USPAS Accelerator Physics 2015
Old Dominion University

Chapter 2: Coordinates and Weak Focusing

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Happy Birthday to James Watt, Henry Bessemer, Edgar Allen Poe, Paul Cezanne, and Janis Joplin!
Happy Martin Luther King Jr. Day!

Overview

- Coordinate and reference systems
- Equations of motion in an azimuthally symmetric field
  - Elucidation of approximations
  - Transverse separability
- Solutions of equations of motion
  - 2^{nd} order linear ODEs as matrix equations
- Momentum offsets and dispersion
- Weak focusing synchrotron
- Momentum compaction
  - Connections to orbital mechanics

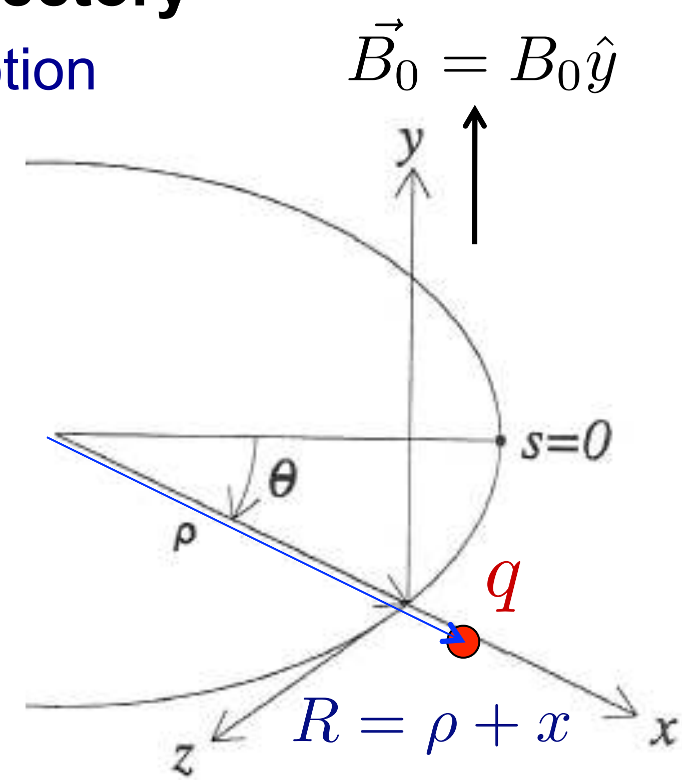
Parameterizing Particle Motion: Coordinates

- Now we derive more general equations of motion
- We need a **local coordinate system** $(\hat{x}, \hat{y}, \hat{z} \equiv \hat{s})$ **relative to the design particle trajectory**
  - s is the direction of design particle motion
  - y is the main magnetic field direction
  - x is the radial direction
  - ρ is not a coordinate, but the **design** bending radius in magnetic field B_0
- Can express total radius R as

$$\boxed{R = \rho + x} \qquad \theta = \frac{s}{R} = \frac{\beta c t}{R}$$

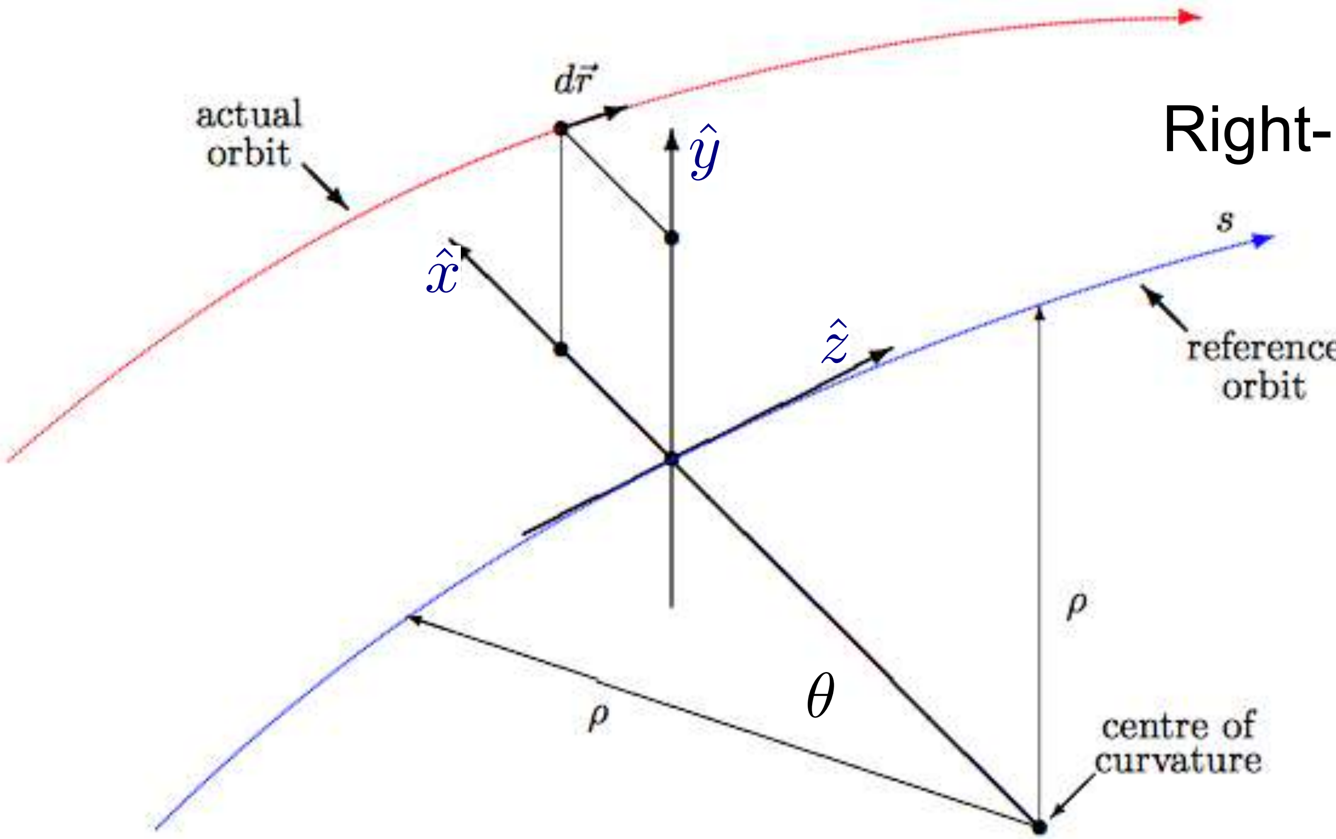
- Also define local trajectory angle

$$\boxed{x' \equiv \frac{dx}{ds} = \frac{1}{R}\frac{dx}{d\theta} \approx \frac{p_x}{p_0}}$$



Example: MAD/madx Coordinate System

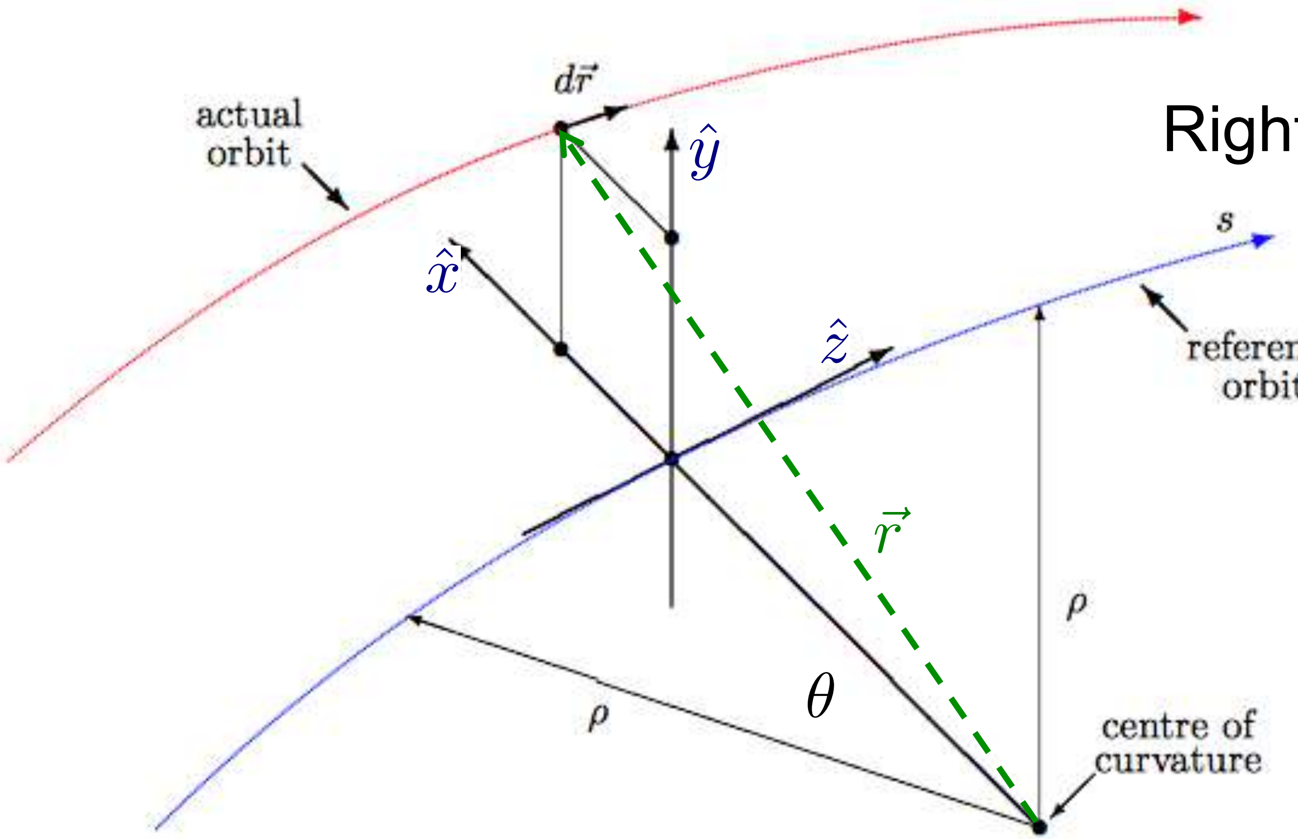
http://madx.web.cern.ch/madx/madX/doc/latexuguide/madxuguide.pdf p.10



Right-handed

Example: MAD/madx Coordinate System

http://madx.web.cern.ch/madx/madX/doc/latexuguide/madxuguide.pdf p.10

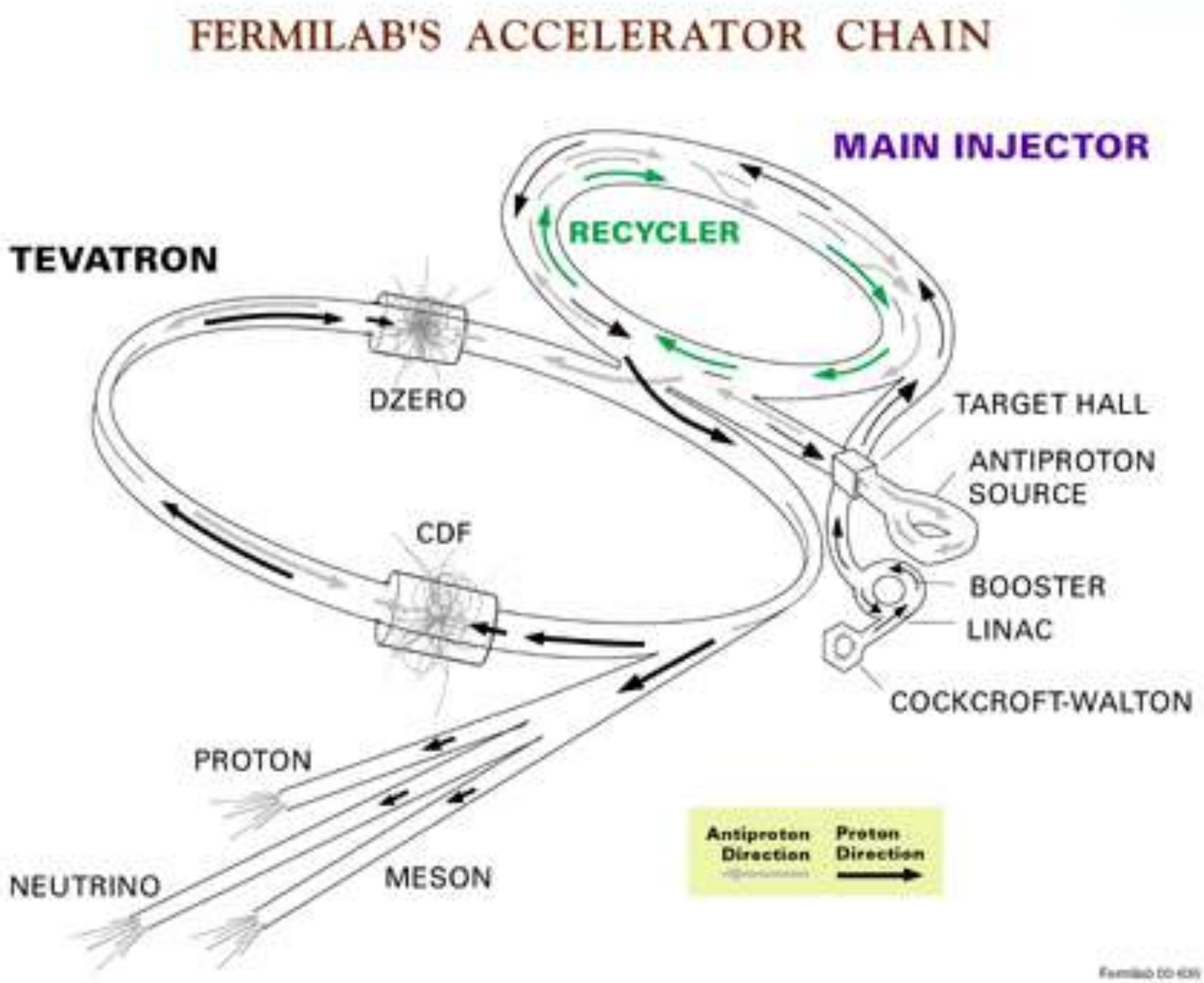


Right-handed

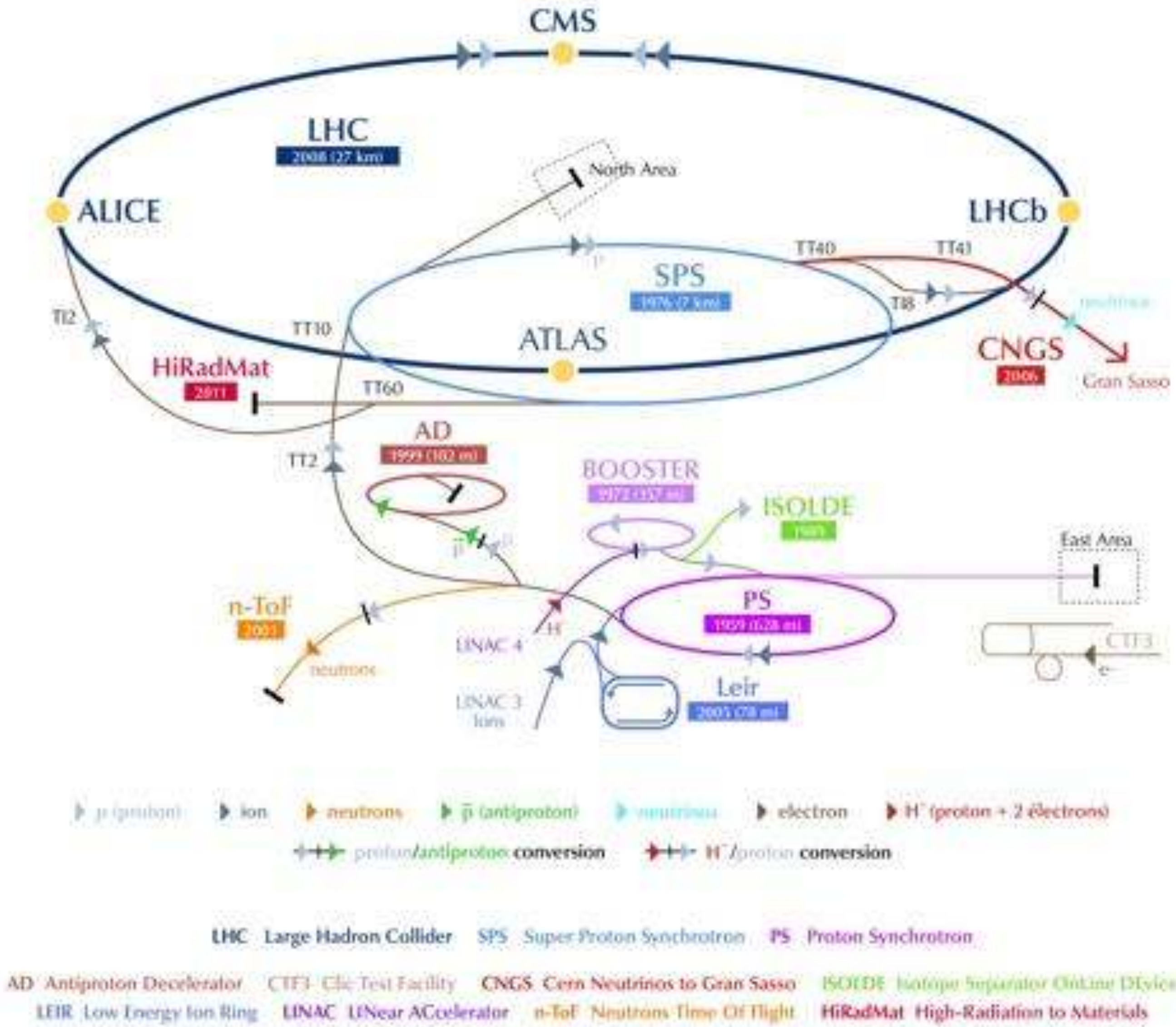
$$\vec{r} = (\rho + x)\hat{x} + y\hat{y} + z\hat{z}$$

$$R = \rho + x \qquad \frac{R}{\rho} = \left(1 + \frac{x}{\rho}\right)$$

Example: FNAL Coordinate Systems



Example: CERN Coordinate Systems



Parameterizing Particle Motion: Approximations

- We will make a few reasonable approximations:

(A0) 0) No local currents (beam travels in a near-vacuum)

(A1) 1) Paraxial approximation: $x', y' \ll 1 \quad \text{or} \quad p_x, p_y \ll p_0$

(A2) 2) Perturbative coordinates: $x, y \ll \rho$

(A3) 3) Transverse uncoupled linear magnetic field:

$$\vec{B} = B_0 \hat{y} + (x\hat{y} + y\hat{x}) \left(\frac{\partial B_y}{\partial x} \right)$$

(A0) Note this obeys Maxwell's equations in free space

(A4) 4) Negligible E field: $\gamma \approx \text{constant}$

- Equivalent to assuming adiabatically changing B fields relative to $dx/dt, dy/dt$

Parameterizing Particle Motion: Acceleration

- Lorentz force equation of motion is

$$q\vec{v}\times\vec{B} = \frac{d(\gamma m\vec{v})}{dt} = \gamma m\dot{\vec{v}} \qquad \text{(A4)}$$

- Calculate velocity and acceleration in our coordinate system

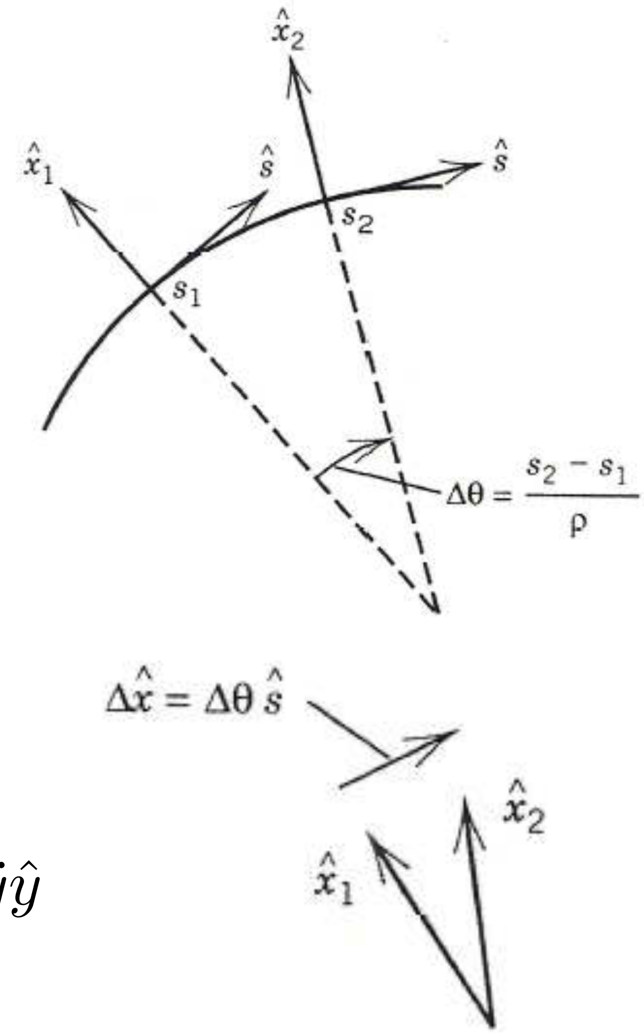
$$\vec{v} = \dot{R}\hat{x} + R\dot{\hat{x}} + \dot{y}\hat{y} = \dot{R}\hat{x} + R\dot{\theta}\hat{s} + \dot{y}\hat{y}$$

$$\dot{\vec{v}} = \ddot{R}\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + R\dot{\theta}\dot{\hat{s}} + \ddot{y}\hat{y}$$

$$\dot{\hat{s}} = -\dot{\theta}\hat{x} = -\frac{v}{R}\hat{x}$$

so $$\dot{\vec{v}} = (\ddot{R} - R\dot{\theta}^2)\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + \ddot{y}\hat{y}$$

$$\dot{\vec{v}} = \boxed{\left(\ddot{x} - \frac{v^2}{R}\right)\hat{x}} + \frac{2\dot{x}v}{R}\hat{s} + \boxed{\ddot{y}\hat{y}}$$



Parameterizing Particle Motion: Eqn of Motion

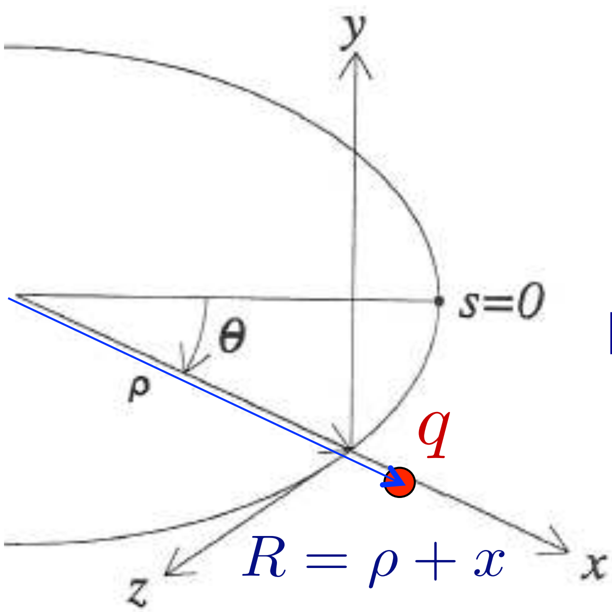
- Component equations of motion

Vertical:

$$F_y = q\beta c B_x = \gamma m \ddot{y} \qquad \ddot{y} - \frac{q\beta c B_x}{\gamma m} = 0$$

$$t = \frac{R}{\beta c}\theta \Rightarrow \frac{d}{dt} = \frac{\beta c}{R}\frac{d}{d\theta} \qquad \frac{d^2 y}{d\theta^2} - \frac{qB_x}{\beta c \gamma m}R^2 = 0$$

$$\boxed{\frac{d^2 y}{d\theta^2} - \frac{qB_x}{p}R^2 = 0}$$



Horizontal:

$$F_x = -q\beta c B_y = \gamma m \left(\ddot{x} - \frac{v^2}{R}\right)$$

$$\frac{d^2 x}{d\theta^2} = -\frac{qB_y R}{p} + R$$

$$\boxed{\frac{d^2 x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0}$$

Equations of Motion

- Apply our paraxial and linearization approximations

$$p = qB_0\rho \qquad R = \rho\left(1 + \frac{x}{\rho}\right) \quad \text{(A2)} \qquad B_y = B_0 + \left(\frac{\partial B_y}{\partial x}\right)x \quad \text{(A3)} \qquad B_x = \left(\frac{\partial B_y}{\partial x}\right)y \quad \text{(A3)}$$

Horizontal: $$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0$$

$$\frac{d^2x}{d\theta^2} + \left[\left(1 + \frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right)\left(1 + \frac{x}{\rho}\right) - 1\right]\rho\left(1 + \frac{x}{\rho}\right) = 0 \quad \text{(A2)}$$

$$\Rightarrow \boxed{\frac{d^2x}{d\theta^2} + (1-n)x = 0} \qquad \text{where} \qquad \boxed{n \equiv -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)}$$

Vertical: $$\frac{d^2y}{d\theta^2} - \frac{qB_x}{p}R^2 = 0 \qquad \Rightarrow \boxed{\frac{d^2y}{d\theta^2} + ny = 0}$$

Things to try sometime

- Expand the horizontal equation of motion to second order in x
  - Does it reduce to the stated equation at first order?
  - Use expansions for R, B that are still first order!

- Expand the horizontal and vertical equations of motion to second order in x, y, δ (p. 25-6 of these slides)
  - Use expansions for R, B that are still first order!

Simple Equations of Motion!

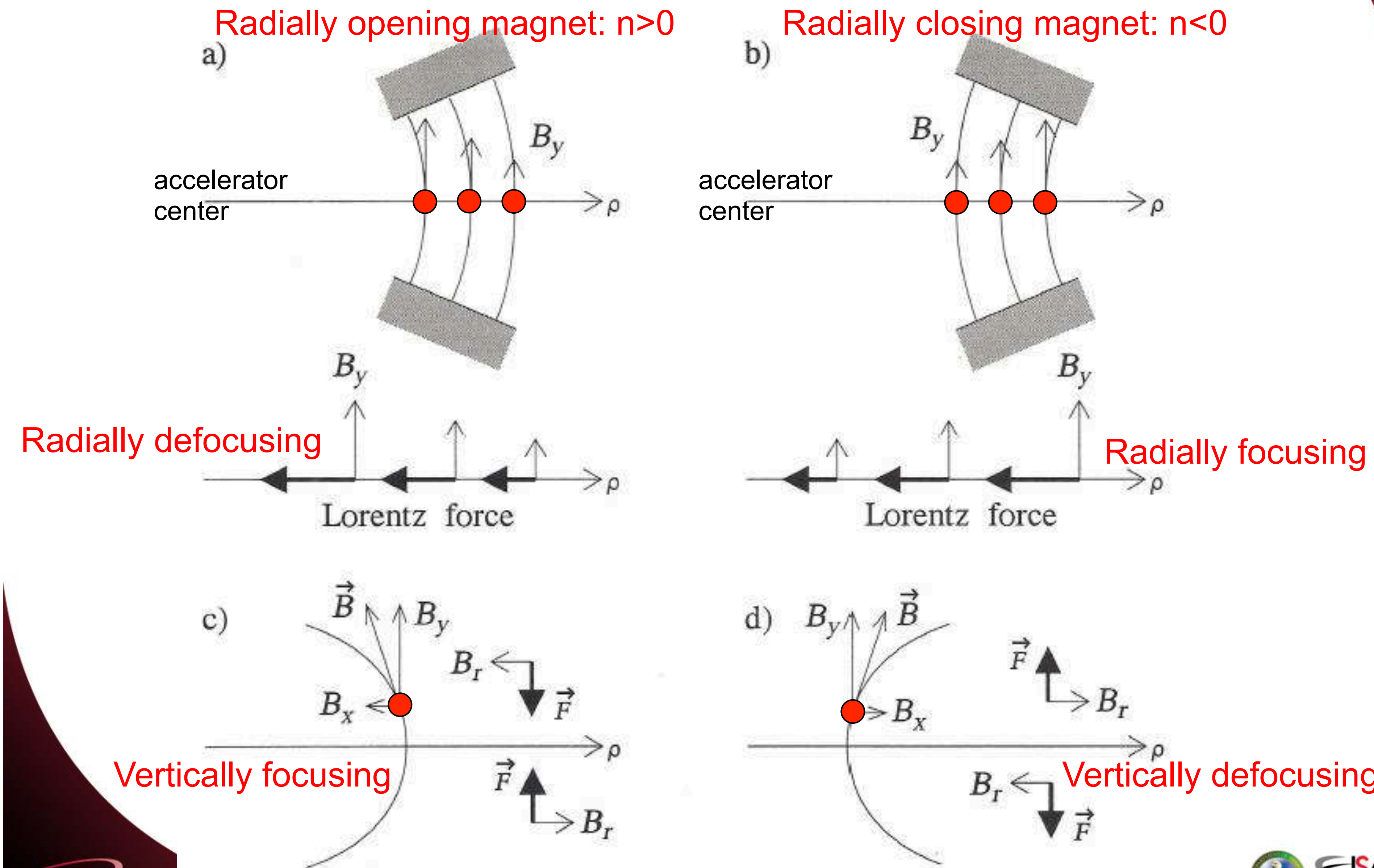
$$\frac{d^2x}{d\theta^2} + (1-n)x = 0 \qquad\qquad \frac{d^2y}{d\theta^2} + ny = 0$$

- These are simple harmonic oscillator equations
  (Not surprising since we linearized 2nd order differential equations)

- These are known as the **weak focusing** equations
  If n does not depend on θ, stability is only possible in both planes if **0<n<1**
  This is known as the **weak focusing criterion**

electron velocity INTO page

Visualization: Weak Focusing Forces



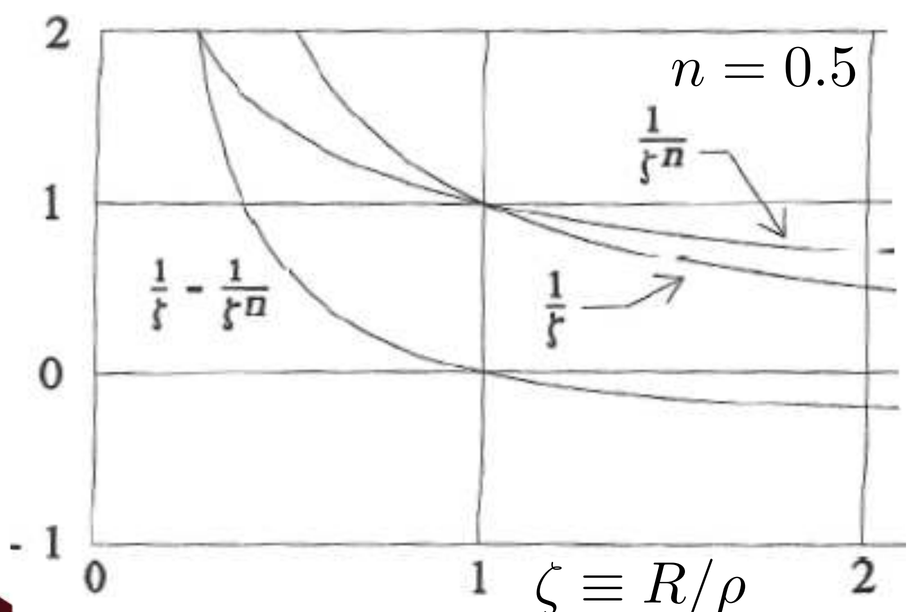
But Wasn't 0<n<1 Stable?

- This seems to indicate n>0 is horizontally unstable!
- Horizontal motion is a combination of two forces

  Centrifugal mv^2/R and centripetal Lorentz qvB_y

  Both forces cancel **by definition** for the design trajectory

$$F_{tot} = \frac{mv^2}{R} - qvB_y \approx \frac{mv^2}{\rho}\left(\frac{\rho}{R}\right) - qvB_0\left(\frac{\rho}{R}\right)^n = qvB_0\left(\frac{1}{\zeta} - \frac{1}{\zeta^n}\right)$$



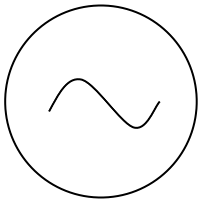
where $\zeta \equiv \frac{R}{\rho}$

$F_{tot} > 0$ for $\zeta < 0$

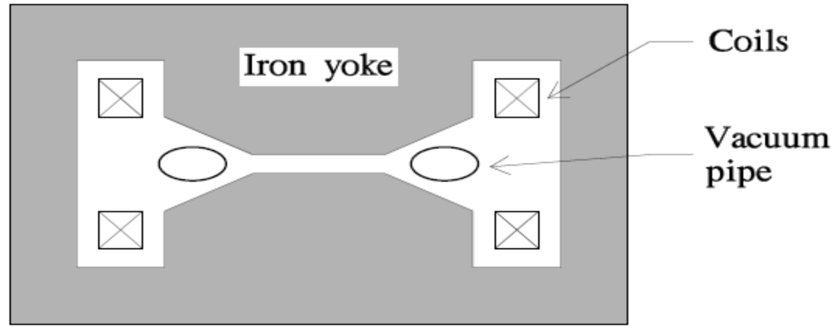
$F_{tot} < 0$ for $\zeta > 0$

Nonlinear too! But we linearize near ζ=1

Example: The Betatron



$I(t)=I_0 \cos(2\pi\omega_l t)$



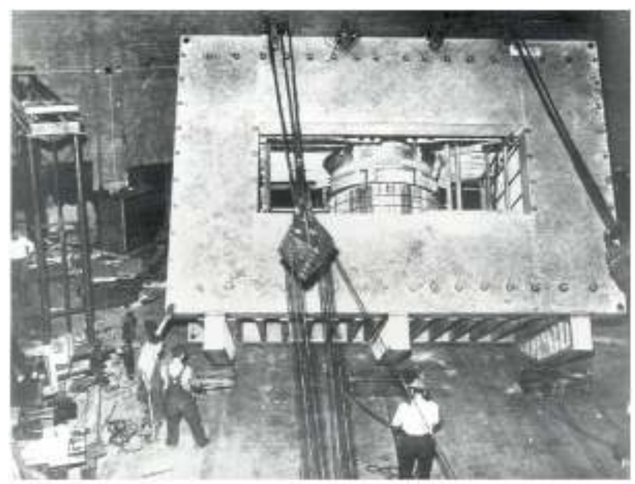
- Weak focusing formalism was originally developed for the Betatron
- Apply Faraday's law with time-varying current in coils
- Beam sees time-varying accelerating electric field too!
- Early proofs of stability: focusing and "betatron" motion

Donald Kerst

UIUC 2.5 MeV

Betatron, 1940

Don't try this at home!!



UIUC 312 MeV

betatron, 1949

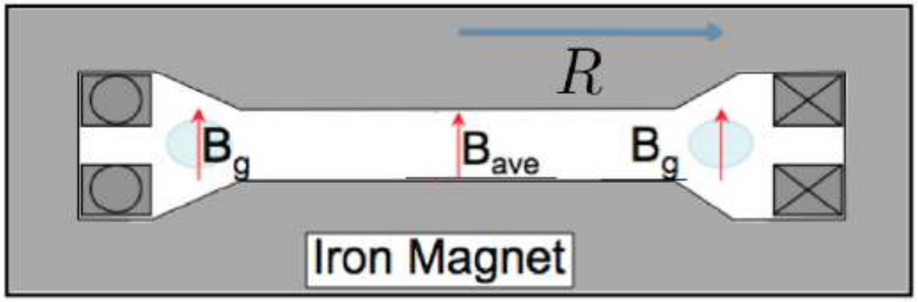
Really don't try this at home!!

Example: The Betatron

apply sinusoidally varying current to toroidal conductors

$$I = I_0 \sin(\omega t)$$

$$B_{\rm ave} = B_0 \sin(\omega t)$$



cylindrically symmetric about center vertical axis

$$\text{Magnetic flux} \quad \phi \approx \pi R^2 B_{\rm ave} = \pi R^2 B_0 \sin(\omega t)$$

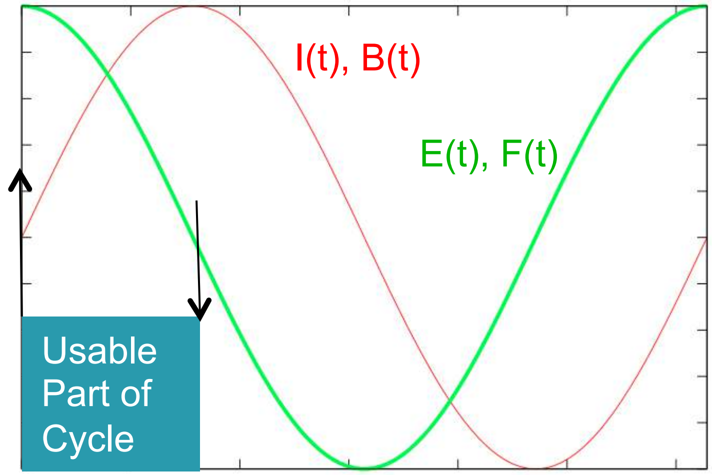
$$\text{Faraday's law} \quad \mathcal{E} = -\frac{d\phi}{dt} \qquad \oint \vec{E}\cdot d\vec{r} = 2\pi R E = -\frac{d\phi}{dt} = -\pi R^2 B_0 \omega \cos(\omega t)$$

$$\text{Force on electron} \quad F = qE = -\frac{qR}{2}\frac{dB_{\rm ave}}{dt} = \frac{eRB_0\omega}{2}\cos(\omega t)$$

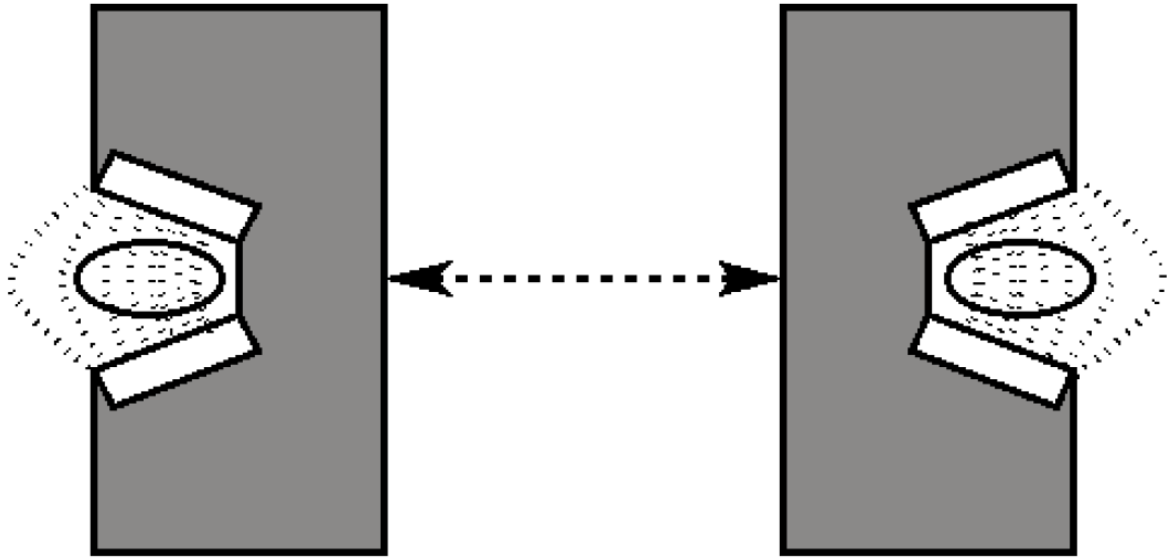
Circular motion

$$p/q = B_g R \quad \Rightarrow \quad F = \frac{dp}{dt} = eR\frac{dB_g}{dt}$$

$$\text{Betatron Field Condition} \quad B_g = \frac{B_0}{2}$$



Example: Synchrotron Weak Focusing



- Separate RF cavities can eliminate the need for central iron (and corresponding huge inductance)
  - Accelerator can be (much) larger (higher energies for same B_0!)
  - But stability equation scaling with ρ is still not good:

$$0 < -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)_{x=0} < 1$$

Back to Solutions of Equations of Motion

$$\frac{d^2x}{d\theta^2} + (1-n)x = 0 \qquad\qquad \frac{d^2y}{d\theta^2} + ny = 0$$

- Assume azimuthal symmetry (n does not depend on θ)
- Solutions are simple harmonic oscillator solutions

$$x(\theta) = A\cos(\theta\sqrt{1-n}) + B\sin(\theta\sqrt{1-n})$$

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})]$$

- Constants A,B are related to initial conditions (x_0, x_0')

$$x_0 = x(\theta=0) = A \qquad\qquad x_0' = \frac{1}{\rho}\left(\frac{dx}{d\theta}\right)(\theta=0) = \frac{\sqrt{1-n}}{\rho}B$$

$$A = x_0 \qquad\qquad B = \frac{\rho}{\sqrt{1-n}}x_0'$$

Solutions of Equations of Motion

- Write down solutions in terms of initial conditions

$$x(\theta) = \cos(\theta\sqrt{1-n})\, x_0 + \frac{\rho}{\sqrt{1-n}} \sin(\theta\sqrt{1-n})\, x_0'$$

$$x'(\theta) = \frac{1}{\rho}\frac{dx}{d\theta} = -\frac{\sqrt{1-n}}{\rho} \sin(\theta\sqrt{1-n})\, x_0 + \cos(\theta\sqrt{1-n})\, x_0'$$

- This can be (very) conveniently written as matrices (including both horizontal and vertical)

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}} \sin(\theta\sqrt{1-n}) \\ -\frac{\sqrt{1-n}}{\rho} \sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}} \sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho} \sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Transport Matrices

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \boxed{\begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix}} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \boxed{M_V(\theta)} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

- M_V here is an example of a **transport matrix**
  - Linear: derived from linear equations
    - Can be concatenated to make further transformations

$$M_V(\theta_1 + \theta_2) = M_V(\theta_1) M_V(\theta_2)$$

  - Depends only on "length" θ, radius ρ, and "field" n
    - Acts to transform or transport coordinates to a new state
    - Our accelerator "lattices" will be built out of these matrices
  - Unimodular: $\det(M_V)=1$
    - More strongly, it's symplectic: $S = M_V^T\, S\, M_V \quad \text{where} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
    - Hamiltonian dynamics, phase space conservation (Liouville)
    - These matrices here are scaled rotations!

Sinusoidal Solutions, Betatron Phases

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

- Sinusoidal simple harmonic oscillators solutions
  - Particles move in transverse **betatron oscillations** around the **design trajectory** $(x, x') = (y, y') = 0$
- We define **betatron phases**

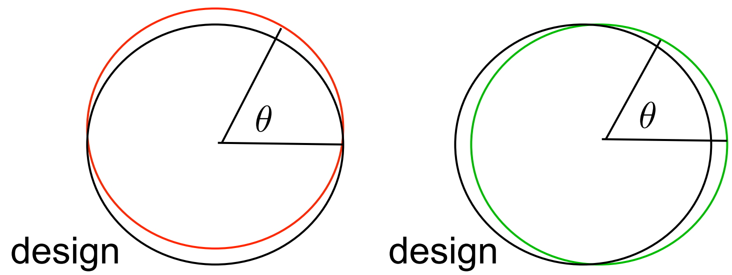
$$\phi_x(s) \equiv \theta\sqrt{1-n} = \frac{s}{\rho}\sqrt{1-n} \qquad \phi_y(s) \equiv \theta\sqrt{n} = \frac{s}{\rho}\sqrt{n}$$

Write matrix equation in terms of s rather than θ

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\phi_x(s) & \frac{\rho}{\sqrt{1-n}}\sin\phi_x(s) \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x(s) & \cos\phi_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

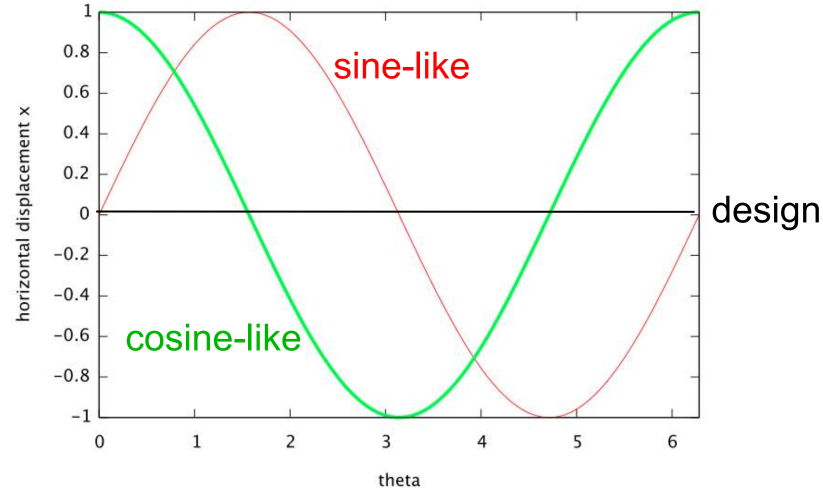
Visualization of Betatron Oscillations

- Simplest case: constant uniform vertical field (n=0)



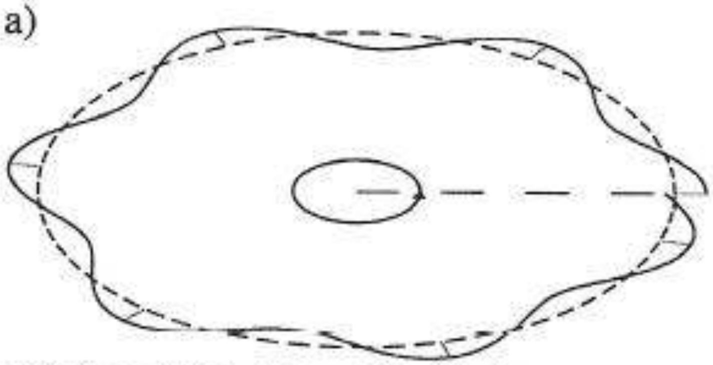
sine-like $x_0 = 0, x_0' \neq 0$

cosine-like $x_0 \neq 0, x_0' = 0$



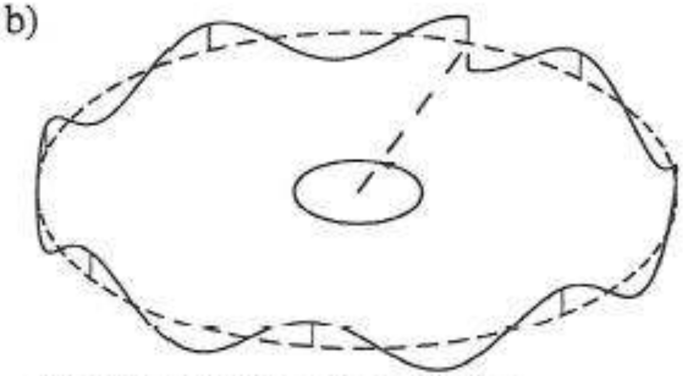
- More complicated **strong focusing**

a)



Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.

b)



Vertical Betatron Oscillation
with tune: $Q_V = 7.5$,
i.e., 7.5 oscillations per turn.

Visualization of Betatron Oscillations, Tunes

- What happens for 0<n<1?
  - Example picture below has 5 “turns” with sin(0.89 θ)
  - The betatron oscillation precesses, not strictly periodic
  - **Betatron tune $Q_{X,Y}$**: number of cycles made for every revolution or turn around accelerator

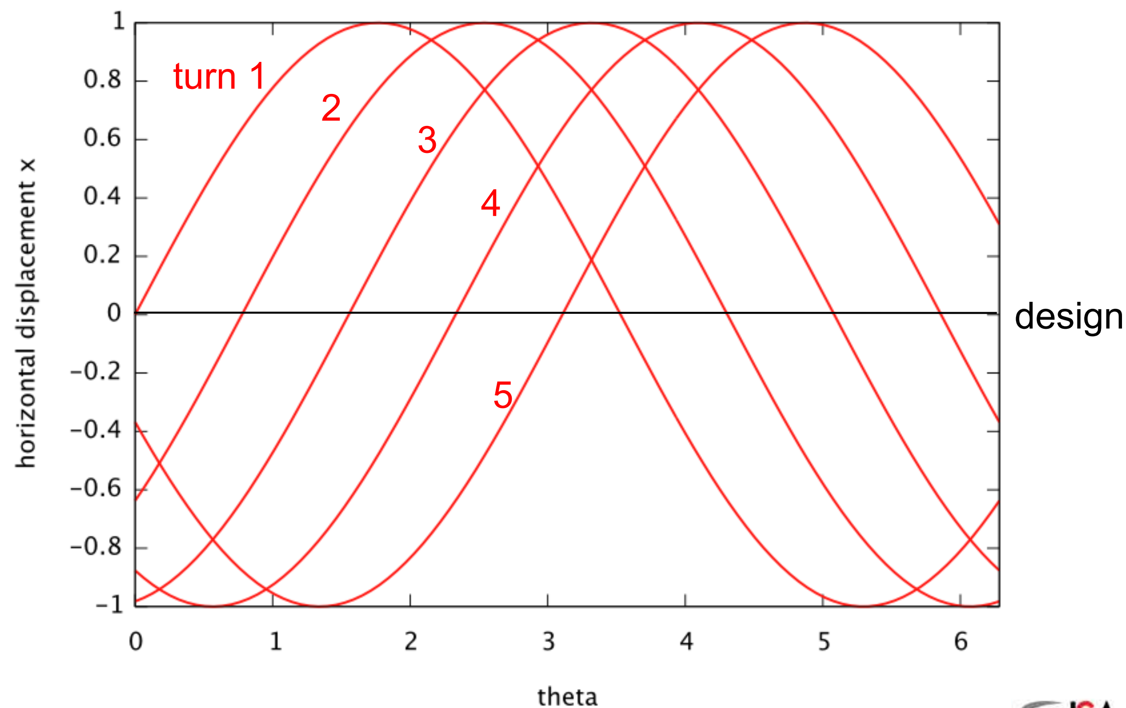
$$Q_x = \frac{1}{2\pi}\sqrt{1-n}(2\pi) = \sqrt{1-n}$$

$$Q_y = \frac{1}{2\pi}\sqrt{n}(2\pi) = \sqrt{n}$$

Frequency of betatron oscillations relative to turns around accelerator

For weak focusing:

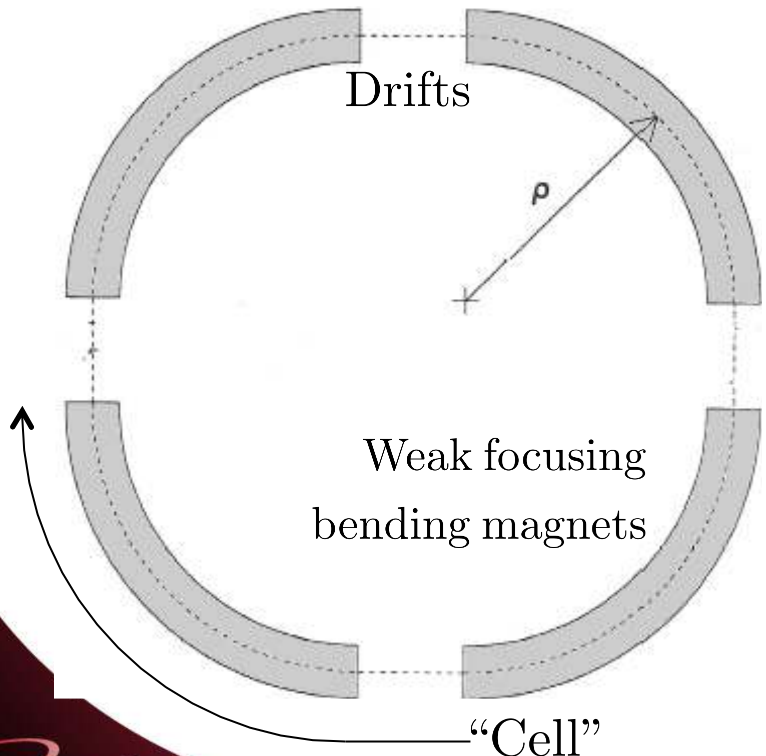
$$Q_x^2 + Q_y^2 = 1$$



Transport Matrices: Piecewise Solutions

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \boxed{\begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix}} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \boxed{M_V(\theta)} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

- Linear transport matrices make piecewise solutions of equations of motion accessible



"Cell" transport matrix:

$$M_{\text{cell}} = M(\theta)\, M_{\text{drift}}$$

"One turn" transport matrix:

$$M_{\text{one turn}} = (M(\theta)\, M_{\text{drift}})^4$$

Build accelerator optics out of "Lego" transport matrices

(everything is awesome!)

Transport Matrices: Accelerator Legos

(A3)

- With linear fields, there are two basic types of Legos
  - Dipoles $$\vec{B} = B_0 \hat{y} + (x\hat{y} + y\hat{x}) \left(\frac{\partial B_y}{\partial x} \right) \qquad B_0 \neq 0$$
    - Often long magnets to bend design trajectory
    - Entrance/exit locations can become important
    - May or may not include focusing ("combined function")
    - Special case: drift when all B components are zero
  - Quadrupoles $$\vec{B} = (x\hat{y} + y\hat{x}) \left(\frac{\partial B_y}{\partial x} \right) \qquad B_0 = 0$$
    - Design trajectory is straight! (no fields at x=y=0)
    - Act to focus particles moving off of design trajectory
    - Special case: "thin lens" approximation
    - We'll talk about quadrupoles tomorrow

Transport Matrices: Dipole

- We have already derived a very general **transport matrix for a dipole magnet with focusing**

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\phi_x(s) & \frac{\rho}{\sqrt{1-n}}\sin\phi_x(s) & 0 & 0 \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x(s) & \cos\phi_x(s) & 0 & 0 \\ 0 & 0 & \cos\phi_y(s) & \frac{\rho}{\sqrt{n}}\sin\phi_y(s) \\ 0 & 0 & -\frac{\sqrt{n}}{\rho}\sin\phi_y(s) & \cos\phi_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

$$\phi_x(s) \equiv \theta\sqrt{1-n} = \frac{s}{\rho}\sqrt{1-n} \qquad \phi_y(s) \equiv \theta\sqrt{n} = \frac{s}{\rho}\sqrt{n}$$

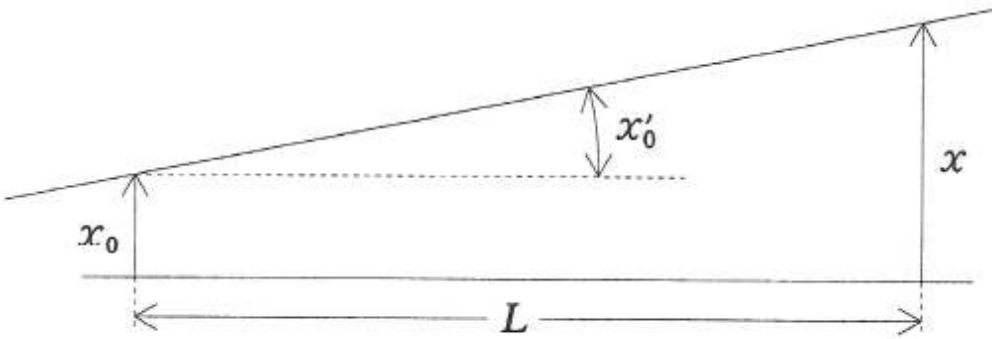
- Taking n->0 (and being careful) gives the **transport matrix for a dipole of bend angle θ without focusing**

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Transport Matrices: Drifts

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

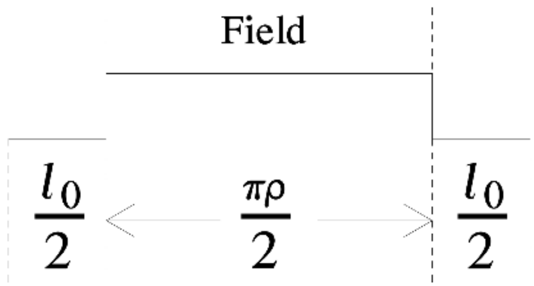
- For n=0, there is no horizontal field or vertical force
  - The vertical transport matrix here is for a **field-free drift**
  - This applies in both x,y planes when there is no field



$$\boxed{\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}}$$

Weak Focusing Synchrotron (text section 2.5)

- Back to our Lego machine
- Pick a more symmetric cell



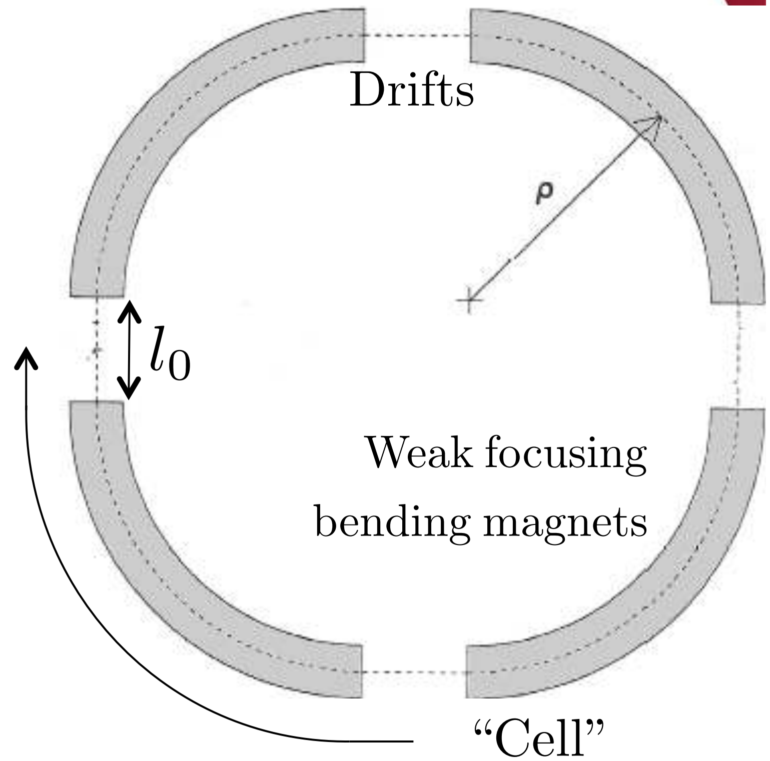
$$M_H = M_{\text{drift}}\left(\frac{l_0}{2}\right) M_{\text{dipole}}\left(\frac{\pi}{2}\right) M_{\text{drift}}\left(\frac{l_0}{2}\right)$$

- For $l_0 \ll \pi\rho$ "one can show"

$$\text{Phase advance per cell}: \mu_H = \left(1+\frac{l_0}{\pi\rho}\right)\frac{\pi\sqrt{1-n}}{2}$$

$$\text{Horizontal tune}: Q_H = \frac{4\mu_H}{2\pi} = \left(1+\frac{l_0}{\pi\rho}\right)\sqrt{1-n}$$

$$\text{Vertical tune}: Q_V = \left(1+\frac{l_0}{\pi\rho}\right)\sqrt{n}$$



Circumference
$$C = 2\pi\rho + 4l_0$$

Weak Focusing Synchrotron Parameterization

- We can write the cell transport matrix M_H in a form that is very similar to a rotation matrix

$$M_H = \begin{pmatrix} \cos\mu_H & \beta_H \sin\mu_H \\ -\frac{1}{\beta_H}\sin\mu_H & \cos\mu_H \end{pmatrix}$$

  - We will investigate this parameterization (and its non-periodic lattice extensions) extensively later this week
  - β_H is a length scale for the betatron oscillations
  - Details in Section 2.5 derive:

$$\beta_H \approx \frac{\rho}{\sqrt{1-n}}\left(1+\frac{l_0}{\pi\rho}\right) \qquad \beta_V \approx \frac{\rho}{\sqrt{n}}\left(1+\frac{l_0}{\pi\rho}\right)$$

  - Note the familiar scaling with the radius of curvature ρ !

What About Momentum?

- So far we have assumed that the design trajectory particle and our particle have the **same momentum**
- How do equations change if we break this assumption?
  - Expect only horizontal motion changes to first order (A2)

$$p = p_0(1+\delta) \ \text{where} \ \delta \equiv \frac{\Delta p}{p_0} \ll 1 \qquad p_0 = \text{design particle momentum}$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0 \quad \Rightarrow \quad \frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0(1+\delta)}R - 1\right)R = 0$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}(1-\delta)R - 1\right)R = 0 \quad \text{(A1)}$$

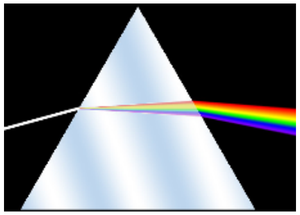
$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}R - 1\right)R = \delta\frac{R^2 qB_y}{p_0} = \rho\delta \quad \text{(A2)}$$

$$\boxed{\frac{d^2x}{d\theta^2} + (1-n)x = \rho\delta}$$

Add inhomogeneous term to original δ=0 equation of motion

Solutions of Dispersive Equations of Motion

$$\frac{d^2x}{d\theta^2} + (1-n)x = \rho\delta \qquad \frac{d^2y}{d\theta^2} + ny = 0$$



- This momentum effect is called **dispersion**
  - Similar to prism light dispersion in classical optics
- Solutions are simple harmonic oscillator solutions
  - But now we add a specific inhomogeneous solution

$$x(\theta) = A\cos(\theta\sqrt{1-n}) + B\sin(\theta\sqrt{1-n}) + \frac{\rho}{1-n}\delta$$

inhomogeneous term!

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})]$$

- Constants A,B again related to initial conditions (x_0, x_0')

$$A = x_0 - \frac{\rho}{1-n}\delta \qquad B = \frac{\rho}{\sqrt{1-n}}x_0'$$

δ is constant (A4)

Solutions of Dispersive Equations of Motion

- Write down solutions in terms of initial conditions

$$x'(\theta) = -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n})x_0 + \cos(\theta\sqrt{1-n})x_0' + \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n})\delta_0$$

$$x'(\theta) = \frac{1}{\rho}\frac{dx}{d\theta} = -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n})x_0 + \cos(\theta\sqrt{1-n})x_0' + \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n})\delta_0$$

$$\delta = \delta_0$$

- This can be now be "conveniently" written in terms of a 3x3 matrix:

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}$$

As usual, this can be simplified for n=0 (pure dipole)

Note that δ has become a "coordinate"!

Sector Dipole Magnets

- This transport matrix applies for **any** angle θ
- So it is also the transport matrix of any section of constant field, and now includes dispersion.
  - In particular, it is the horizontal transport for a **combined function sector dipole** of length $L = \rho\theta$
  - With n=0 this is the horizontal transport for a **sector dipole** of length $L = \rho\theta$, bend angle θ, and radius of curvature ρ

$$\mathbf{M}_{\mathrm{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

  - Sector dipole: design trajectory at entrance and exit of magnet is perpendicular to magnet face

Example: 180 Degree Dipole Magnet

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

$$n = 0 \quad \Rightarrow \quad M_H(\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

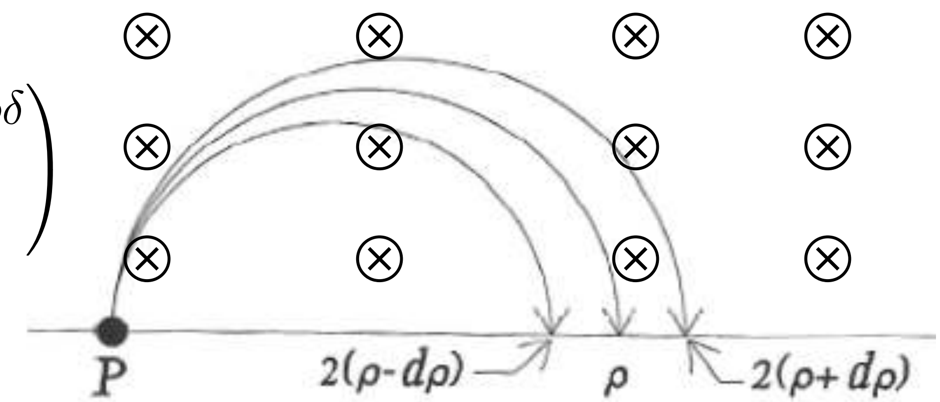
$$n = 0,\ \theta = \pi \quad \Rightarrow \quad M_H(\theta) = \begin{pmatrix} -1 & 0 & 2\rho \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{For initial coordinates } \begin{pmatrix} 0 \\ 0 \\ \pm\delta \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2\rho \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \pm\delta \end{pmatrix} = \begin{pmatrix} \pm 2\rho\delta \\ 0 \\ \pm\delta \end{pmatrix}$$

This makes sense from p/q=Bρ!

electrons moving through uniform vertical B field



Momentum Compaction

- Different momenta particles will have different path lengths $L \equiv \oint ds$ around the accelerator
  - Naively $\text{larger } p \to \text{larger } \rho \to \text{larger } L$
  - This is not necessarily true (as we'll see!)

- This is quantified by a quantity called **momentum compaction**
  - Ratio of fractional change in pathlength to fractional change in momentum

$$\boxed{\alpha_p \equiv \frac{(dL/L)}{(dp/p)} = \frac{p}{L}\frac{dL}{dp}}$$

recall $\delta \equiv \dfrac{dp}{p}$

Transition Energy

- Relativistic particle motion in a periodic accelerator (like a synchrotron) creates some weird effects
  - For particles moving around with frequency ω in circumference C

$$\omega = \frac{2\pi\beta c}{C} \quad \Rightarrow \quad \frac{d\omega}{\omega} = \frac{d\beta}{\beta} - \frac{dC}{C} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{\mathrm{tr}}^2}\right)\delta$$

momentum compaction $\alpha_p \equiv \frac{dC}{C}/\delta = \frac{p}{C}\frac{dC}{dp}$ transition gamma $\gamma_{\mathrm{tr}} \equiv \frac{1}{\sqrt{\alpha_p}}$

  - At "transition", $\gamma = \gamma_{\mathrm{tr}}$ and **particle revolution frequency does not depend on its momentum**

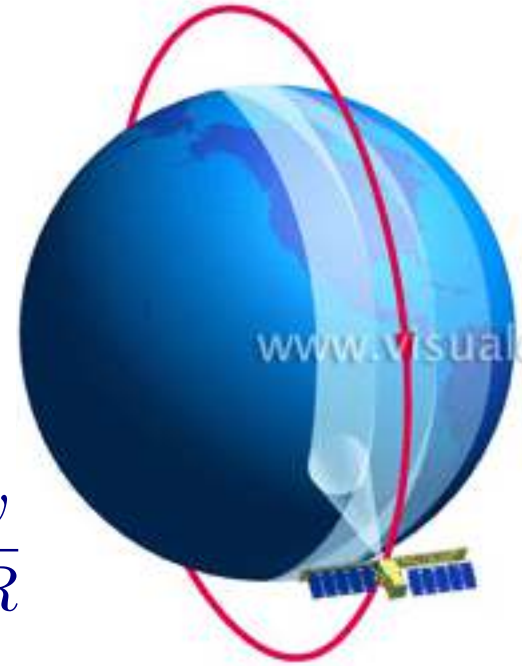
    Reminiscent of a cyclotron but now we're strong focusing and at constant radius!

electron ring	At $\gamma_r > \gamma_{\mathrm{tr}}$ higher momentum gives **lower** revolution frequency
electron linac	At $\gamma_r < \gamma_{\mathrm{tr}}$ higher momentum gives **higher** revolution frequency

Hadron synchrotrons can accelerate through transition!

Momentum Compaction: A Classic Example

- Consider a satellite in circular orbit
- A classical gravity/centripetal force problem



$$F = \frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$\frac{GM}{R} = v^2 \qquad \omega = 2\pi f = 2\pi \left(\frac{v}{2\pi R}\right) = \frac{v}{R}$$

momentum compaction $\quad \alpha_p \equiv \frac{dC}{C} \Big/ \frac{dp}{p} = \frac{dR}{R} \Big/ \frac{dv}{v} = \frac{v}{R} \Big/ \frac{dv}{dR}$

$$2v\, dv = -\frac{GM}{R^2} dR \quad \Rightarrow \quad \frac{dv}{dR} = -\frac{v}{2R}$$

$$\boxed{\text{momentum compaction } \alpha_p = -2}$$

Raising momentum p **lowers** orbit radius, raises angular frequency ω

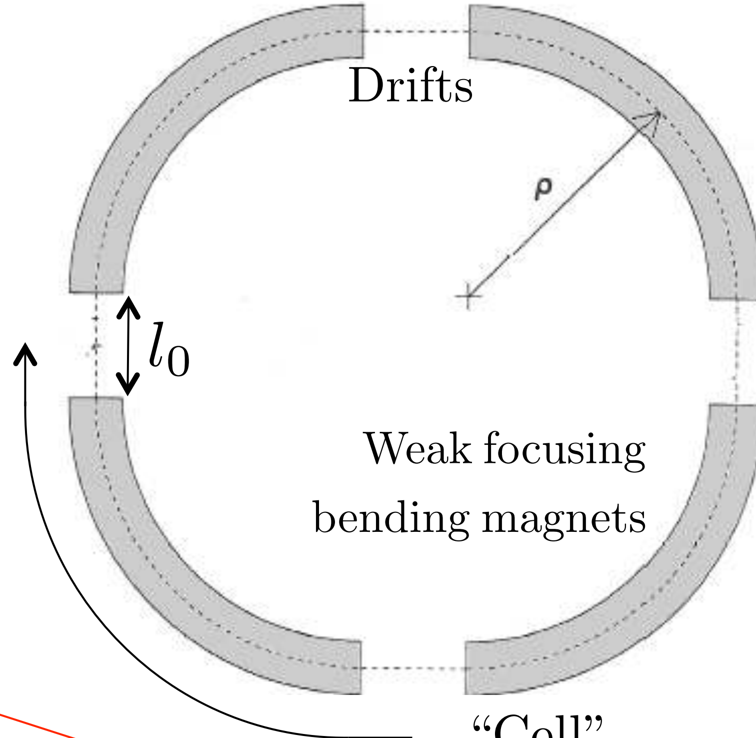
Momentum Compaction: Weak Focusing Synchrotron

- Recall $\frac{p}{q} = B\rho$

$$\frac{dp}{p} = \delta = \frac{d\rho}{\rho} + \frac{dB}{B} = \left(1 + \frac{\rho}{B}\frac{\partial B}{\partial \rho}\right)\frac{d\rho}{\rho}$$

$$\delta = (1-n)\frac{d\rho}{\rho}$$

$$n \equiv -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)$$



- Now

$$\alpha_p \equiv \frac{dC/C}{dp/p} = \frac{(dC/d\rho)(d\rho/C)}{\delta}$$

$$= \frac{2\pi\rho}{C(1-n)} = \left(1 + \frac{2l_0}{\pi\rho}\right)^{-1}(1-n)^{-1}$$

Circumference
$C = 2\pi\rho + 4l_0$

$$\text{Horizontal tune}: Q_H = \frac{4\mu_H}{2\pi} = \left(1 + \frac{l_0}{\pi\rho}\right)\sqrt{1-n}$$

$$\boxed{\alpha_p \approx Q_H^{-2}}$$