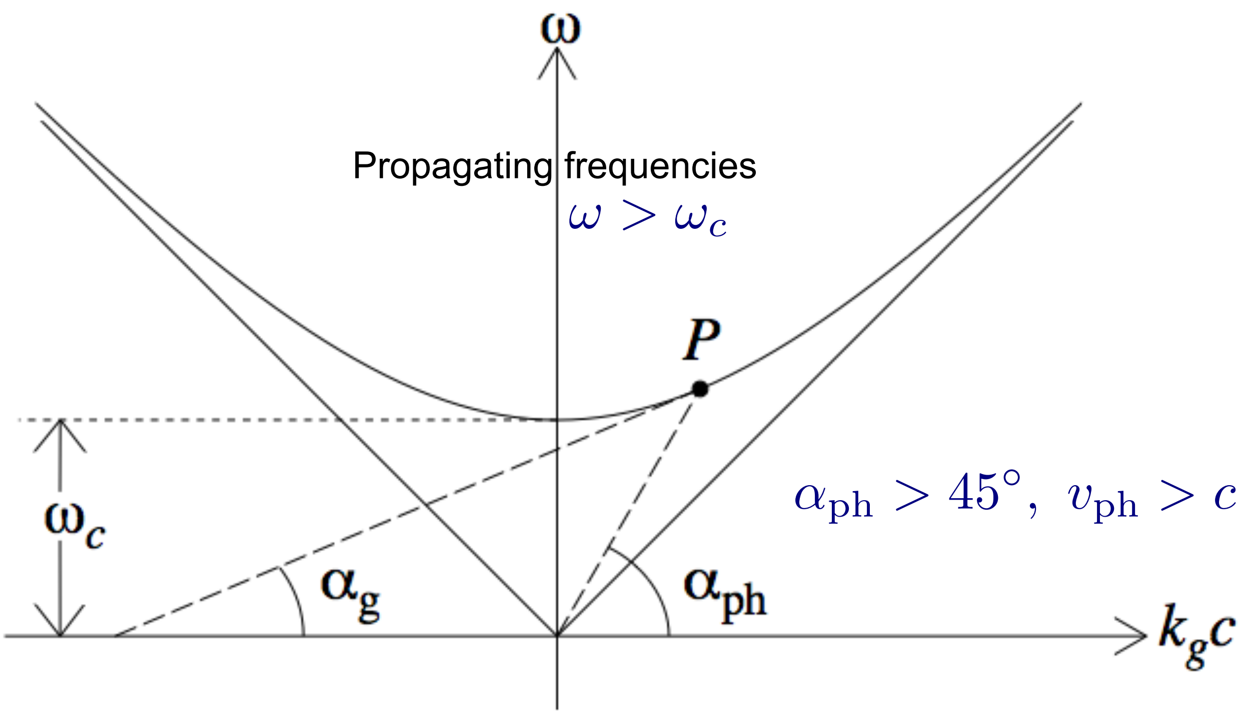
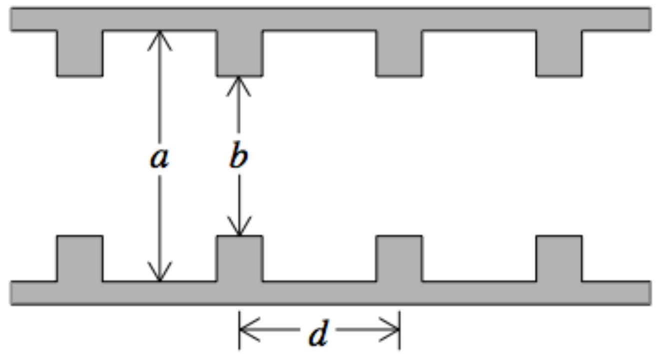


Figure. 9.1 The dispersion or Brillouin diagram for a uniform cylindrical waveguide of radius a. The angles $\alpha_{\rm ph}$ and α_g for the point P are defined by the relations $\tan \alpha_{\rm ph} = v_{\rm ph}/c$, and $\tan \alpha_g = v_g/c$. The asymptotic diagonal lines are the dispersion curve for a wave traveling in free space.

- Circular waveguides above cutoff have phase velocity >c
- Cannot easily be used for particle acceleration over long distances

Iris-Loaded Waveguides

- Solution: Add impedance by varying cylinder radius
  - This changes the dispersion condition by loading the waveguide



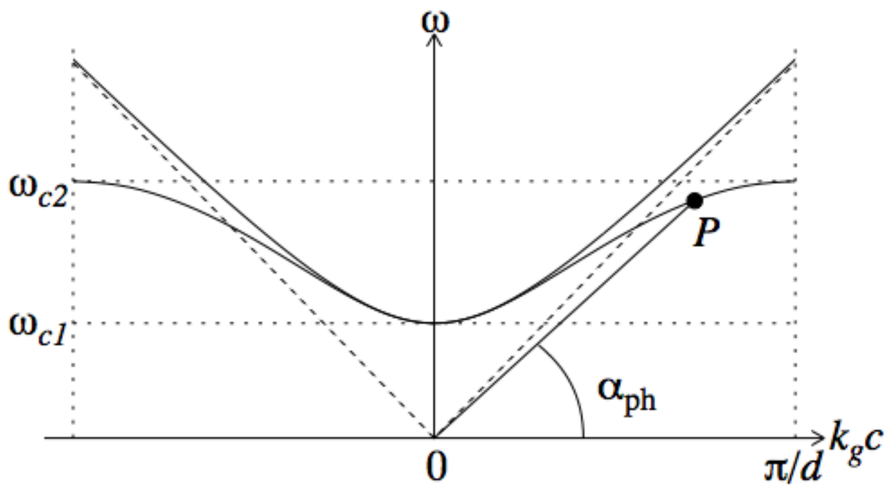
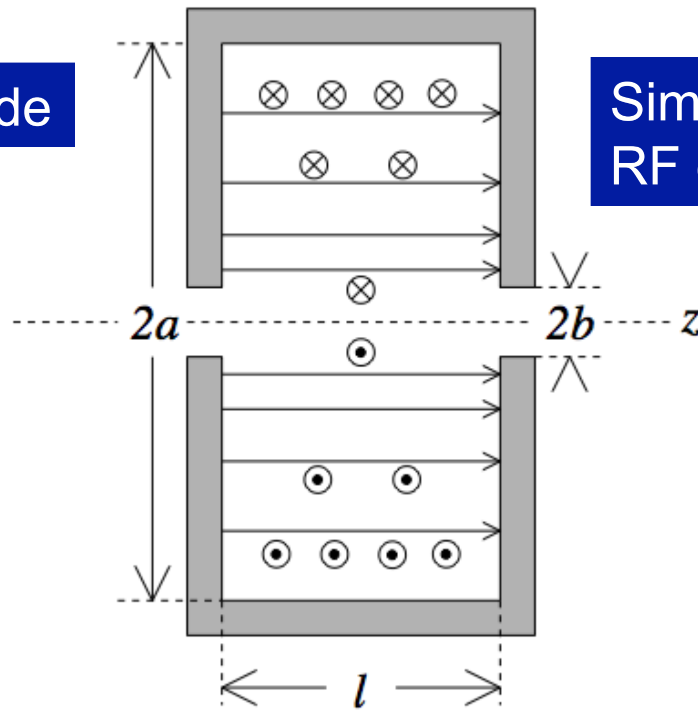


Figure. 9.3 Brillouin diagram for a loaded cylindrical waveguide. The point P has phase velocity less than c, i. e., $\alpha_{\rm ph} < 45°$. As $k_g c$ approaches π/d, the group velocity goes to zero.

Cylindrical RF Cavities

- What happens if we close two ends of a waveguide?
  - With the correct length corresponding to the longitudinal wavelength, we can produce longitudinal standing waves
  - Can produce long-standing waves over a full linac
  - Modes have another index for longitudinal periodicity

TM010 mode



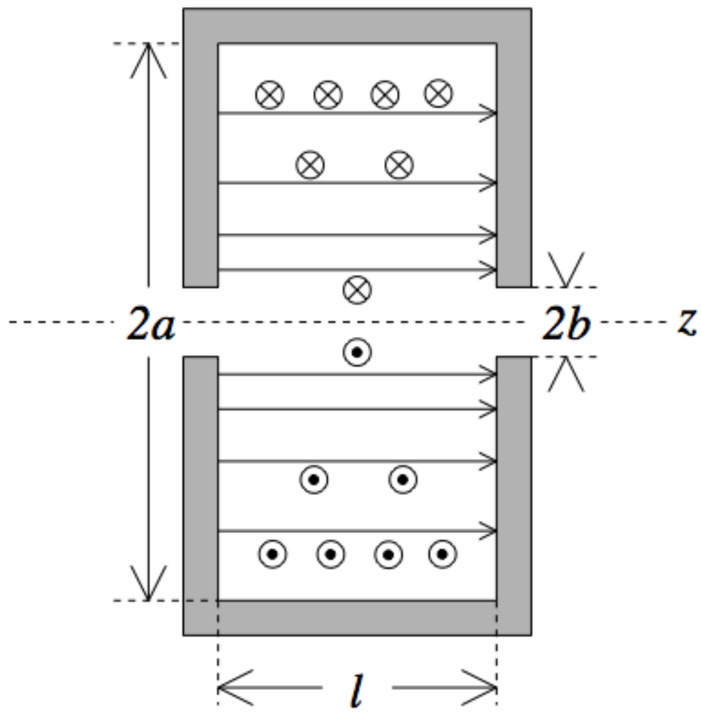
Simplified “tuna fish can” RF cavity model

Boundary Conditions

- Additional boundary conditions at end-cap conductors at z=0 and z=l

$$H_z(r,\theta,0) = H_z(r,\theta,l) = 0$$

$$E_r(r,\theta,0) = E_\theta(r,\theta,0) = E_r(r,\theta,l) = E_\theta(r,\theta,l) = 0$$



TE_{nmj} Modes

$$E_z = 0 \text{ everywhere}$$

$$H_z(r,\theta,z) \sim J_n(k_c r)(C_1 \cos n\theta + C_2 \sin n\theta) \sin\left(\frac{m\pi z}{l}\right)$$

$$k_c^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{l}\right)^2$$

Longitudinal periodicity

$$J_n'(k_c a) = 0 \qquad X_{nj}' \text{ is j}^{\text{th}} \text{ root of } J_n'$$

$$f_{nmj} = \frac{c}{2\pi}\sqrt{\left(\frac{X_{nj}'}{a}\right)^2 + \left(\frac{m\pi}{l}\right)^2}$$

TM_{nmj} Modes

$$H_z = 0 \text{ everywhere}$$

$$E_z(r,\theta,z) \sim J_n(k_c r)(C_1 \cos n\theta + C_2 \sin n\theta) \cos\left(\frac{m\pi z}{l}\right)$$

Longitudinal periodicity

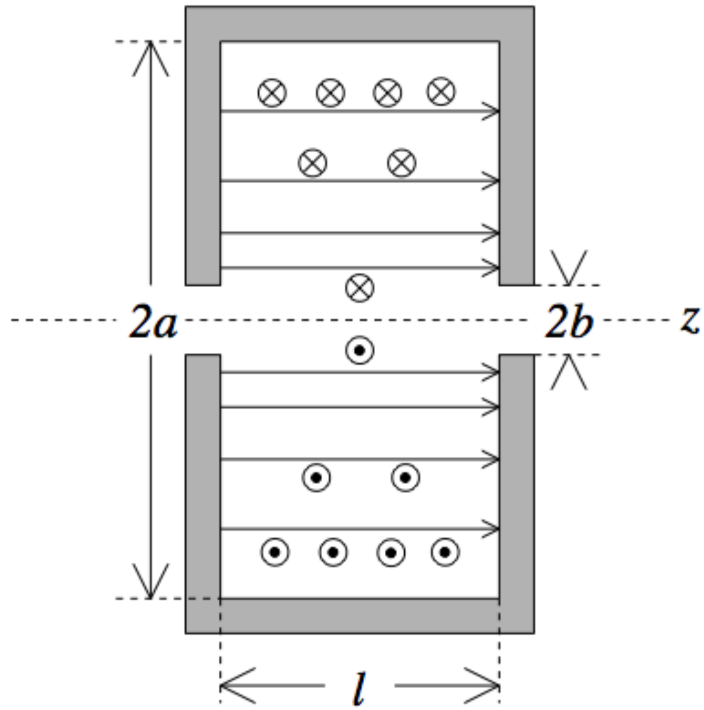
$$J_n(k_c a) = 0 \qquad X_{nj} \text{ is the j}^{\text{th}} \text{ root of } J_n$$

$$f_{nmj} = \frac{c}{2\pi}\sqrt{\left(\frac{X_{nj}}{a}\right)^2 + \left(\frac{m\pi}{l}\right)^2}$$

TM_{010} is (again) the preferred acceleration mode

Equivalent Circuit

- A TM_{010} cavity looks much like a lumped LRC circuit
  - Ends are capacitive
  - Stored magnetic energy is inductive
  - Currents move over resistive walls
- E, H fields 90 degrees out of phase



- Stored energy

$$U = \frac{\epsilon_0}{2} \iiint |\vec{E}(r,\theta,z,t)|^2 d^3r + \frac{\mu_0}{2} \iiint |\vec{H}(r,\theta,z,t)|^2 d^3r$$

$$U = \frac{\epsilon_0}{2} E_0^2 \int_0^l \int_0^{2\pi} \int_0^a \left[J_n \left(\frac{X_{01}}{a} r \right) \right]^2 r\, dr\, d\theta\, dz$$
$$= \frac{\epsilon_0}{2} E_0^2\, \pi a^2 l\, [J_1(X_{01})]^2,$$

C&M 9.82

TM_{010} Average Power Loss

$$\langle P_{\text{loss}} \rangle = \frac{1}{2\sigma\delta} \left\{ 2 \times 2\pi \left(\frac{E_0}{\mu_0 c} \right)^2 \int_0^a \left[J_1 \left(\frac{X_{01}}{a} r \right) \right]^2 r \, dr \right.$$

ends

$$\left. + \pi a l \left(\frac{E_0}{\mu_0 c} \right)^2 [J_1(X_{01})]^2 \right\}$$

sides

$$= \frac{\pi a(a+l)}{\sigma\delta} \left(\frac{E_0}{\mu_0 c} \right)^2 [J_1(X_{01})]^2,$$

- Compare to stored energy
  - Both vary like square of field, square of Bessel root
  - Dimensional area vs dimensional volume in stored energy

$$U = \frac{\epsilon_0}{2} E_0^2 \pi a^2 l [J_1(X_{01})]^2$$

Quality Factor

- Ratio of average stored energy to average power lost (or energy dissipated) during one RF cycle
  - How many cycles does it take to dissipate its energy?
  - High Q: nondissipative resonator
  - Copper RF: Q~10^3 to 10^6
  - SRF: Q up to 10^{11}

$$Q = \frac{U\omega}{\langle P_{\mathrm{loss}} \rangle}$$

$$U(t) = U_0 \mathrm{e}^{-\omega_0 t/Q}$$

$$\text{Pillbox cavity} \quad Q = \frac{al}{\delta(a+l)}$$

Niobium Cavity SRF "Q Slope" Problem