An accelerator lattice with non-linear transverse motion integrable in polar coordinates

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2. Concept of 2D paraxial nonlinear integrable optics

Desired spread of frequencies can be achieved by adding to the Hamiltonian an additional nonlinear potential:

$$\mathcal{K}[p_x, p_z, x, z; s] = \underbrace{\sum_{q=x,z} \left[\frac{p_q^2}{2} + g_q(s)\frac{q^2}{2} \right]}_{\mathcal{K}_0[p_x, p_z, x, z; s]} + \boxed{V(x, z, s)}.$$

In general, the new equations of motion do not necessarily provide two (and even a one) analytic invariants as it was in the case of linear lattice. Below, we will consider one of the possibile ways of how to modify a paraxial Hamiltonian preserving the integrability at the same time **[Danilov, Nagaitsev]**.

2.1 First integral of motion

Consider an accelerator lattice which provides axially symmetric linear focusing, i.e. $\forall\, s : g_x(s) = g_z(s)$

Step 1: Change of independent variable

The betatron phase advance $\psi(s)$ can be choosen as a new independent variable. The new Hamiltonian $\tilde{\mathcal{K}}$ would yield the same physical motion as $\mathcal{K}$ if their gradients are proportional:

$$\frac{\partial \tilde{\mathcal{K}}}{\partial p_q} = \lambda^{-1}(s)\frac{\partial \mathcal{K}}{\partial p_q}, \qquad \frac{\partial \tilde{\mathcal{K}}}{\partial q} = \lambda^{-1}(s)\frac{\partial \mathcal{K}}{\partial q}, \qquad \text{which gives}$$

$$\tilde{\mathcal{K}}[p_x, p_z, x, z; \psi] = \beta\,[s(\psi)]\;\mathcal{K}[p_x, p_z, x, z; s(\psi)].$$

Step 2: Transformation to normalized coordinates

Subsequent canonical transformation, $(p, q) \rightarrow (\mathcal{P}_q, \eta_q)$, moves the time dependence into the nonlinear term.

$$\eta_q = q/\sqrt{\beta_q},$$

$$\mathcal{P}_q = p_q\sqrt{\beta_q} - q\,\frac{\beta_q'}{2\,\beta_q^{3/2}},$$

where $' \stackrel{\text{def}}{=} d/d\psi$.

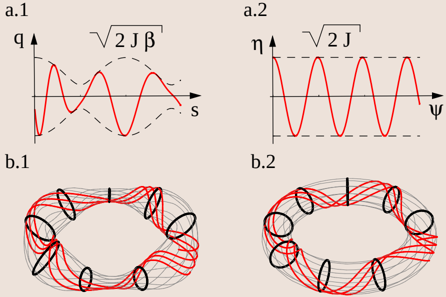


Figure: Particle trajectory (a.1,2) and the evolution of the phase-space ellipse along the accelerator circumference (b.1,2) in old and new canonical variables respectively.

Step 3: Special “time”-dependence

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + \beta\left[s(\psi)\right]\; V(\mathbf{q}(\eta, \psi), \psi)$$

One can see that at least one integral of motion, the Hamiltonian by itself, can be ensured by a special “time”-dependence of the nonlinear potential which compensate a modulation by the β-function:

$$\boxed{\beta\left[s(\psi)\right]\; V(\mathbf{q}(\eta, \psi), \psi) = U(\eta_x, \eta_z).}$$

2.2 Second integral of motion

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + U(\eta_x, \eta_y)$$

A presence of a second integral can be guaranteed by the choice of new generalized coordinates where variables can be separated.

Harmonic condition

Additional constraint on a potential $U(\eta_x, \eta_z)$ to satisfies the Laplas equation essentially reduce the number of possible choises among the whole possible functions: only three different families of such a integrable lattices were found for the invariant in the form

$$W = A(x,z)\mathcal{P}_x^2 + B(x,z)\mathcal{P}_x\mathcal{P}_z + C(x,z)\mathcal{P}_z^2 + D(x,z).$$

3. Variables separation in polar coordinates

In the normalized polar coordinates (r, θ):

$$\eta_x = r\cos\theta, \qquad \mathcal{P}_x = p_r\cos\theta - \frac{p_\theta}{r}\sin\theta,$$
$$\eta_z = r\sin\theta, \qquad \mathcal{P}_z = p_r\sin\theta + \frac{p_\theta}{r}\cos\theta,$$

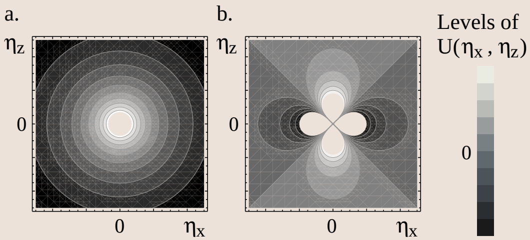
the variables separation is possible for the potentials in the form:

$$U(r,\theta) = f(r) + \frac{h(\theta)}{r^2}$$

where $f(r)$ and $h(\theta)$ are arbitrary functions.

Harmonic potentials in polar normalized coordinates

- $B \ln r$ — potential of the straight wire carries constant current **(special "time"-dependence can not be ensured)**
- $\dfrac{A\sin(2\theta+\varphi)}{r^2}$ — point-like magnetic quadrupole **(initialy s-independent potential remains so after the transformation to the normalized coordinates and time)**



4. Transverse motion model

Finally we have a Hamiltonian

$$\mathcal{H}[p_r, p_\theta, r, \theta; \psi] = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{r^2}{2} + \frac{A\sin(2\theta+\varphi)}{r^2},$$

with two invariants of motion:

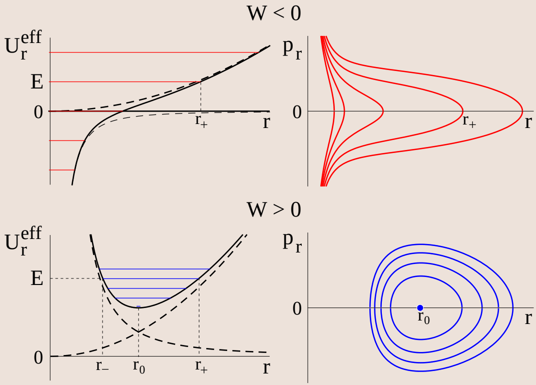
- energy

$$\boxed{E = \frac{p_r^2 + r^2}{2} + \frac{W}{r^2}}$$

- effective angular momentum

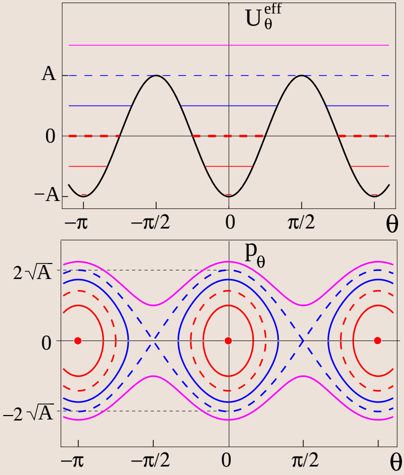
$$\boxed{W = \frac{p_\theta^2}{2} + A\sin(2\theta+\varphi)}$$

Radial motion



$$J_r(E) = \frac{1}{2\pi}\oint p_r\,dr = \frac{E-\sqrt{2W}}{2} \qquad \omega_r = \frac{\partial\mathcal{H}}{\partial J_r} = 2$$

Angular motion



Falling to the center:

$W = -A$

$-A < W < 0$

$W = 0$

Libration:

$0 < W < A$

Separatrix:

$W = A$

Rotation around singularity:

$W > A$

Trajectories classification

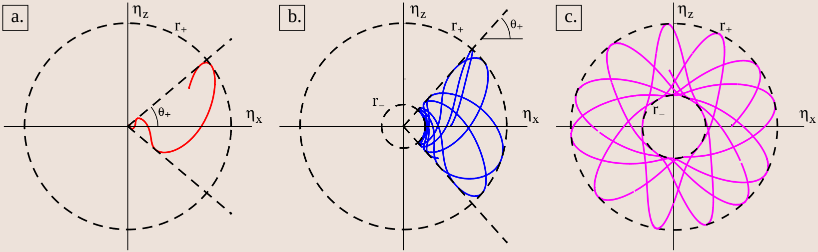
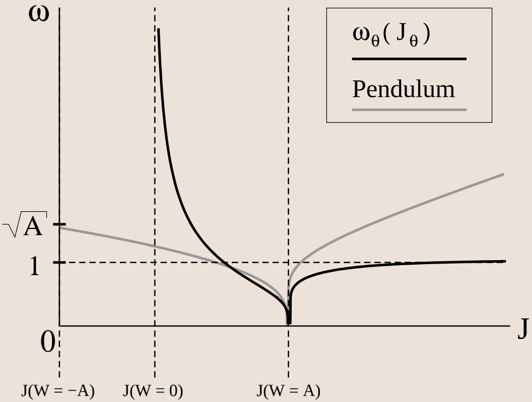


Figure: Particle trajectory in the normalized coordinates for
(a.) falling to the center ($-A < W < 0$)
(b.) libration ($0 < W < A$)
(c.) rotation around the origin ($W > A$).

Frequency dependence of the amplitude for the angular motion

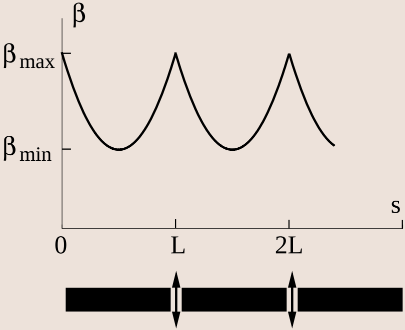


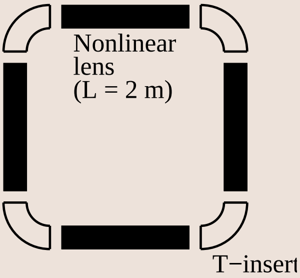
$$\omega_\theta = \frac{1}{\sqrt{2W}} \left(\frac{\partial J_\theta}{\partial W} \right)^{-1} = \frac{1}{\sqrt{2W}} \times \begin{cases} 2\ \omega_{\text{pend}} \\ \omega_{\text{pend}} \end{cases} ,$$

5. Monochromatic beam simulation

Numerical example for $E_{\mathrm{KIN}} = 1.9$ MeV proton beam

Linear lattice consists of 4 superperiods





$\beta_{\mathrm{min}} \approx 100$ cm, $\beta_{\mathrm{max}} \approx 200$ cm, $\nu \in (0;2)$

$$V = -\frac{\mu_0 I}{\pi[B\rho]}\left(\left(\frac{b}{\rho}\right)^2 \cos 2\theta + \frac{1}{3}\left(\frac{b}{\rho}\right)^6 \cos 6\theta + \cdots\right), \ \rho = \sqrt{x^2+z^2}$$

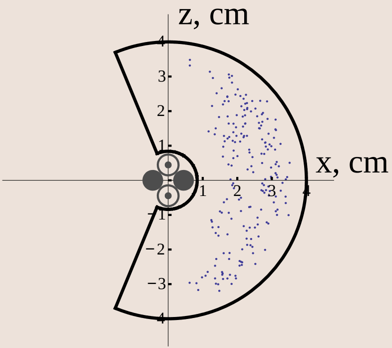
Nonlinear lens parameters

I — current through
2800 (500) A

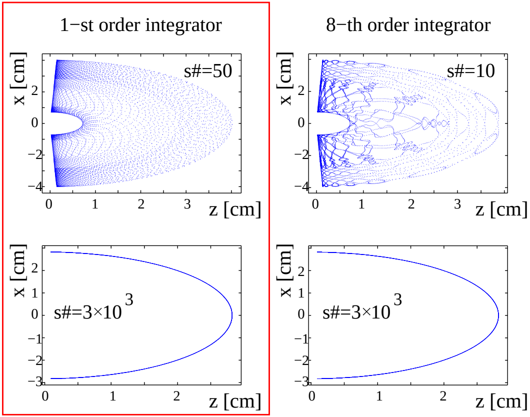
b — wire displacement
4.2 (5.6) mm

D — wire diameter
6 (8) mm

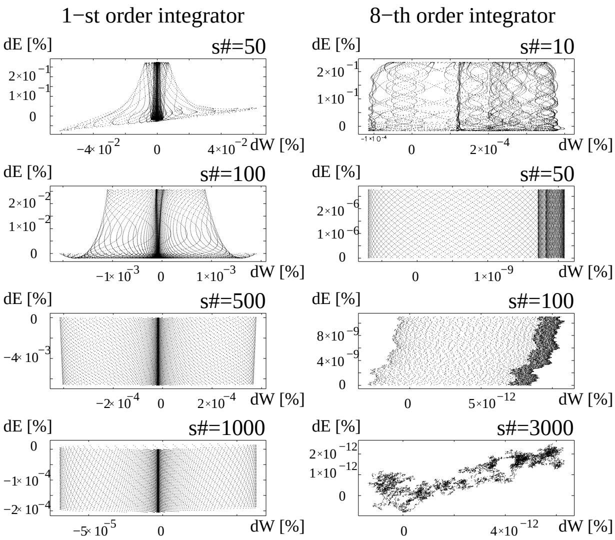
ρ_I — current density
100 (10) A/mm^2



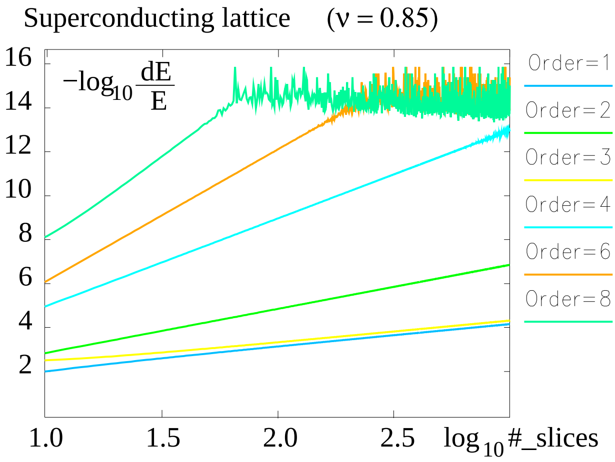
Q: Why do we see the trajectory smear for $\nu = 0.5$, when particle should have the same radial displacement for turn-by-turn map?
Q: Does this problem disappear with the increase of the number of slices?



Answer: the proper symplectic integrator should be used

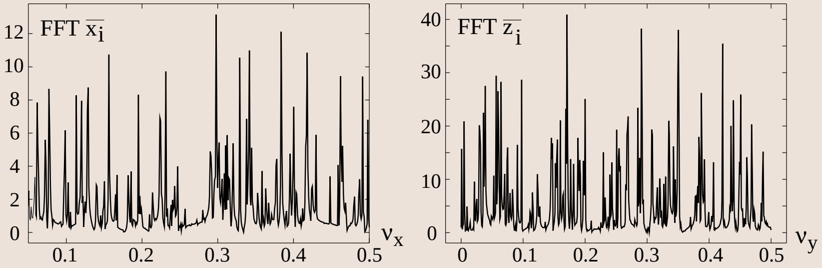


Answer: the proper symplectic integrator should be used

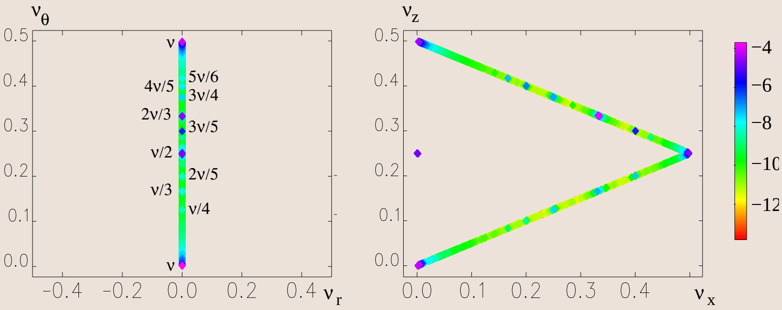


Q: What is the frequency spread in cartesian coordinates?

Answer: Schottky noise analysis



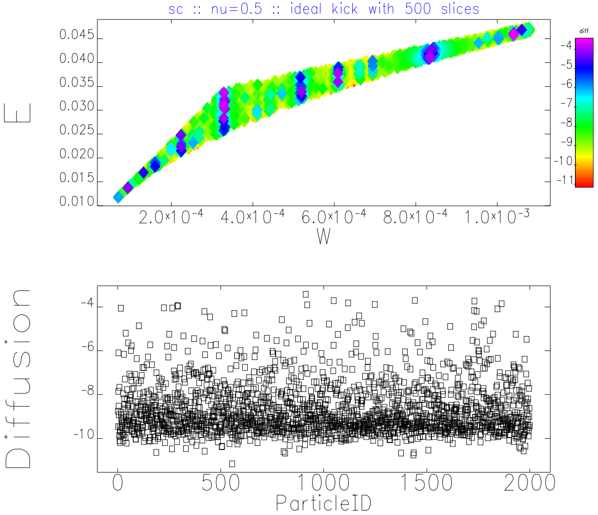
FMA - Frequency Map Analysis



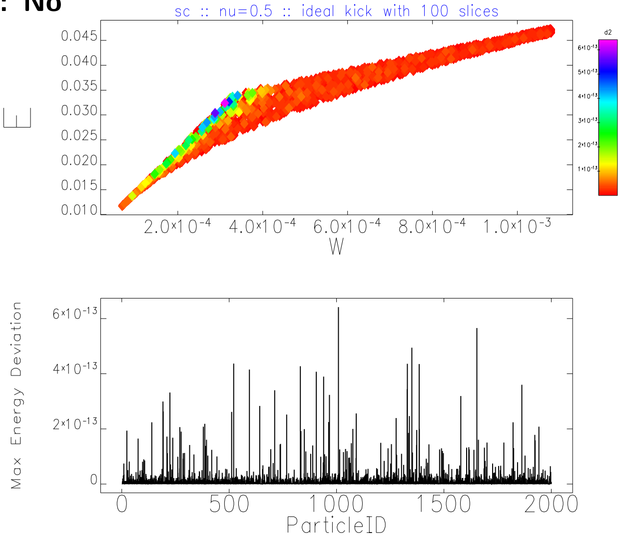
The FMA shows

- the beam footprint in the space of ferquencies (amplitudes);
- diffusion of the frequency: $\log \sqrt{\Delta\nu_1^2 + \Delta\nu_2^2}$.

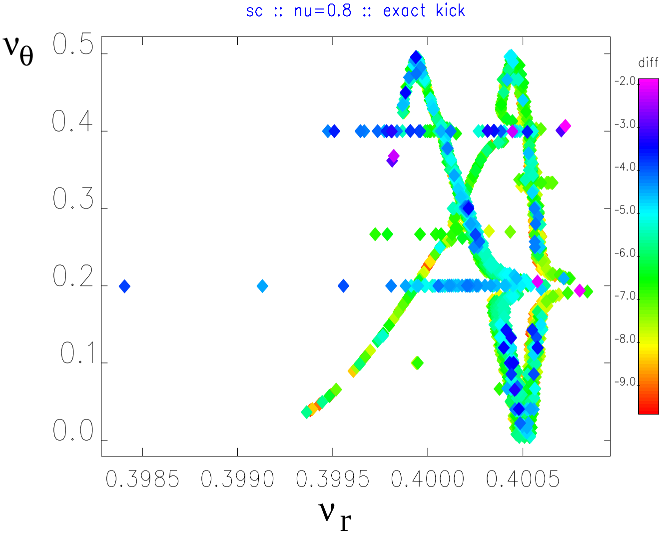
Q: Whether these are real resonances?



Answer: No


sc :: nu=0.5 :: ideal kick with 100 slices
E
0.045
0.040
0.035
0.030
0.025
0.020
0.015
0.010
2.0x10-4
4.0x10-4
6.0x10-4
8.0x10-4
1.0x10-3
W
d2
Max Energy Deviation
6x10-13
4x10-13
2x10-13
0
500
1000
1500
2000
ParticleID

Example of perturbed system



Conclusions and further plans

Thank you for your attention.