USPAS Accelerator Physics 2013
Duke University

Chapter 10: More Nonlinear Dynamics
"Lots More Equations" with the occasional pretty picture

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10.4: Krylov-Bogoliubov(-Mitropolsky) Averaging

- Waldo had started with a simplified model of one-dimensional transverse dynamics earlier

$$\frac{d^2x}{d\theta^2} + Q_{\rm H}^2 x = f(\theta) \qquad\qquad (\theta \text{ is like time})$$

and evaluated it in terms of harmonics of the driving term

- He started to look at nonlinear forcing terms too
- An approach for phase-averaging nonlinear resonators **close to resonance** was developed by Krylov and Bogoliubov

Following C&M

$$\frac{d^2x}{d\theta^2} + (Q_{\rm res}^2 + \delta)x = \epsilon F(x, \theta)$$

$$\delta = Q_{\rm H}^2 - Q_{\rm res}^2 \approx 2Q_{\rm res}(Q_{\rm H} - Q_{\rm res})$$

δ, ϵ are perturbative terms

Krylov and Bogoliubv, "*Methodes approchees de la mecanique non-lineaire dans leurs application a l'Aeetude de la perturbation des mouvements periodiques de divers phenomenes de resonance s'y rapportant*. Kiev: Academie des Sciences d'Ukraine (1935)

Ansatz Solution

$$\frac{d^2x}{d\theta^2} + (Q_{\rm res}^2 + \delta)x = \epsilon F(x, \theta)$$

- For $\epsilon = 0$ the solution is a SHO: $x(\theta) = a \sin(Q_{\rm H}\theta + \phi)$
- For $\epsilon \ll 1$ we approximate the solution as quasiperiodic

$$x(\theta) \approx a(\theta) \sin \psi(\theta) \qquad \psi(\theta) \equiv Q_{\rm H}\theta + \phi(\theta)$$

- $a(\theta)$ and $\phi(\theta)$ vary slowly with respect to θ ($\psi(\theta)$ does not!)
- We want the derivative first derivative of x to be simple

$$\frac{dx}{d\theta} = \frac{da}{d\theta}\sin\psi + \frac{d\psi}{d\theta} a\cos\psi = \frac{da}{d\theta}\sin\psi + Q_{\rm res} a\cos\psi + \frac{d\phi}{d\theta} a\cos\psi$$

Unperturbed "fast" resonator

Small "slow" perturbations

Slow vs Fast

- Constrain the slow terms to cancel in the first derivative of x

$$\frac{dx}{d\theta} = \frac{da}{d\theta}\sin\psi + \frac{d\psi}{d\theta}a\cos\psi = \frac{da}{d\theta}\sin\psi + Q_{\rm res}a\cos\psi + \frac{d\phi}{d\theta}a\cos\psi$$

Unperturbed "fast" resonator

Small "slow" perturbations

$$\boxed{\frac{da}{d\theta}\sin\psi + \frac{d\phi}{d\theta}a\cos\psi = 0}$$

$$\frac{dx}{d\theta} = Q_{\rm res}a\cos\psi$$

- This gives one first-order differential equation for a, ϕ
- We get another from the full second-order equation of motion derived from this reduced first-order derivative

Back to Equation of Motion

$$\frac{dx}{d\theta} = Q_{\mathrm{res}} a \cos\psi$$

$$\frac{d^2x}{d\theta^2} = -Q_{\mathrm{res}}^2 a \sin\psi + Q_{\mathrm{res}} \frac{da}{d\theta}\cos\psi - Q_{\mathrm{res}} \frac{d\phi}{d\theta} a \sin\psi$$

$$\Rightarrow \quad \boxed{\frac{da}{d\theta}\cos\psi - \frac{d\phi}{d\theta} a \sin\psi = -\frac{a\delta}{Q_{\mathrm{res}}}\sin\psi + \frac{\epsilon}{Q_{\mathrm{res}}} F\left(a\sin\psi, \frac{\psi-\phi}{Q_{\mathrm{res}}}\right)}$$

- This is breaking down our second order differential equation into two first order differential equations
  - We can solve the two blue boxed equations for $da/d\theta$, $d\phi/d\theta$

$$\frac{da}{d\theta} = -\frac{\delta a}{Q_{\mathrm{res}}}\sin\psi\cos\psi + \frac{\epsilon}{Q_{\mathrm{res}}} F(\psi)\cos\psi \equiv f(\psi)$$

$$\frac{d\phi}{d\theta} = \frac{\delta}{Q_{\mathrm{res}}}\sin^2\psi - \frac{\epsilon}{Q_{\mathrm{res}} a} F(\psi)\sin\psi \equiv g(\psi)$$

And Now For The Trick

$$\frac{da}{d\theta} = -\frac{\delta a}{Q_{\rm res}} \sin\psi\cos\psi + \frac{\epsilon}{Q_{\rm res}} F(\psi)\cos\psi \equiv f(\psi)$$

$$\frac{d\phi}{d\theta} = \frac{\delta}{Q_{\rm res}} \sin^2\psi - \frac{\epsilon}{Q_{\rm res} a} F(\psi)\sin\psi \equiv g(\psi)$$

- We assumed $a(\theta)$ and $\phi(\theta)$ are slowly varying with θ or ψ
- We then approximate these as nearly periodic in ψ

$$f(\psi) \approx \sum_{n=-\infty}^{\infty} f_n\, \mathrm{e}^{in\psi} \qquad g(\psi) \approx \sum_{n=-\infty}^{\infty} g_n\, \mathrm{e}^{in\psi}$$

$$f_n \equiv \frac{1}{2\pi}\int_0^{\infty} f(\psi)\, \mathrm{e}^{-in\psi}\, d\psi \qquad g_n \equiv \frac{1}{2\pi}\int_0^{\infty} g(\psi)\, \mathrm{e}^{-in\psi}\, d\psi$$

Fourier coefficients

- Approximate the derivatives as their averages over one cycle in ψ

$$\frac{da}{d\theta} \approx \left\langle \frac{da}{d\theta} \right\rangle = f_0 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{da}{d\theta}\right) d\psi$$

$$\frac{d\phi}{d\theta} \approx \left\langle \frac{d\phi}{d\theta} \right\rangle = g_0 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{d\phi}{d\theta}\right) d\psi$$

The Trick's Reveal

$$\frac{da}{d\theta} = -\frac{\delta a}{Q_{\mathrm{res}}} \sin\psi \cos\psi + \frac{\epsilon}{Q_{\mathrm{res}}} F(\psi) \cos\psi \equiv f(\psi)$$

$$\frac{d\phi}{d\theta} = \frac{\delta}{Q_{\mathrm{res}}} \sin^2\psi - \frac{\epsilon}{Q_{\mathrm{res}} a} F(\psi) \sin\psi \equiv g(\psi)$$

$$\frac{da}{d\theta} \approx \left\langle \frac{da}{d\theta} \right\rangle = f_0 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{da}{d\theta} \right) d\psi$$

$$\frac{d\phi}{d\theta} \approx \left\langle \frac{d\phi}{d\theta} \right\rangle = g_0 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d\phi}{d\theta} \right) d\psi$$

$$\boxed{\begin{aligned} h_1 \equiv \frac{da}{d\theta} &= \frac{\epsilon}{2\pi Q_{\mathrm{res}}} \int_0^{2\pi} F(\psi)\ \cos\psi\, d\psi \\ h_2 \equiv \frac{d\phi}{d\theta} &= \frac{\delta}{2Q_{\mathrm{res}}} - \frac{\epsilon}{2\pi Q_{\mathrm{res}} a} \int_0^{2\pi} F(\psi)\ \sin\psi\, d\psi \end{aligned}}$$

$F\left(a\sin\psi, \frac{\psi - \phi}{Q_{\mathrm{res}}}\right)$ so this is still nontrivial!

10.5: Half-Integer Resonance

- Consider the half-integer resonance, $Q_{\mathrm{H}} = m/2 \ (m \text{ odd})$
- We had assumed

$$x(\theta) \approx a(\theta) \sin\psi(\theta) \qquad\qquad \psi(\theta) \equiv Q_{\mathrm{H}}\theta + \phi(\theta)$$

- So $$F(x,\theta) = x\cos(2\theta) = a\sin\psi\cos[2(\psi - \phi)]$$

- Our KB(M) integrals give first order differential equations

$$\frac{da}{d\theta} = \frac{\epsilon a}{4}\sin(2\phi)$$

$$\frac{d\phi}{d\theta} = \frac{\delta}{2} + \frac{\epsilon}{4}\cos(2\phi)$$

Awww, Our First Invariant – Isn't It Cute?

$$\frac{1}{d\theta} = \frac{\epsilon a}{4} \sin(2\phi) \frac{1}{da} \qquad \frac{d\phi}{d\theta} = \frac{\delta}{2} + \frac{\epsilon}{4} \cos(2\phi)$$

$$-\frac{d}{da}[\cos(2\phi)] = \frac{4\delta}{\epsilon a} + \frac{2}{a}\cos(2\phi) \qquad \Rightarrow \qquad \cos(2\phi) = \frac{A}{a^2} - \frac{2\delta}{\epsilon}$$

- Here A is a constant of integration that must be based on initial conditions
  - But we can also express it completely in terms of our dynamical variables and correspondingly, their initial conditions

$$\boxed{A = a^2 \cos(2\phi) + \frac{2\delta}{\epsilon} = a_0^2 \cos(2\phi_0) + \frac{2\delta}{\epsilon}}$$

- A is an example of a **dynamical invariant**
  - These are the **holy grails** of dynamical analysis
  - Also known as an integral of the motion

It Blow'd Up Real Good

- We'll skip down a bit in the text's derivation to get to another important result: finding the solution for oscillation amplitude a

$$a^2 = \frac{1}{4\alpha}\left[W_0 \boxed{\exp\left(\sqrt{\alpha}\frac{\varepsilon}{m}\theta\right)} - 2b + \frac{4A_0^2}{W_0}\exp\left(-\sqrt{\alpha}\frac{\varepsilon}{m}\theta\right)\right]$$

$$\boxed{\alpha = 1 - \left(\frac{2\delta}{\epsilon}\right)^2}, \quad b = \frac{4\delta}{\varepsilon}A_0, \quad \text{and} \quad c = -A_0^2$$

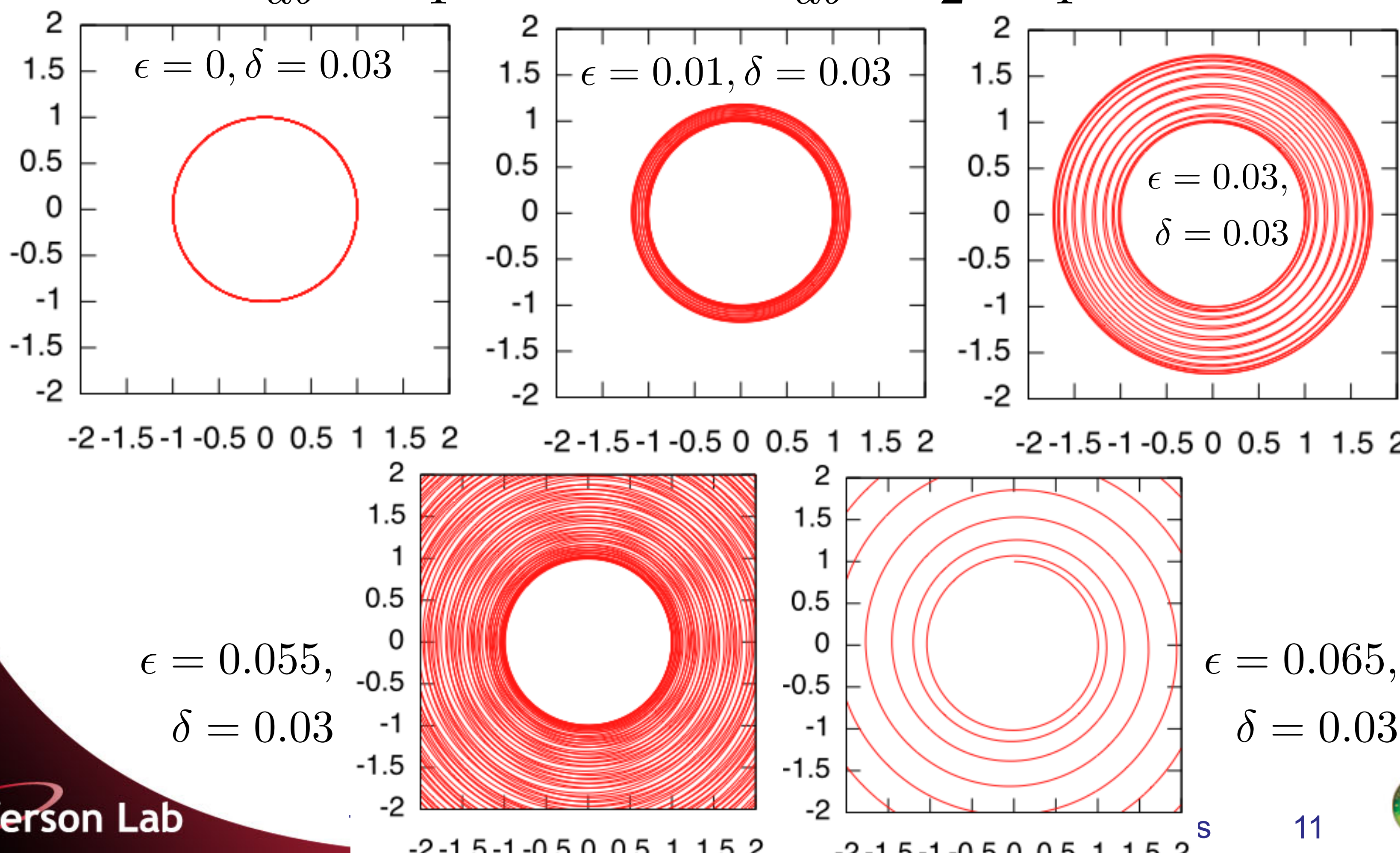
$$W_0 = 2\sqrt{\alpha}\sqrt{\alpha a_0^4 + b a_0^2 + c} + 2\alpha a_0^2 + b$$

- The amplitude grows exponentially if $\alpha > 0$ or $\rightarrow |\delta| < \left|\frac{\epsilon}{2}\right|$
- Conversely, the amplitude oscillates stably if $\rightarrow |\delta| > \left|\frac{\epsilon}{2}\right|$
- This is known as a **resonance stop-band**
  - In an accelerator with this type of resonance, tunes cannot get closer than the resonance width from, e.g., Q=1/2

But You Promised Us Pictures

- It's not too hard to write a primitive tracking program to evaluate motion under this resonance

$$\frac{da}{d\theta} = \frac{\epsilon a}{4}\sin(2\phi) \qquad \frac{d\phi}{d\theta} = \frac{\delta}{2} + \frac{\epsilon}{4}\cos(2\phi)$$



10.6: Third Integer Resonance

- Consider the third-integer resonance, $Q_{\rm H} = m/3$
- We had assumed

$$x(\theta) \approx a(\theta) \sin\psi(\theta) \qquad\qquad \psi(\theta) \equiv Q_{\rm H}\theta + \phi(\theta)$$

- So

$$F(x,\theta) = x\cos(3\theta) = a\sin\psi\cos[3(\psi-\theta)]$$

- Our KB(M) integrals give first order differential equations

$$\frac{da}{d\theta} = -\frac{3}{8m}\epsilon a^2 \cos(3\phi)$$

$$\frac{d\phi}{d\theta} = \frac{3\delta}{2m} + \frac{3}{8m}\epsilon a \sin(3\phi)$$

Stroboscopic Representation

- How do we plot these coordinates?
- We can still plot “position”

$$x(\theta) = a(\theta) \sin\left[\frac{m}{3}\theta + \phi(\theta)\right]$$

- And we can plot that position vs the normalized “angle”

$$\hat{x}(\theta) = \frac{3}{m}\frac{dx}{d\theta} = a(\theta) \cos\left[\frac{m}{3}\theta + \phi(\theta)\right]$$

- Unperturbed motion in these coordinates is just simple harmonic oscillators, or circles.

Nontrivial Fixed Points

$$\frac{da}{d\theta} = -\frac{3}{8m}\epsilon a^2 \cos(3\phi) \qquad \frac{d\phi}{d\theta} = \frac{3\delta}{2m} + \frac{3}{8m}\epsilon a \sin(3\phi)$$

- These differential equations are a description of a dynamical system that has fixed points where both derivatives are equal to zero
  - One natural fixed point is a=0 (no surprise)
  - But we also find

$$\frac{da}{d\theta} = 0 \quad \Rightarrow \quad \cos(3\phi_{\mathrm{FP}}) = 0 \quad \Rightarrow \quad \sin(3\phi_{\mathrm{FP}}) = \pm 1$$

$$\frac{d\phi}{d\theta} = 0 \quad \Rightarrow \quad \delta \pm \frac{\epsilon a_{\mathrm{FP}}}{4} = 0 \quad \Rightarrow \quad a_{\mathrm{FP}} = \mp \frac{4\delta}{\epsilon}$$

  - This produces three fixed points that are locally stable (elliptical), and three fixed points that are locally unstable (hyperbolic)

Linearizing Around The Fixed Points

- Linearizing motion around the fixed points gives simple harmonic oscillator equations

$$a = a_{\mathrm{FP}} + \Delta a \quad \Rightarrow \quad \frac{d(\Delta a)}{d\theta} \approx \pm \frac{9\epsilon a_{\mathrm{FP}}^2}{8m} \Delta\phi$$

$$\phi = \phi_{\mathrm{FP}} + \Delta\phi \quad \Rightarrow \quad \frac{d(\Delta\phi)}{d\theta} \approx \pm \frac{3\epsilon}{8m} \Delta a$$

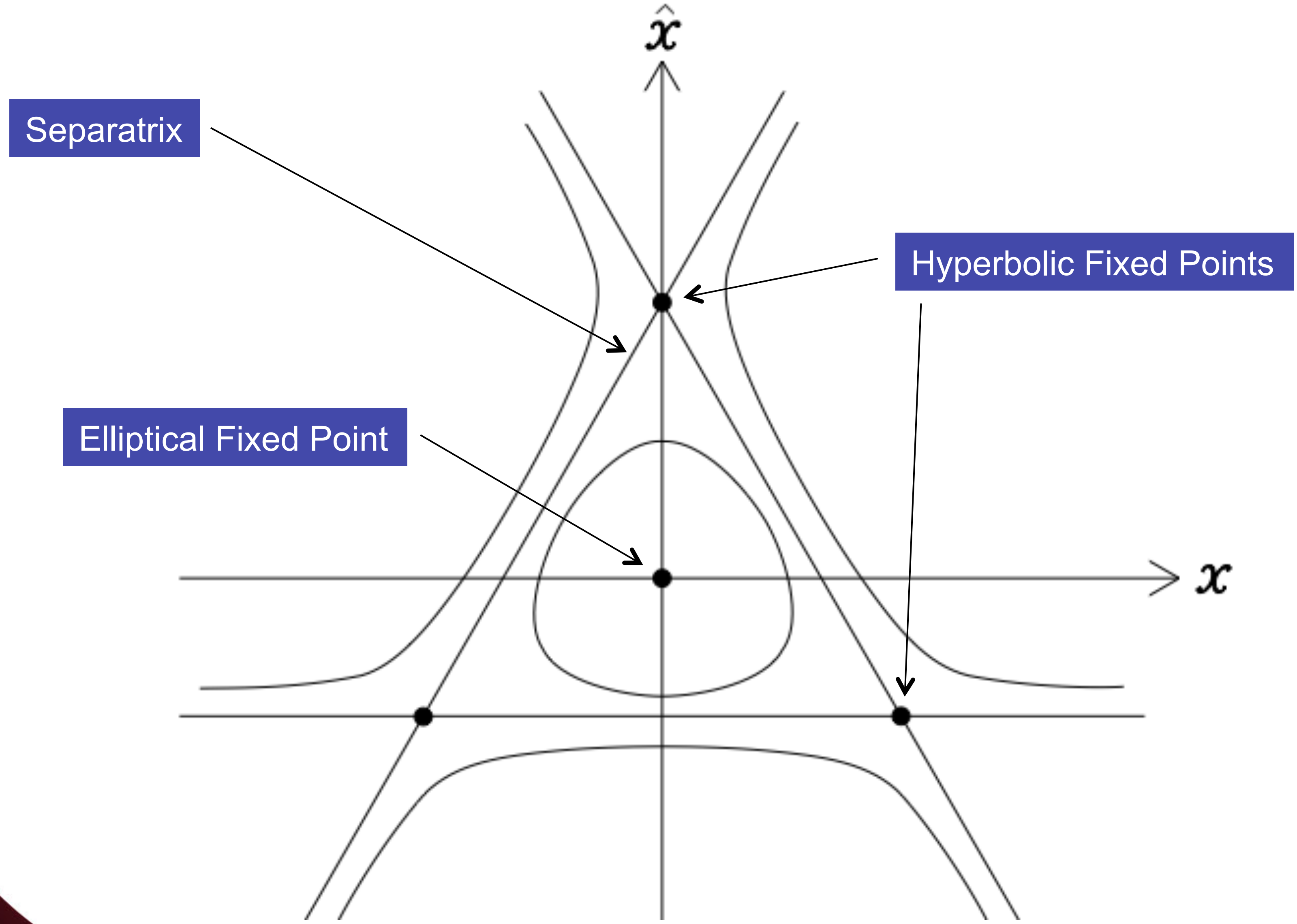
$$\frac{d^2(\Delta a)}{d\theta} - k^2 \Delta a = 0 \qquad \frac{d^2(\Delta\psi)}{d\theta} - k^2 \Delta\psi = 0$$

Simple Harmonic Oscillators

$$k^2 = \frac{27\epsilon^2 a_{\mathrm{FP}}^2}{64m^2} \qquad k = \frac{3\sqrt{3}\,\epsilon a_{\mathrm{FP}}}{8m}$$

$$a_{\mathrm{FP}} = \mp \frac{4\delta}{\epsilon}$$

Third Integer Phase Space



Congratulations: Another Invariant!

- Doing the invariant trick from the half-integer resonance gives us

$$\frac{1}{d\theta} = -\frac{3}{8m}\epsilon a^2 \cos(3\phi) da \qquad \frac{d\phi}{d\theta} = \frac{3\delta}{2m} + \frac{3}{8}\epsilon a \sin(3\phi)$$

$$\frac{d\phi}{da} = -\frac{a_{\mathrm{FP}} + a\sin(3\phi)}{a^2\cos(3\phi)} \qquad \sin(3\phi) = \frac{A}{a^3} - \frac{3a_{\mathrm{FP}}}{2a}$$

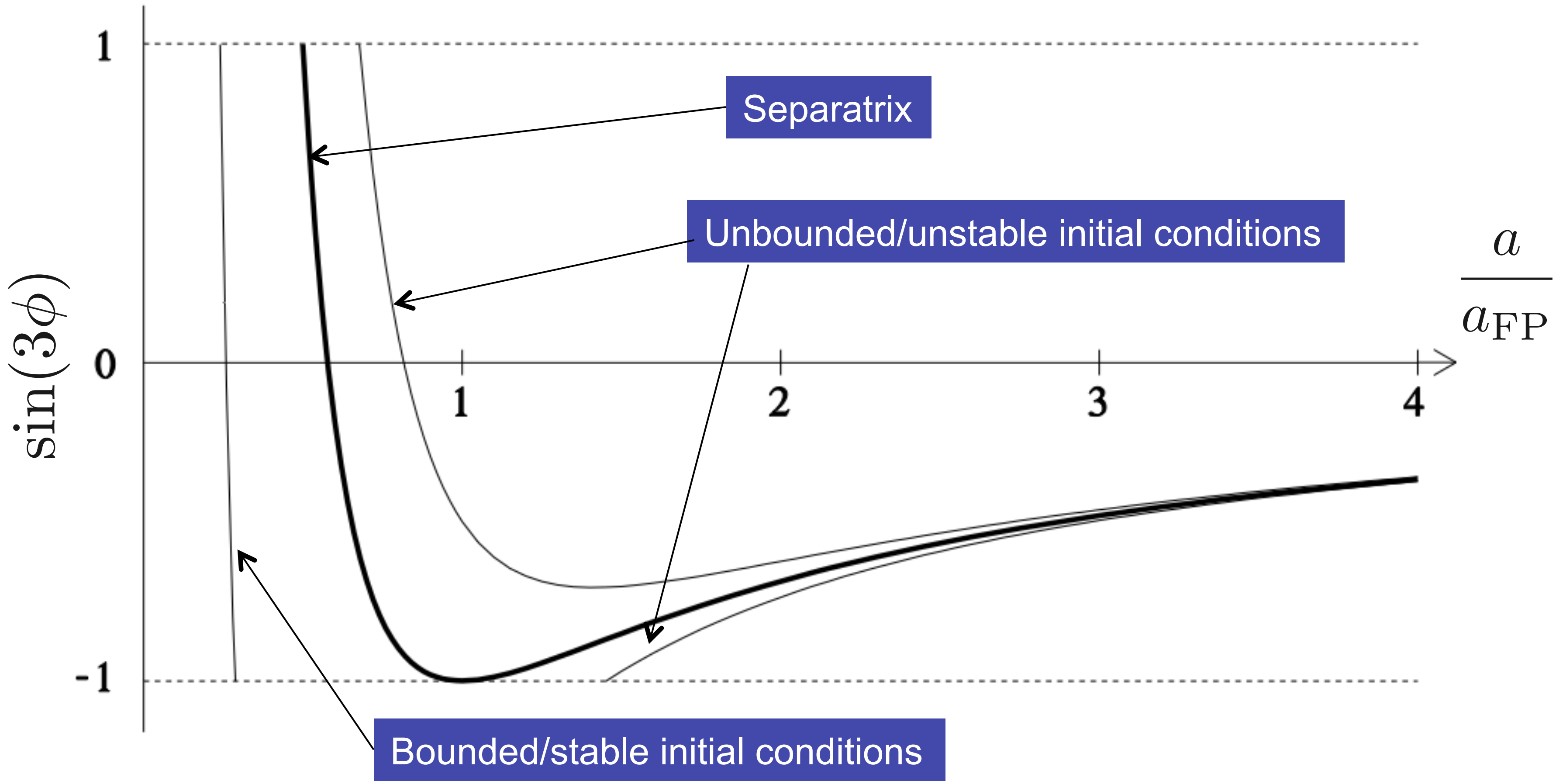
$$A = a^3\left(\sin(3\phi) + \frac{3a_{\mathrm{FP}}}{2a}\right) = a_0^3\left(\sin(3\phi_0) + \frac{3a_{\mathrm{FP}}}{2a_0}\right)$$

  - The lines for the separatrix are given by factoring the invariant evaluated at the nontrivial fixed points

$$\sin(3\phi_{\mathrm{FP}}) = 1 = \frac{1}{2}\left(\frac{a_{\mathrm{FP}}}{a}\right)^3 - \frac{3a_{\mathrm{FP}}}{2a}$$

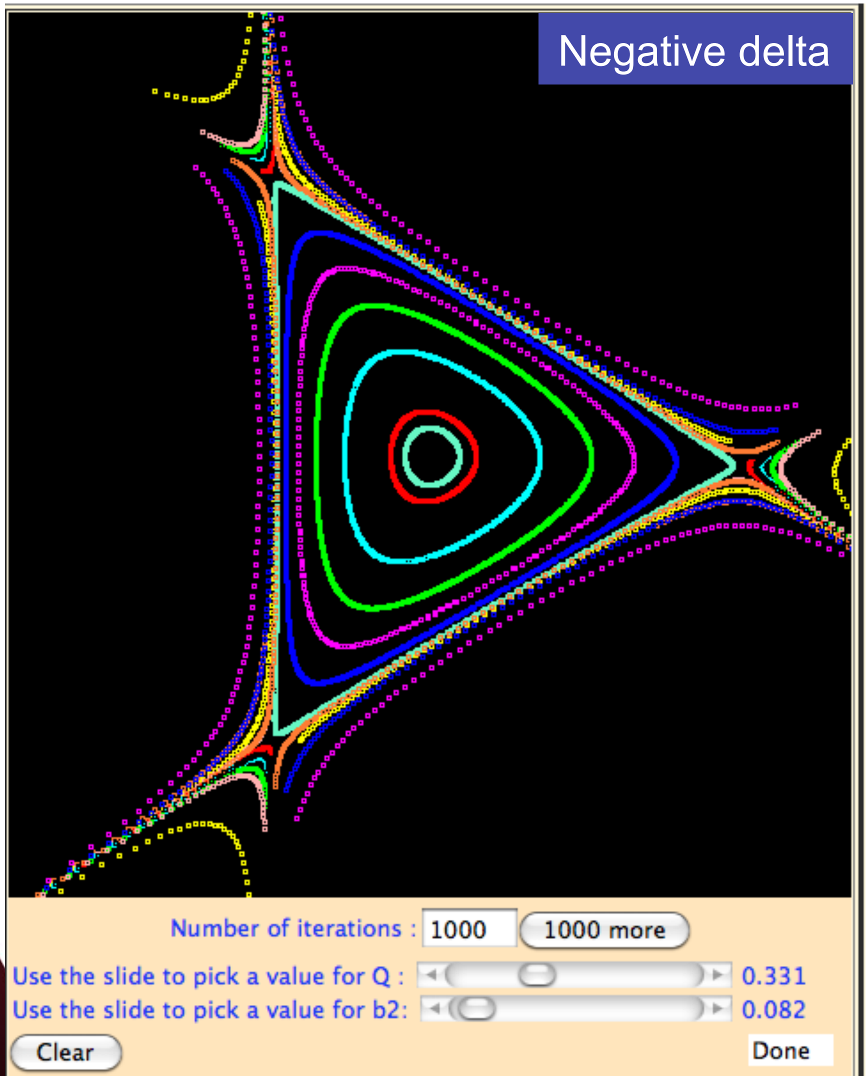
  - This invariant is really like a Hamiltonian (system energy)

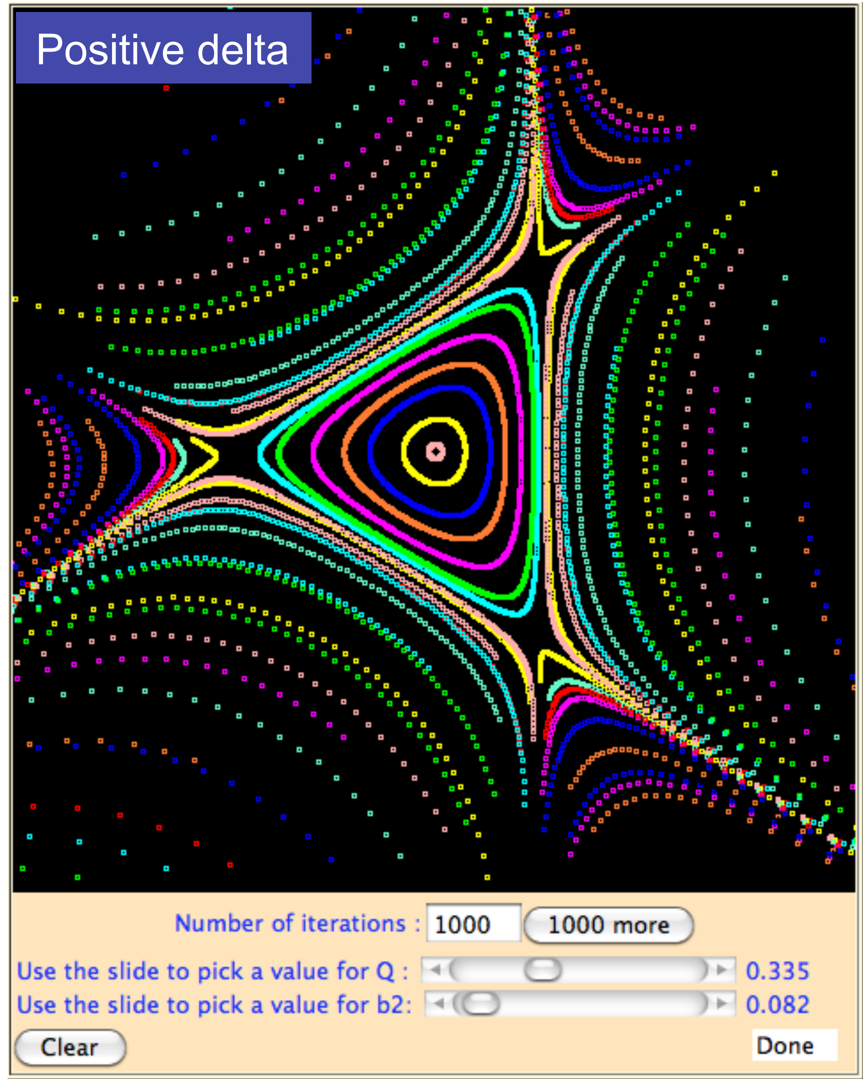
Invariant Parameter Space



Examples from an Online Lab

- Java program lab at http://www.toddsatogata.net/2013-USPAS/labNL.pdf





Fun With Nonlinear Dynamics (USPAS 2011)