Sync rad: Radiated power

- "Circular" orbits:

$$P_\gamma = \frac{2}{3} r_e m c^3 \frac{\gamma^4 \beta^4}{\rho^2}, \qquad r_e = \frac{e^2}{4\pi\epsilon_0\, mc^2}.$$

Radiation in forward direction with opening angle $\propto \gamma^{-1}$

Classical radii of a few species:

$$\begin{aligned} r_e &= 2.82 \times 10^{-15} \text{ m} \\ r_{\mathrm{p}} &= 1.53 \times 10^{-18} \text{ m} \\ r_{\mathrm{Au}} &= 4.88 \times 10^{-17} \text{ m} \\ r_{\mathrm{Pb}} &= 4.98 \times 10^{-17} \text{ m} \end{aligned}$$

$$P_\gamma \propto \left(\frac{q}{m}\right)^4 U^2 B^2, \quad \text{since} \quad \rho = \frac{p}{qB_\perp} \quad \text{for fixed radius.}$$

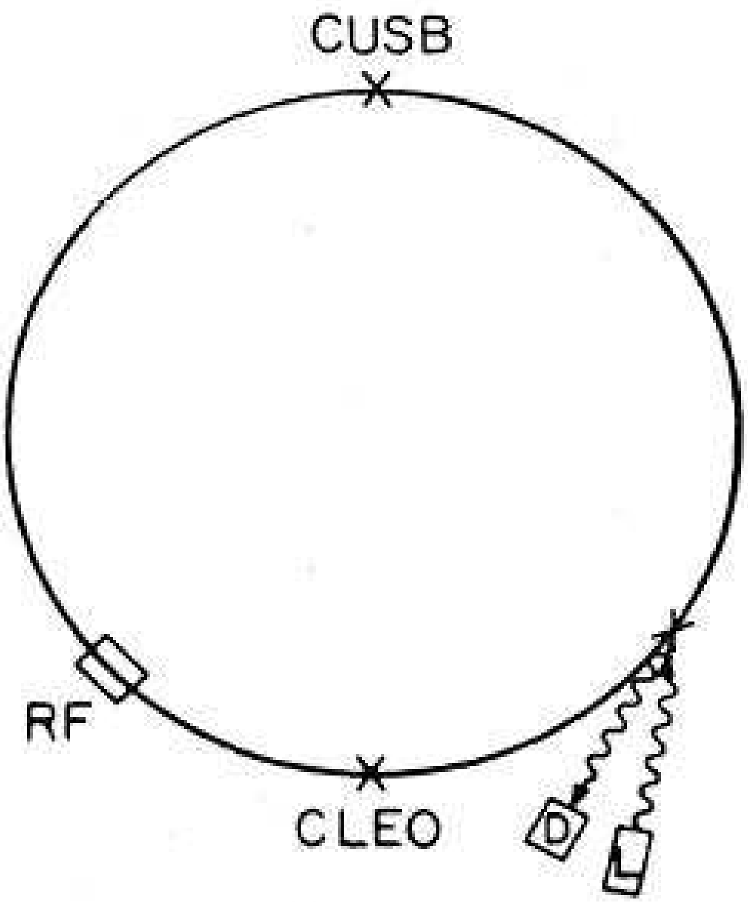
Energy loss per turn

- Energy loss per turn:

$$U_\gamma = \oint \frac{P_\gamma}{c} ds$$
$$\simeq \frac{C_\gamma U^4}{2\pi} \oint \frac{ds}{\rho^2},$$
$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} = 8.85 \times 10^{-5} \frac{\mathrm{m}}{(\mathrm{GeV})^3}$$

If all dipoles are identical, then $U_\gamma = \frac{C_\gamma U^4}{\rho}$.

CESR circa 1982



$e^+ \; e^-$ collider.

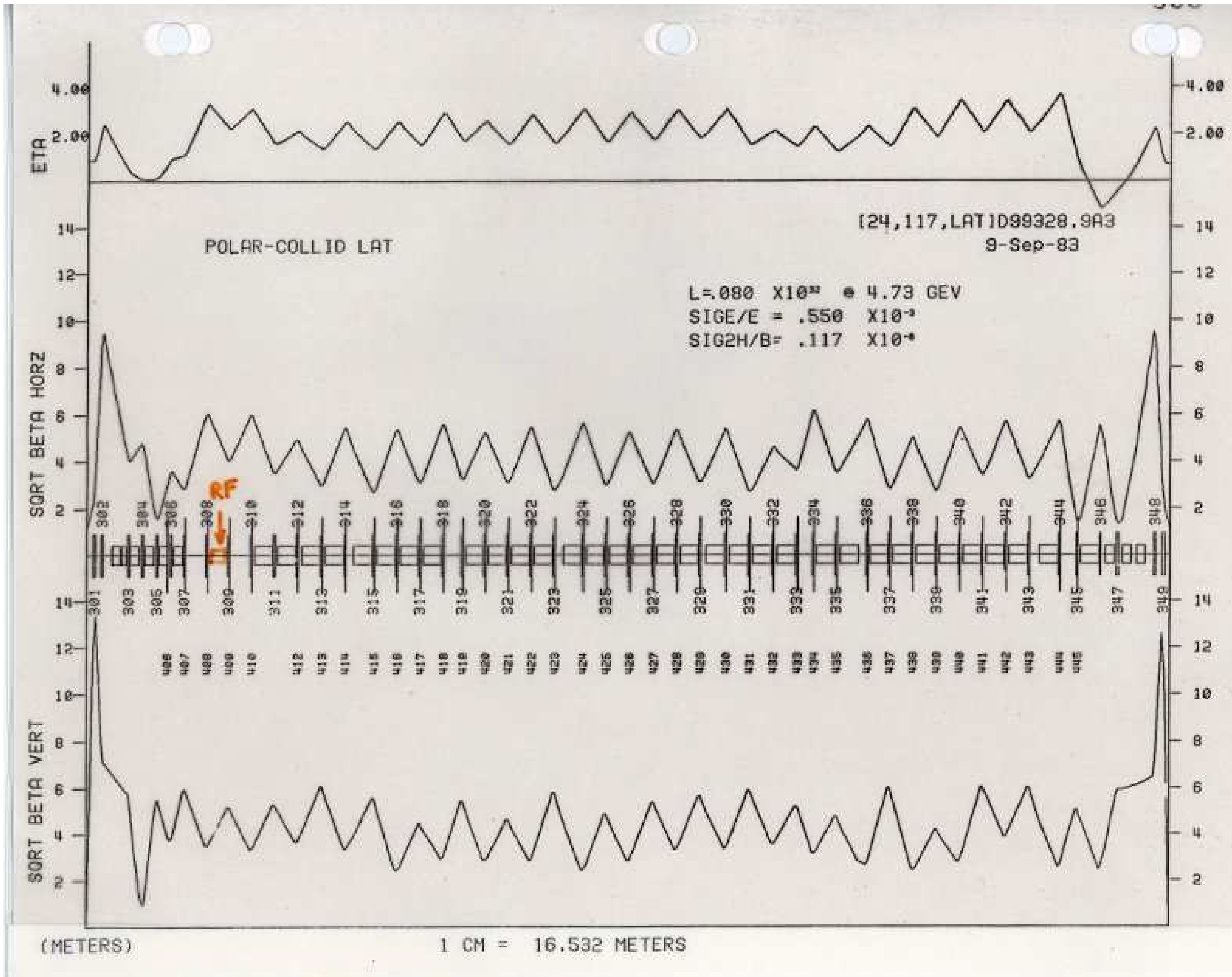
$L = 768$ m.

$U \sim 4 \to 5.6$ GeV.

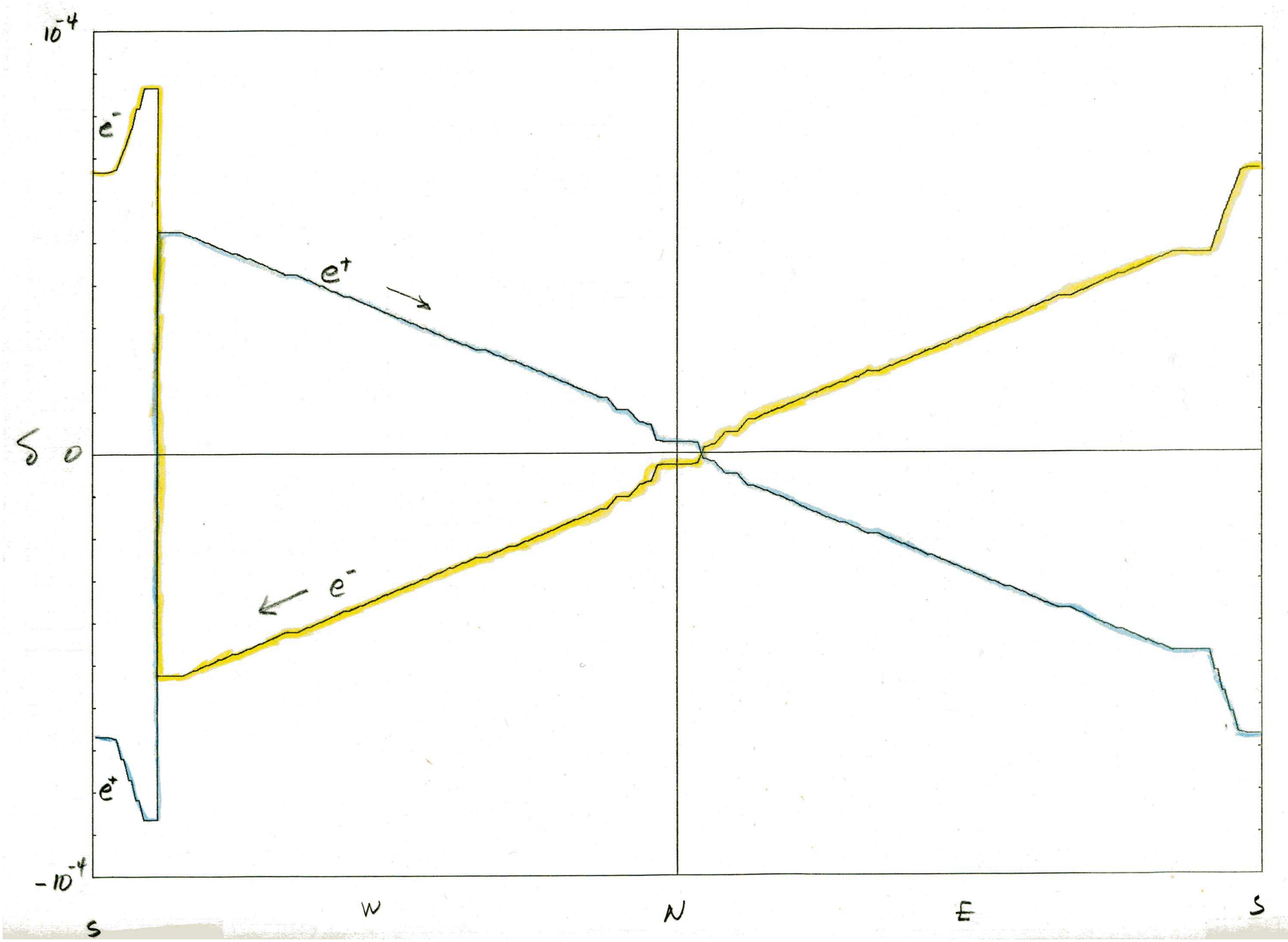
$f_s = 390$ kHz.

- positrons — clockwise.
- electrons — counterclockwise.

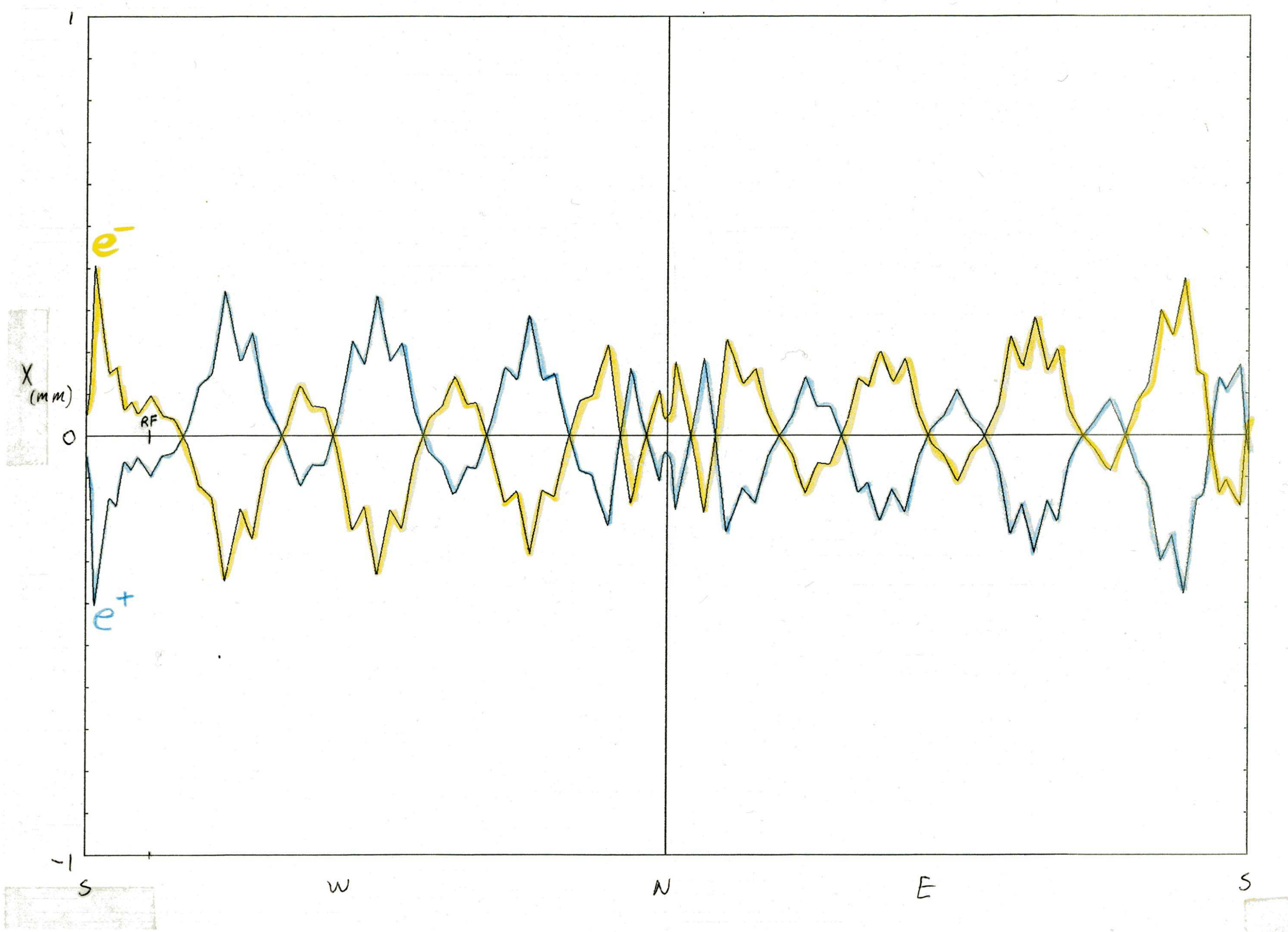
CESR optics (half ring)



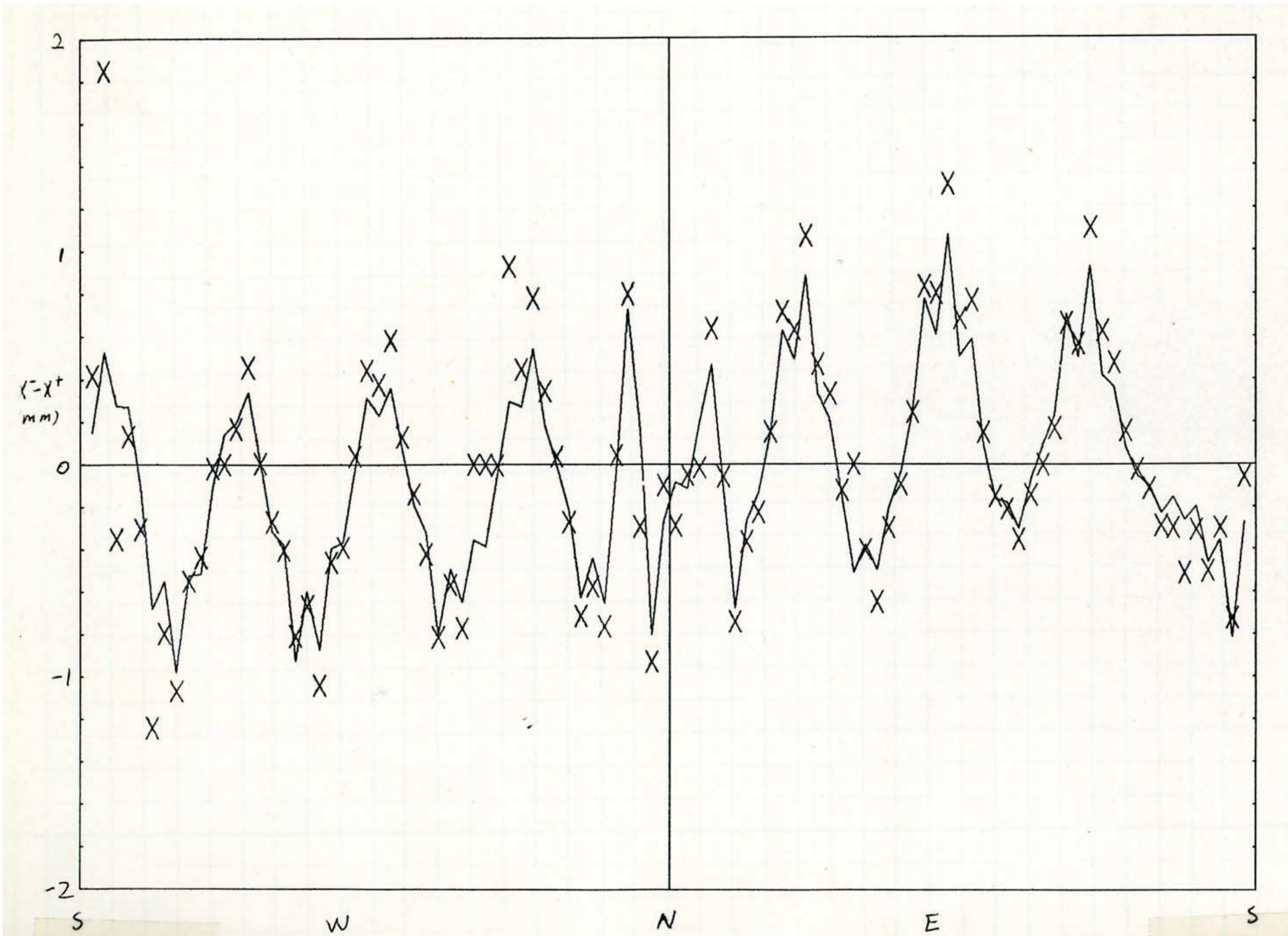
Calculated energy loss



Calculated orbits



Calculated and measured orbit difference



Radiation damping of Energy Oscillations

- energy deviation of particle in question: $u = U - U_s$
- Difference equations of motion:

$$\frac{du}{dt} \simeq \frac{\omega_\mathrm{s}}{2\pi} qV \cos\phi_s\,\varphi - \frac{1}{\tau_s}[U_\gamma(U_s + u) - U_\gamma(U_s)],$$
$$\frac{d\varphi}{dt} \simeq -\frac{\omega_\mathrm{rf}\eta_\mathrm{tr}}{U_s} u,$$

- Expand energy loss about the synchronous energy U_s:

$$U_\gamma(U) = U_\gamma(U_s) + \left(\frac{dU_\gamma}{dU}\right)_s u + \ldots.$$

- Combining these yields damped oscillator equation:

$$\frac{d^2u}{dt^2} + \frac{\omega_\mathrm{s}}{2\pi}\left(\frac{dU_\gamma}{dU}\right)_s \frac{du}{dt} + \Omega_s^2 u = 0, \quad \text{with} \quad \Omega_s = \omega_\mathrm{s}\sqrt{\frac{qVh\eta_\mathrm{tr}\cos\phi_s}{2\pi U_s}}$$

- Of course $\eta_\mathrm{tr} = -\alpha_p$ for most electron rings with $\gamma \gg \gamma_\mathrm{tr}$.

Damped solution

$$u(t) = u_0 e^{-t/\tau_u} \sin(\Omega_s' t + \psi_0),$$

with damping rate and modified frequency:

$$\frac{1}{\tau_u} = \frac{1}{2\tau_s}\left(\frac{dU_\gamma}{dU}\right)_s \quad \text{and} \quad \Omega_s' = \sqrt{\Omega_s^2 - \frac{1}{\tau_u^2}},.$$

The derivative, $(dU_\gamma/dU)_s$, can be calculated from the formula:

$$U_\gamma = \oint P_\gamma \frac{dt}{ds} ds = \frac{1}{c}\oint P_\gamma \left(1 + \frac{\eta}{\rho}\frac{u}{U_s}\right) ds,$$

since the particle has velocity: $c = \dfrac{d\sigma}{dt} = \left(1 + \dfrac{\eta}{\rho}\dfrac{u}{U_s}\right)\dfrac{ds}{dt}.$ (see CM:§5.6)

- Differentiating with respect to energy gives:

$$\left(\frac{dU_\gamma}{dU}\right)_s = \frac{1}{c}\oint\left[\left(\frac{dP_\gamma}{dU}\right)_s + P_\gamma\frac{\eta}{\rho U_s} + \left(\frac{dP_\gamma}{dU}\right)_s \frac{\eta}{\rho}\frac{u}{U_s}\right] ds.$$

  - The 3rd term is negligible: $\Omega_s >> \frac{1}{\tau_u}$.

    e.g. CESR at 5 GeV: $\Omega_s/2\pi \simeq 20$ khz, whereas damping time $\sim$ 10ms.

- On average: $\frac{dx}{dU} = \frac{\eta}{U_s}$, and $P_\gamma \propto U^2 B^2$, so

$$\left(\frac{dP_\gamma}{dU}\right)_s = 2\frac{P_\gamma}{U_s} + 2\frac{P_\gamma}{\rho}\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)_s \frac{\eta}{U_s}.$$

- Integrating: $\left(\frac{dU_\gamma}{dU}\right)_s = \frac{U_\gamma}{U_s}(2+\mathcal{D})$, where

$$\mathcal{D} = \frac{1}{cU_\gamma}\oint P_\gamma \eta \frac{1-2n}{\rho} ds, \quad \text{with} \quad n = -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)_0$$

- Damping time for synchrotron oscillations:

$$\tau_u = \frac{2}{2+\mathcal{D}} \frac{U_s}{U_\gamma} \tau_s.$$

- For separated function lattice ($n = 0$ in dipoles), we should note that

$$\mathcal{D} \simeq \frac{1}{cU_\gamma} \oint P_\gamma \frac{\eta}{\rho} ds.$$

Damping of vertical oscillations

- trajectory: $y = \sqrt{\mathcal{W}\beta}\cos\psi$
- Courant-Snyder invariant: $\mathcal{W} = \beta y'^2 + 2\alpha y y' + \gamma y^2$
- For simplicity (following Matt Sands), let's set $\alpha = 0$, $\Rightarrow$ $\gamma = 1/\beta$.

  (The gory details with nonconstant α are worked out in the book and end up giving the same result.)
- So we have for a given trajectory:

$$y = A\sqrt{\beta}\cos\psi, \quad \text{and} \quad y' = -\frac{A}{\sqrt{\beta}}\sin\psi$$

$$\mathcal{W} = \beta y'^2 + \frac{y^2}{\beta} = \beta\,\frac{A^2}{\beta}\sin^2\psi + \frac{1}{\beta}\,A^2\beta\cos^2\psi = A^2$$

- For damping, we ignore quantum fluctuations and take the momentum loss in the direction of motion:
- When a photon is radiated, the electron loses Δu of energy, but: $\Delta y = 0$ (always), and $\Delta y' = 0$ (ignoring quant. fluct.)
- before the photon is radiated: $p_{y0} = p_0 y_0'$.
- after the photon is radiated: $p_{y1} = p_0 y_0' \left(1 - \frac{\Delta u}{U_s}\right)$
- integrating the loss around the ring gives: $p_y \simeq p_0 y' \left(1 - \frac{U_\gamma}{U_s}\right)$
- longitudinal component after one turn is $p_z \simeq p_0 \left(1 - \frac{U_\gamma}{U_s}\right)$
- add rf kick to recover energy: $p_z \simeq p_0 \left(1 - \frac{U_\gamma}{U_s}\right)\left(1 + \frac{U_\gamma}{U_s}\right)$
- after complete revolution: $y'_{\mathrm{rf}} \simeq y' \left(1 - \frac{U_\gamma}{U_s}\right)$.

- per turn we have: $\Delta y' = -y' \frac{U_\gamma}{U_s}$, and $\Delta y = 0$.
- How does our invariant change?

$$(A + \Delta A)^2 - A^2 = 2A\,\Delta A + (\Delta A)^2 \simeq 2A\,\Delta A,$$

if we assume the $(\Delta A)^2$ term is small compared to $2A\,\Delta A$.

$$\begin{aligned}
2A\,\Delta A &\simeq \beta(y_0' + \Delta y')^2 + \frac{(y_0 + 0)^2}{\beta} - \left(\beta y_0'^2 + \frac{y_0^2}{\beta}\right) \\
&\simeq 2\beta y_0' \Delta y' + \beta(\Delta y')^2 \simeq 2\beta y_0' \Delta y' \\
A\,\Delta A &\simeq -\beta y_0'^2 \frac{U_\gamma}{U_s} \\
&\simeq -\beta \left\langle \left(\frac{A}{\sqrt{\beta}} \sin\psi\right)^2 \right\rangle_\psi \frac{U_\gamma}{U_s} \qquad \text{(average over all betatron-phases)} \\
&= \frac{1}{2} A^2 \frac{U_\gamma}{U_s}
\end{aligned}$$

$$\frac{dA}{dt} \simeq \frac{\Delta A}{\tau_s} \simeq -\frac{1}{2}\,\frac{U_\gamma}{U_s\tau_s}\,A$$

$$A = A_0 e^{-t/\tau_y} \quad \text{with}$$

$$\tau_y = 2\frac{U_s}{U_\gamma}\,\tau_s = 2\,\frac{U_s}{\langle P_\gamma\rangle},$$

where $\langle P_\gamma \rangle$ radiated power averaged over one turn.

Damping or horizontal oscillations

- trajectory equations:

$$
\begin{aligned}
x &= x_\beta + x_p, = x_\beta + \eta\,\frac{u}{U_s},\\
x_\beta &= A\sqrt{\beta}\cos\psi,\\
x' &= x'_\beta + x'_p = x'_\beta + \eta'\,\frac{u}{U_s},\\
x'_\beta &= \frac{A}{\sqrt{\beta}}\sin\psi, \qquad (\alpha = 0 \text{ approx.})
\end{aligned}
$$

- When electron radiates, $dx = 0$ and $dx' = 0$ just like vertical, but

$$dx_\beta = -\eta\,\frac{-du}{U_s}, \quad \text{and} \quad dx'_\beta = -\eta'\,\frac{-du}{U_s}$$

- Courant-Snyder invariant: $\mathcal{W} = A^2 = \beta x_\beta'^2 + \frac{x_\beta^2}{\beta}$

$$2A\,d(\Delta A) + [d(\Delta A)]^2 = \beta(x_\beta' + dx_\beta')^2 + \frac{1}{\beta}(x_\beta + dx_\beta)^2 - \left(\beta x_\beta'^2 + \frac{x_\beta^2}{\beta}\right)$$
$$\simeq 2\beta x'\beta\,dx_\beta' + 2\beta x\beta\,dx_\beta + \mathcal{O}(d?^2)$$

$$A\,d(\Delta A) = \beta\eta'\,\frac{dU}{U_s}\,x_\beta' + \frac{1}{\beta}\,x_\beta\eta\,\frac{dU}{U_s}$$
$$= \left(\beta\eta' x_\beta' + \frac{\eta x_\beta}{\beta}\right)\frac{dU}{U_s}$$

Recall: $P_\gamma \propto U^2 B^2$, so

$$\frac{dU_\gamma}{ds} = \frac{1}{c}\,\frac{dU_\gamma}{dt}\left(1+\frac{x}{\rho}\right) = \frac{P_\gamma}{c}\left(1+\frac{2}{B}\,\frac{\partial B_y}{\partial x}\,x\right)\left(1+\frac{x}{\rho}\right)$$

$$= \frac{P_\gamma}{c}\left(1-\frac{2n}{\rho}\,x\right)\left(1+\frac{x}{\rho}\right)$$

$$= \frac{P_\gamma}{c}\left[1-\frac{2n}{\rho}(x_\beta+x_p)\right]\left(1+\frac{x_\beta+x_p}{\rho}\right)$$

$$= \frac{P_\gamma}{c}\left[1+\frac{1-2n}{\rho}\,x_\beta-\frac{4n}{\rho^2}\,x_\beta x_p-\frac{2n}{\rho^2}\,x_\beta^2+\left(1+\frac{x_p}{\rho}-\frac{2n}{\rho^2}\,x_p^2\right)\right]$$

Average over all betatron and synchrotron phases:

$$\langle A\,d(\Delta A)\rangle_\psi \to \frac{P_\gamma}{cU_s}\left\langle\left(\beta\eta' x'_\beta+\frac{\eta x_\beta}{\beta}\right)x_\beta\left[\frac{1-2n}{\rho}-\frac{4n}{\rho^2}\,\frac{u}{U_s}\right]\right\rangle_\psi ds$$

Since

$$\langle x'_\beta x_\beta \rangle = 0, \quad \text{and} \quad \langle x_\beta^2 \rangle = \frac{A^2\beta}{2},$$

we find

$$\langle d(\Delta A)\rangle_\psi = \frac{A}{2cU_s}\, P_\gamma \eta\, \frac{1-2n}{\rho}\, ds,$$

and integrating around the ring, the contribution from radiation is

$$(\Delta A)_{\rm rad} = \frac{A}{2cU_s} \oint P_\gamma \eta \frac{1-2n}{\rho}\, ds = \frac{A}{2}\,\frac{U_\gamma}{U_s}\,\mathcal{D}$$

- Now for the rf contribution to ΔA.
  Just in front of the cavity:

$$x_1 = x_{\beta 1} + \eta \frac{u_1}{U_s}, \quad \text{and} \quad x_1' = x_{\beta 1}' + \eta' \frac{u_1}{U_s}$$

Right after the cavity we have

$$x_2 = x_{\beta 2} + \eta \frac{u_2}{U_s} = x_1, \quad \text{and} \quad x_2' = x_{\beta 2}' + \eta' \frac{u_2}{U_s} = \left(1 - \frac{U_\gamma}{U_s}\right) x_1'.$$

So with $u_2 = u_1 + U_\gamma$ from the rf kick

$$(\Delta x_\beta)_{\mathrm{rf}} = x_{\beta 2} - x_{\beta 1} = -\eta \frac{U_\gamma}{U_s},$$

$$x_2' - x_1' = \Delta x_\beta + \eta' \frac{U_\gamma}{U_s} = -\frac{U_\gamma}{U_s} x_1', \quad \text{or on rearranging}$$

$$(\Delta x_\beta')_{\mathrm{rf}} = x_{\beta 2}' - x_{\beta 1}' = -x_{\beta 1} \frac{U_\gamma}{U_s} - \eta' \frac{U_\gamma}{U_s} - \eta' \frac{U_\gamma u_1}{U_s^2}$$

For the variation of the amplitude, we have the first order terms:

$$2A\,\Delta A = 2\left(\beta x'_\beta\,\Delta x'_\beta + \frac{x_\beta\,\Delta x_\beta}{\beta}\right) + \mathcal{O}(\Delta^2),$$

but averaging over all betatron phases:

$$\langle x_\beta\,\Delta x_\beta\rangle \propto \langle x_\beta\rangle \to 0, \quad \text{and} \quad \langle x'_\beta\,\Delta x'_\beta\rangle \to -\frac{U_\gamma}{U_s}\langle x'^2_\beta\rangle = \frac{A^2}{2\beta}\frac{U_\gamma}{U_s}.$$

$$A(\Delta A)_{\mathrm{rf}} = -\frac{U_\gamma}{U_s}\beta\,\langle x'^2_\beta\rangle = -\frac{A^2}{2}\frac{U_\gamma}{U_s}$$

- Adding the radiation and rf parts together yields:

$$\Delta A = (\Delta A)_{\mathrm{rf}} + (\Delta A)_{\mathrm{rad}} = -\frac{A}{2}\frac{U_\gamma}{U_s} + \frac{A}{2}\frac{U_\gamma}{U_s}\mathcal{D}$$

Dividing by the revolution period, we get the differential equation:

$$\frac{dA}{dt} = \frac{\Delta A}{\tau_s} = -\frac{U_\gamma}{2U_s\tau_s}(1-\mathcal{D})\,A$$

which has the solution

$$A = A_0 e^{-t/\tau_x},$$

with damping time

$$\tau_x = \frac{2}{1-\mathcal{D}}\,\frac{U_s}{U_\gamma}\,\tau_s.$$

To summarize the damping rates, define

$$\frac{1}{\tau_0} = \frac{U_\gamma}{2U_s\tau_s}.$$

$$\begin{aligned}
\frac{1}{\tau_x} &= \frac{1}{\tau_0}\,(1-\mathcal{D}), \\
\frac{1}{\tau_y} &= \frac{1}{\tau_0}, \\
\frac{1}{\tau_u} &= \frac{1}{\tau_0}\,(2+\mathcal{D}).
\end{aligned}$$

The sum of the three rates is

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_u} = \frac{4}{\tau_0}.$$

Partition numbers: $J_x = 1 - \mathcal{D}, \quad , J_y = 1 \quad \text{and} \quad J_u = 2 + \mathcal{D}$

$$\sum_i J_i = 4$$

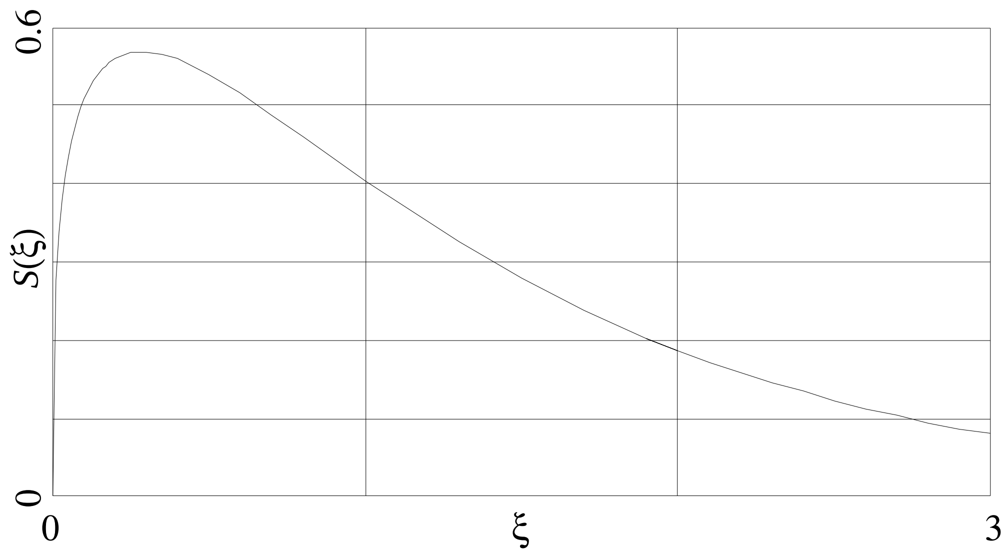
Some Properies of Synchrotron Radiation

- Power spectrum:

$$dP_\gamma = \frac{P_\gamma}{\omega_c}\, S\left(\frac{\omega}{\omega_c}\right)\, d\omega$$

$$S(\xi) = \frac{9\sqrt{3}}{8\pi}\, \xi \int_\xi^\infty K_{\frac{5}{3}}(\xi')\, d\xi'$$

$$\int_0^\infty S(\xi)\, d\xi = 1$$



- Critical energy: half the power is radiated by photons less than the critical energy, and the other half, above.

$$u_c = \hbar\omega_c = \frac{3\hbar c}{2\rho}\gamma^3$$

- Number of photons per second:

$$N_\gamma = \int_0^{U_{\max}} n_\gamma(u_\gamma)\, du_\gamma = \frac{5}{2\sqrt{3}} \frac{\alpha_{\mathrm{f}} c}{\rho} \gamma$$

here: $\alpha_{\mathrm{f}} = 1/137$

- Number of photons per radian:

$$N_r = \frac{5\alpha_{\mathrm{f}}}{2\sqrt{3}} \gamma$$

- Average photon energy and 2$^{\text{nd}}$ moment:

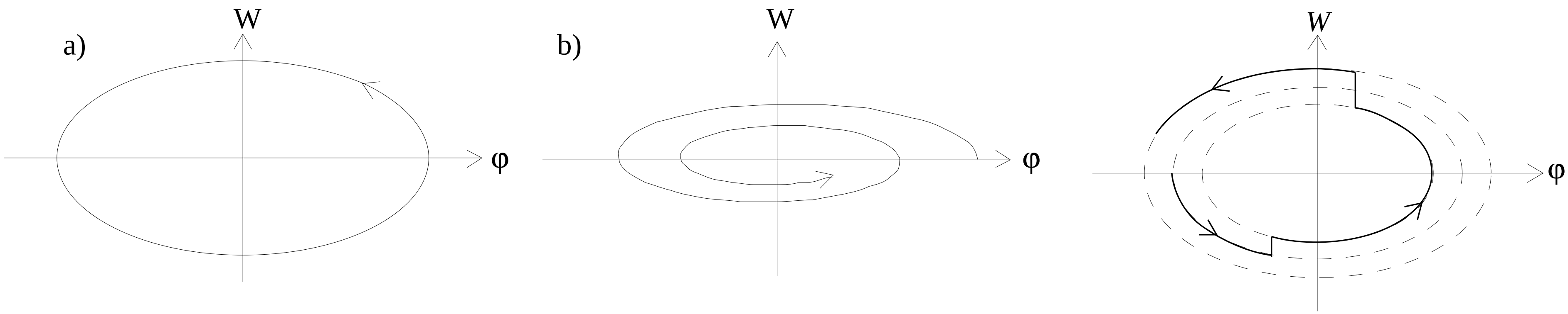
$$\langle u_\gamma \rangle = \frac{1}{N_\gamma} \int_0^{U_{\max}} u\, n_\gamma(u)\, du = \frac{8}{15\sqrt{3}} u_c \simeq 0.32 u_c$$

$$\langle u_\gamma^2 \rangle = \frac{1}{N_\gamma} \int_0^{U_{\max}} u^2 n_\gamma(u)\, du = \frac{11}{27\sqrt{3}} u_c^2 \simeq 0.41 u_c^2$$

Table I. Estimate of radiation for several rings.

Machine	U_s[GeV]	$\langle u_\gamma \rangle$[eV]	N_γ[s^{-1}]	N_r
Adone	1.5	4.6×10^2	1.8×10^9	31
SPEAR	4.5	4.9×10^3	2.2×10^9	93
CESR	5.0	5.0×10^2	4.0×10^8	160
PETRA	19	2.5×10^4	6.1×10^8	390
LEP I	50	1.4×10^4	5.0×10^7	1030
LEP II	100	1.1×10^5	1.0×10^8	2060

Energy spread and bunch length



No radiation — Continuous damping : Quantum fluctuation effect

- Equilibruim: damping rate = growth rate from quantum excitations.

- Recall the longitudinal phase space variables: W and φ

$$W = -\frac{u}{\omega_{\rm rf}} = -\frac{U_s}{\omega_{\rm rf}^2 \alpha_p}\frac{d\varphi}{dt}, \qquad (\eta_{\rm tr} \to -\alpha_p \quad \text{for ultrarelativistic electrons.})$$

- Longitudinal phase space variables:

$$u = -\omega_{\rm rf} W = -A\cos[\Omega_s(t-t_0)],$$
$$\varphi = \frac{\omega_{\rm rf}\alpha_p}{U_s\Omega_s}\,A\sin[\Omega_s(t-t_0)].$$

For simplicity, rescale the rf phase to

$$\xi = \frac{U_s\Omega_s}{\omega_{\rm rf}\alpha_p}\,\varphi.$$

Then we have the Courant-Snyder type invariant:

$$\xi^2 + u^2 = A^2.$$

- When a photon of energy u_γ is emitted, this invariant changes by

$$\Delta(A^2) = [(u-u_\gamma)^2 + \xi^2] - [u^2 + \xi^2] = -2uu_\gamma + u_\gamma^2.$$

- Summing over all emissions around the ring:

$$\Delta(A^2) \simeq -2u \oint N_\gamma \langle u_\gamma \rangle \frac{ds}{c} + \oint N_\gamma \langle u_\gamma^2 \rangle \frac{ds}{c}.$$

- averaging over syncrhotron phase: $< u_\gamma >= 0.$

$$\left(\frac{d(A^2)}{dt}\right)_{\rm QF} \simeq \frac{\Delta(A^2)}{\tau_s} = \frac{1}{c\tau_s} \oint N_\gamma \langle u_\gamma^2 \rangle \, ds$$

- Damping: $A = A_0 e^{-t/\tau_u} \qquad \Rightarrow \qquad \left(\frac{d(A^2)}{dt}\right)_{\rm damping} = -\frac{2}{\tau_u} A^2$

- Equilibrium: $\quad 0 = -\frac{2}{\tau_u} \langle A^2 \rangle + \frac{1}{c\tau_s} \oint N_\gamma \langle u_\gamma^2 \rangle \, ds$

$$\sigma_u^2 = \frac{\langle A^2 \rangle}{2} = \frac{\tau_u}{4c\tau_s} \oint N_\gamma \langle u_\gamma^2 \rangle \, ds.$$

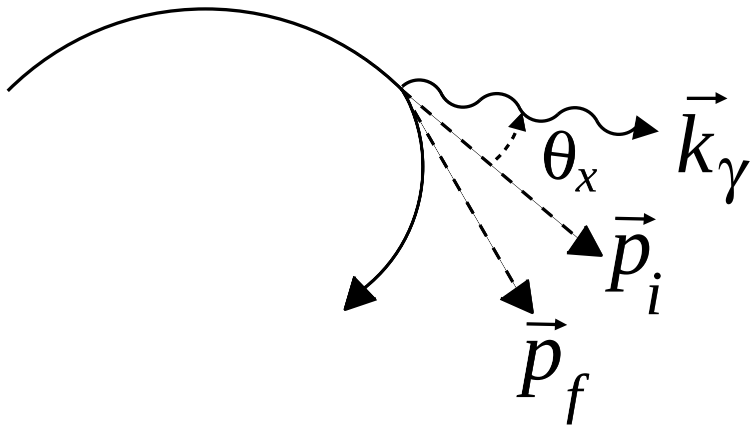
- Energy spread: $$\sigma_u = \sqrt{\frac{C_q}{J_u \rho}} \gamma^2 mc^2$$

with $C_q = 3.8 \times 10^{-8}$ m and $J_u \sim 2 + \mathcal{D}$.

Table II. Energy spread of several rings.

Machine	U_s[GeV]	σ_u[MeV]	σ_u/U_s
Adone	1.5	0.86	5.7×10^{-4}
SPEAR	4.5	4.9	1.1×10^{-3}
CESR	5.0	3.0	5.8×10^{-4}
PETRA	19	27.3	1.2×10^{-3}
LEP I	50	55	1.1×10^{-3}
LEP II	100	220	2.2×10^{-3}

Transverse excitations and emittances



$$k_\gamma = \frac{u_\gamma}{c}$$
$$\vec{p}_f = \vec{p}_i - \vec{k}_\gamma$$
$$\langle\theta_x^2\rangle \sim \frac{1}{\gamma^2}$$

- Projection of recoil angle:

$$\Delta p_x \simeq -\theta_x u_\gamma$$
$$\Delta x' \simeq \frac{\Delta p_x}{p} \simeq -\theta_x \frac{u_\gamma}{U_s}$$

$$\delta x_\beta = \eta \frac{u_\gamma}{U_s},$$
$$\delta x'_\beta = (\eta' - \theta_x) \frac{u_\gamma}{U_s},$$

- For horizontal the dispersion $(\eta u/U_s)$ typically dominates the $\langle\theta_x^2\rangle$.

  e.g, $\eta \sim 1$ to 2 m with quad separation of 10 m, $\Rightarrow \eta' \sim 0.05$ to 0.1.

  1 GeV electrons have $\gamma \simeq 2\times 10^3$ and $\theta_{x,\mathrm{rms}} \sim 5\times 10^{-4} \ll \eta'$

$$
\begin{aligned}
\Delta(A^2) &= \beta_{\mathrm{H}}(x'_\beta + \delta x'_\beta)^2 + 2\alpha_{\mathrm{H}}(x_\beta + \delta x_\beta)(x'_\beta + \delta x'_\beta) + \gamma_{\mathrm{H}}(x_\beta + \delta x_\beta)^2 \\
&\quad - [\beta_{\mathrm{H}} x'^2_\beta + 2\alpha_{\mathrm{H}} x_\beta x'_\beta + \gamma_{\mathrm{H}} x^2_\beta] \\
&= 2[\beta_{\mathrm{H}} x'_\beta\, \delta x'_\beta + \alpha_{\mathrm{H}}(x_\beta \delta x'_\beta + x'_\beta \delta x_\beta) + \gamma_{\mathrm{H}} x_\beta \delta x_\beta] && \text{(Line 1)} \\
&\quad + \beta_{\mathrm{H}}(\delta x'_\beta)^2 + 2\alpha_{\mathrm{H}} \delta x_\beta \delta x'_\beta + \gamma_{\mathrm{H}}(\delta x_\beta)^2 && \text{(Line 2)}
\end{aligned}
$$

- Line 1 is linear in u_γ and θ_x.
  - The linear θ_x terms average to zero.
  - The linear u_γ term just averages to give the previous damping contribution.
- Line 2 is quadratic in u_γ and θ_x and will give positive averages.

$$\Delta(A^2) = \left(\frac{u_\gamma}{U_s}\right)^2 [\beta_{\mathrm{H}}\eta'^2 + 2\alpha_{\mathrm{H}}\eta\eta' + \gamma_{\mathrm{H}}\eta^2] + \frac{1}{\gamma^2}\frac{\langle u_\gamma^2\rangle}{U_s^2}\beta_{\mathrm{H}}$$

- It's usual to define:

$$\mathcal{H}(s) = \beta_{\mathrm{H}}(s)\left(\eta'(s)\right)^2 + 2\alpha_{\mathrm{H}}(s)\eta(s)\eta'(s) + \gamma_{\mathrm{H}}(s)\left(\eta(s)\right)^2.$$

  - Some people mistakenly refer to $\mathcal{H}$ as "the dispersion invariant".
    - $\mathcal{H}(s)$ remains constant in straight sections, but
    - $\mathcal{H}(s)$ varies in bends.
  - I call it the "curly H-function".

- Excitation rate:

$$\left(\frac{d(A^2)}{dt}\right)_{\mathrm{QF}} \simeq \frac{1}{c\tau_s}\oint \frac{\langle u_\gamma^2\rangle}{U_s}\mathcal{H}(s)N_\gamma\, ds.$$

For equilibrium, sum of the damping and excitation rates must be zero:

$$-\left(\frac{d(A^2)}{dt}\right)_{\rm damping} = \frac{2}{\tau_x}\left\langle A^2\right\rangle \simeq \frac{1}{c\tau_s}\oint \frac{\langle u_\gamma^2\rangle}{U_s^2}\,\mathcal{H}(s)N_\gamma\,ds.$$

or

$$< A^2 >= \frac{\tau_x}{2\tau_s c}\oint \frac{\langle u_\gamma^2\rangle}{U_s^2}\,\mathcal{H}(s)N_\gamma\,ds.$$

$$\sigma_{x_\beta}^2 = \frac{1}{2}\beta < A^2 >= \beta_{\rm H}\epsilon_{\rm rms},$$

with

$$\epsilon_{\rm rms} = \frac{\tau_x}{4LU_s^2}\oint \langle u_\gamma^2\rangle\,\mathcal{H}(s)N_\gamma\,ds.$$

- In high energy $e^{\pm}$ rings, it is more usual to quote rms emittances which are unnormalized.

- For flat rings, vertical dispersion is zero, therefore the vertical emittance is typically much smaller than the horizontal.

$$\langle \Delta(A^2) \rangle \simeq \frac{1}{\gamma^2} \frac{u_\gamma^2}{U_s^2} \beta_{\rm V}.$$

- In most rings, there is a little xy-coupling, so the vertical emittance is frequently dominated by the coupling.
- $\epsilon_{\rm V}/\epsilon_{\rm H}$ may be only a few percent or less in a well aligned light source, but in e^+e^- colliders, the ratio is frequently 10—20%.
- For low emittance lattices: minimize $\mathcal{H}$.
  - FODO lattices not optimized.
  - Chasman-Green style lattices with double bend or tripple-bend achromats are much better fro low emittance.

Emittance as temperature (see §12.1)

- In the rest system of the design particle (average CM of beam):

$$\beta_\perp^* = \frac{p_\perp^* c}{U^*} = \frac{p_\perp c}{U/\gamma_0} = \gamma_0 \beta_\perp.$$

Note: $U^* \simeq mc^2$ is practically at rest ($\beta^* \ll 1$).

- Recall from thermodynamics: $\frac{1}{2}k_B T$ for each degree of freedom.
  - $\langle p_\perp^2 \rangle = \langle p_x^2 + p_y^2 \rangle = p^2(\sigma_{x'}^2 + \sigma_{y'}^2)$
- The transverse temperature $T_\perp$ of the beam may be defined as

$$k_B T_\perp = \left\langle \frac{1}{2} m v_\perp^{*2} \right\rangle = \left\langle \frac{1}{2} mc^2 \beta_\perp^{*2} \right\rangle = \left\langle \frac{1}{2} mc^2 \gamma_0^2 \beta_\perp^2 \right\rangle$$

$$
\begin{aligned}
\beta_\perp &\simeq \beta_0 \frac{p_\perp}{p} \\
k_B T_\perp &= \left\langle \frac{1}{2} mc^2 (\beta_0\gamma_0)^2 \frac{p_\perp^2}{p^2} \right\rangle \\
\langle p_\perp^2 \rangle &= p^2 \left(\frac{\epsilon_{\mathrm{H,rms}}}{\beta_{\mathrm{H}}} + \frac{\epsilon_{\mathrm{V,rms}}}{\beta_{\mathrm{V}}} \right) \\
k_B T_\perp &= \frac{1}{2} mc^2 (\beta_0\gamma_0)^2 \left(\frac{\epsilon_{\mathrm{H,rms}}}{\beta_{\mathrm{H}}} + \frac{\epsilon_{\mathrm{V,rms}}}{\beta_{\mathrm{V}}} \right), \quad \text{unnormalized emittance} \\
&= \frac{1}{2} mc^2 (\beta_0\gamma_0) \left(\frac{{\epsilon_{\mathrm{H,rms}}}^*}{\beta_{\mathrm{H}}} + \frac{{\epsilon_{\mathrm{V,rms}}}^*}{\beta_{\mathrm{V}}} \right), \quad \text{normalized emittance}
\end{aligned}
$$

- So $\dfrac{k_B T_\perp}{\beta_0\gamma_0}$ is an invariant.

- Longitudinal temperature: $\frac{1}{2}k_B T_\perp = \left\langle \frac{1}{2}mc^2 \beta_\parallel^{*2} \right\rangle$
- Boosting $\beta_\parallel^*$ from the lab system: $\beta_\parallel^* = \frac{\beta_\parallel - \beta_0}{1-\beta_\parallel\beta_0}$
- Define

$$\Delta\beta = \beta_\parallel - \beta_0$$

$$\Delta\gamma = \gamma - \gamma_0$$

- Recall

$$\gamma = (1-\beta^2)^{-1/2}$$

$$\Delta\gamma \simeq \beta(1-\beta^2)^{-3/2}\Delta\beta \simeq \beta_0\gamma_0^3\Delta\beta$$

$$\beta_\parallel^* = \frac{\Delta\beta}{1-(\beta_0+\Delta\beta)\beta_0} \simeq \frac{\Delta\beta}{1-(\beta_0+\Delta\beta)\beta_0} \simeq \frac{\Delta\beta}{\gamma_0^{-2} - \gamma_0^{-3}\Delta\gamma}$$

$$\simeq \gamma_0^2\Delta\beta, \qquad \text{assuming } \frac{\Delta\gamma}{\gamma_0} \simeq \frac{\Delta p}{p} \ll 1.$$

$$\simeq \frac{\Delta\gamma}{\beta_0\gamma_0},$$

$$\frac{1}{2}\,k_B T_\perp \simeq \frac{1}{2}\,mc^2 \left\langle \left(\frac{\Delta\gamma}{\beta_0\gamma_0}\right)^2 \right\rangle \simeq \frac{1}{2}\,mc^2\beta_0^2 \left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle,$$
$$\simeq \frac{1}{2}\,mc^2\beta_0^2 \left(\frac{\sigma_p}{p}\right)^2$$

since

$$\frac{dU}{U} = \beta^2\,\frac{dp}{p}.$$

One is tempted to write for the overall temperature:

$$\frac{3}{2}k_B T = \frac{mc^2}{2}(\beta_0\gamma_0)^2 \left[\frac{\epsilon_{\mathrm{H,rms}}}{\beta_{\mathrm{H}}} + \frac{\epsilon_{\mathrm{V,rms}}}{\beta_{\mathrm{V}}} + \frac{1}{\gamma_0^2}\,\left(\frac{\sigma_p}{p}\right)^2\right].$$