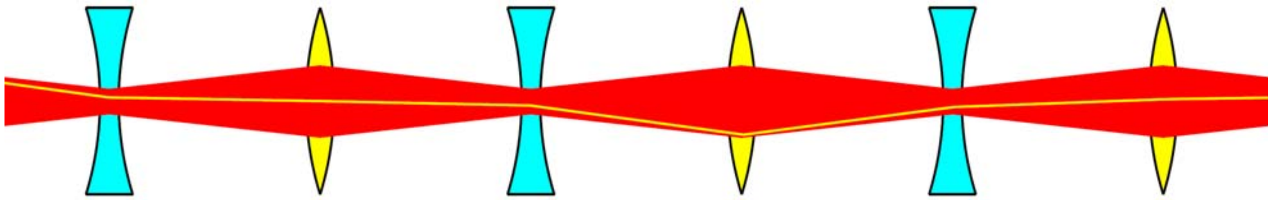
Weak vs. Strong Focusing

- Weak focusing rings tend to have beams with large transverse dimensions
- 69,000 ton yoke of Dubna’s 10 GeV Synchrophasotron vs. NICA booster

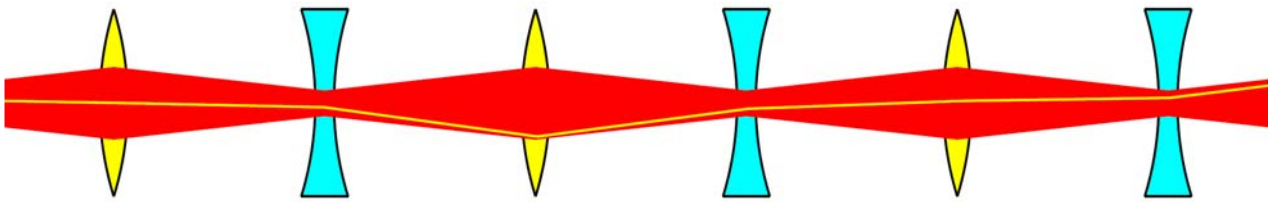
Strong Focusing

- Using alternating focusing and defocusing lenses can provide stable transverse dynamics with stronger focusing and therefore higher betatron tunes.
- Lattice can be built using combined-function magnets with alternating gradients (AGS, Serpukhov) or separate dipole and quadrupole magnets (separated-function lattices, e.g. RHIC).

Horizontal plane:



Vertical plane:



Stability of Linear Motion

- The transfer matrix of a periodic cell can be constructed by successively multiplying the matrices of its constituting elements: dipoles, quadrupoles, and drifts. It could be the matrix of the whole ring or of the periodic section of the ring.

$$\vec{z}_1 = M\vec{z}_0$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Let us examine its eigenvalues to learn about the stability of the linear motion.

  Characteristic polynomial of an n×n matrix:

$$P_M(\lambda) = \det(M - \lambda I) = \sum_{i=0}^{n} A_i \lambda^i = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

$$A_0 = P_M(0) = \underline{\det(M) = \prod_{i=1}^{n} \lambda_i}, \quad A_1 = -\mathrm{tr}(M), \quad A_n = 1$$

Characteristic equation:

$$P_M(\lambda) = 0$$

Stability of Linear Motion

- Characteristic equation of a 2×2 matrix:

$$\lambda^2 - \mathrm{tr}(M)\lambda + 1 = \lambda^2 - (a+b)\lambda + 1 = 0$$

- Since

$$\det(M) = 1 = \lambda_1\lambda_2 \quad \Rightarrow$$

$$\lambda_1 = \frac{1}{\lambda_2}$$

- Either both roots are real or

$$\lambda_1 = \lambda_2^*$$

then

$$\lambda_1 = \lambda_2^* = \frac{1}{\lambda_2} \quad \Rightarrow \quad \lambda_2\lambda_2^* = 1 \quad \Rightarrow \quad |\lambda_2| = |\lambda_1| = 1$$

- The roots of the characteristic equation

$$\lambda_1 = \frac{1}{2}\mathrm{tr}(M) + \sqrt{\left(\frac{1}{2}\mathrm{tr}(M)\right)^2 - 1}, \quad \lambda_1 = \frac{1}{2}\mathrm{tr}(M) - \sqrt{\left(\frac{1}{2}\mathrm{tr}(M)\right)^2 - 1}$$

Liapunov Stability

- A point $\vec{x}_0$ is called a fixed point of the map M if $M\vec{x}_0 = \vec{x}_0$. A system is said to be Liapunov stable about a fixed point $\vec{x}_0$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\left\|\vec{x} - \vec{x}_0\right\| < \delta$ then $\left\|M^n\vec{x} - M^n\vec{x}_0\right\| < \varepsilon$ for all $0 < n < \infty$.
- I.e. there is some neighborhood about the fixed point which remains bounded for all time.
- For the linear transfer matrix

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Cases with degenerate eigenvalues

$$\lambda_1 = \lambda_2 = +1, \quad \lambda_1 = \lambda_2 = -1$$

  are either not bounded such as

$$\text{drift:} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \quad \text{thin quad:} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

  or bounded but unstable (an integer or half-integer betatron tune).
- Cases with non-generate eigenvalues but with zero off-diagonal terms, i.e. $b = c = 0$ are also obviously unstable, e.g.

$$\begin{pmatrix} 1.1 & 0 \\ 0 & 1/1.1 \end{pmatrix} \text{ or } \begin{pmatrix} 0.9 & 0 \\ 0 & 1/0.9 \end{pmatrix}$$

Liapunov Stability

- Assuming eigenvalues are not degenerate and $b \neq 0$ we can find two linearly independent eigenvectors (case of $b = 0$ and $c \neq 0$ can be considered similarly)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b \\ x_{1,2} \end{pmatrix} = \begin{pmatrix} ab + bx_{1,2} \\ cb + dx_{1,2} \end{pmatrix} = \lambda_{1,2}\begin{pmatrix} b \\ x_{1,2} \end{pmatrix} \quad \Rightarrow$$

$$x_{1,2} = \lambda_{1,2} - a$$

$$\vec{\upsilon}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \quad \vec{\upsilon}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}$$

- Expand a vector in terms of the eigenvectors

$$\vec{x} = A\vec{\upsilon}_1 + B\vec{\upsilon}_2$$

- The trajectory deviation after *n* turns

$$D_n = \left\| M^n \vec{x} - M^n \vec{x}_0 \right\| = \left\| A\lambda_1^n \vec{\upsilon}_1 + B\lambda_2^n \vec{\upsilon}_2 \right\| < |\lambda_1^n| \, |A| \left\| \vec{\upsilon}_1 \right\| + |\lambda_2^n| \, |B| \left\| \vec{\upsilon}_2 \right\|$$

Since $\lambda_1 = \dfrac{1}{\lambda_2}$ the motion can be unstable if either $|\lambda_i| \neq 1$

Liapunov Stability

- If $|\operatorname{tr}(M)| < 2$

$$\operatorname{tr}(M) = 2\cos\mu \qquad \text{for some real angle } \mu$$

then

$$\lambda_{1,2} = \frac{1}{2}\left[\operatorname{tr}(M) \pm \sqrt{[\operatorname{tr}(M)]^2 - 4}\right] = \cos\mu \pm i\sin\mu = e^{\pm i\mu}$$

$$D_n < |\lambda_1^n|\,|A|\left\|\vec{\upsilon}_1\right\| + |\lambda_2^n|\,|B|\left\|\vec{\upsilon}_2\right\| = |A|\left\|\vec{\upsilon}_1\right\| + |B|\left\|\vec{\upsilon}_2\right\| \quad \text{bounded for all } n$$

- If $\operatorname{tr}(M) > 2$

$$\operatorname{tr}(M) = 2\cosh\mu \qquad \text{for some real } \mu > 0$$

$$\lambda_{1,2} = \cosh\mu \pm \sinh\mu = e^{\pm\mu}$$

$$D_n \sim (e^{\mu})^n\,|A|\left\|\vec{\upsilon}_1\right\| \quad \text{goes to infinity as } n \to \infty$$

- If $\operatorname{tr}(M) < -2$

$$\operatorname{tr}(M) = -2\cosh\mu \qquad \text{for some real } \mu > 0$$

$$\lambda_{1,2} = -\cosh\mu \pm \sinh\mu = -e^{\mp\mu}$$

$$D_n \sim (e^{\mu})^n\,|B|\left\|\vec{\upsilon}_2\right\| \quad \text{goes to infinity as } n \to \infty$$

- $\operatorname{tr}(M) = 2$ corresponds to the case of generate eigenvalues

Parameterization of Stable Matrix

- Consider the case of stable motion

$$-2 < \mathrm{tr}(M) = 2\cos\mu < 2$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The diagonal elements can be written as

$$a = \cos\mu + k, \quad d = \cos\mu - k \quad \text{where } k = (a-d)/2$$

$$bc = ad - \det(M) = \cos^2\mu - k^2 - 1 = -k^2 - \sin^2\mu$$

- Since $|\cos\mu| < 1, \;\; \sin\mu \neq 0$ and we can define

$$k = \alpha\sin\mu \qquad \text{where } \alpha \text{ is real}$$

$$bc = -(1+\alpha^2)\sin^2\mu$$

- Partition bc as

$$b = \beta\sin\mu$$

$$c = -\underbrace{\frac{1+\alpha^2}{\beta}}_{\equiv\gamma}\sin\mu = -\gamma\sin\mu$$

Matrix Parameterization

- A stable transfer matrix can be written as

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\mu + \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}\sin\mu$$

$$= I\cos\mu + J\sin\mu$$

- Since the sign of μ is ambiguous, we may choose β (and therefore γ) to be positive.

$$\det(J) = \gamma\beta - \alpha^2 = 1$$

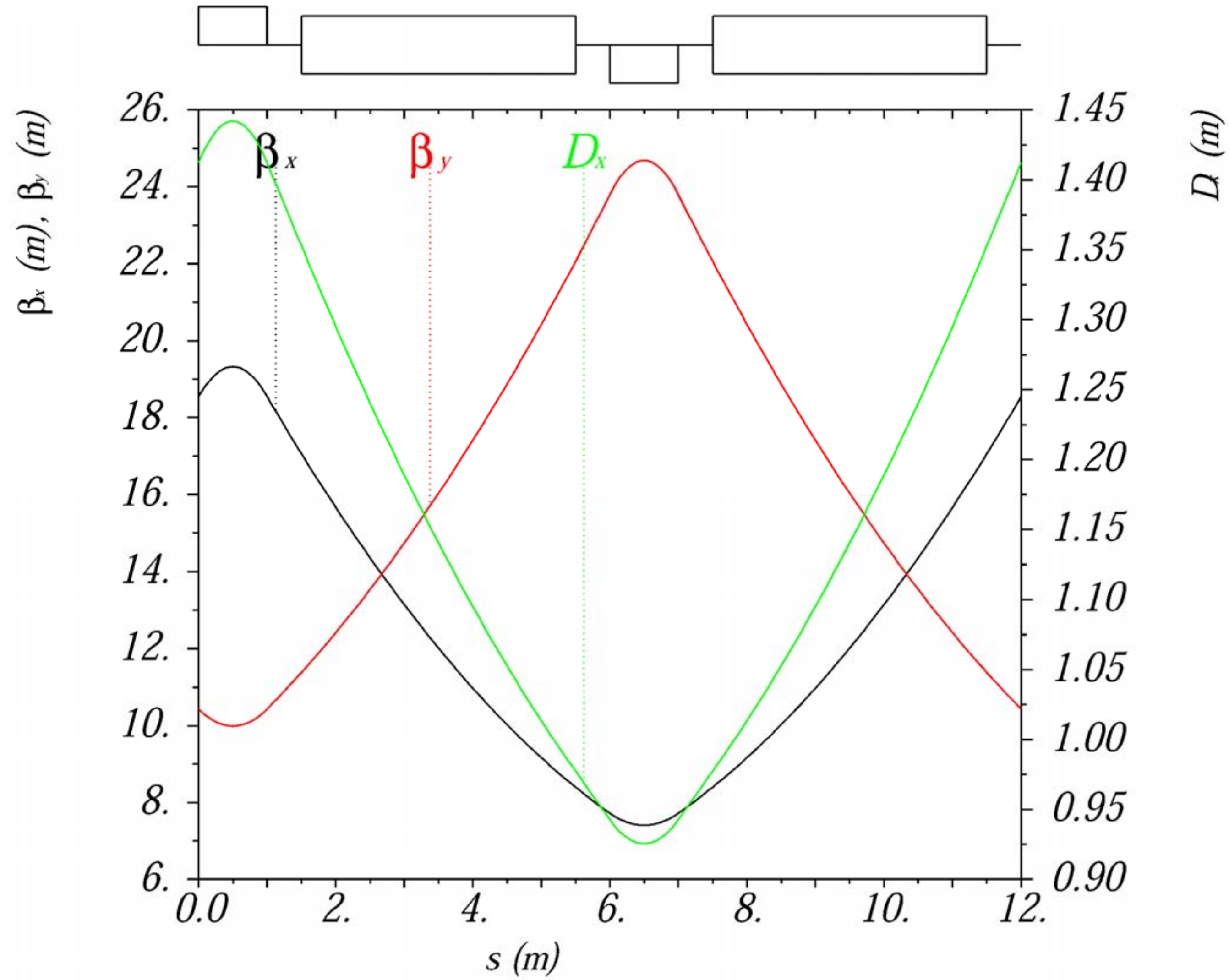
$$J^2 = \begin{pmatrix} \alpha^2 - \gamma\beta & 0 \\ 0 & \alpha^2 - \gamma\beta \end{pmatrix} = -I$$

$$M = I\cos\mu + J\sin\mu = e^{J\mu}$$

- α, β, and γ are often called the Twiss parameters.

FODO Cell Example

- One of the simplest periodic building blocks



FODO Cell Example

- Let's examine its transport matrix for horizontal motion

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$= \begin{pmatrix} -0.8250051897 & 16.06225553 \\ -0.1559954483 & 1.82500519 \end{pmatrix}$$

$$\cos\mu_x = \frac{1}{2}\mathrm{tr}(M) = \frac{1}{2}(-0.8250051897 + 1.82500519) = 0.5 \quad \Rightarrow$$

$$\mu_x = \cos^{-1}(0.5) = \pi/3, \quad Q_x = \mu_x/(2\pi) = 1/6$$

$$\sin\mu_x = \sqrt{3}/2$$

$$\alpha_x = (M_{11} - \cos\mu_x)/\sin\mu_x = (-0.8250051897 - 0.5)/(\sqrt{3}/2) = -1.53$$

$$\beta_x = M_{12}/\sin\mu_x = 16.06225553/(\sqrt{3}/2) = 18.55$$

$$\gamma_x = -M_{21}/\sin\mu_x = 0.1559954483/(\sqrt{3}/2) = 0.18$$

Analytic Approach

- Equations of motion

$$\tilde{H} = -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\left(1+\delta-\frac{1}{2}(x'^2+y'^2)+\ldots\right)+1+\delta$$

$$\frac{dx}{ds} = \frac{\partial \tilde{H}}{\partial x'}, \quad \frac{dx'}{ds} = -\frac{\partial \tilde{H}}{\partial x}$$

$$\frac{dx'}{ds} = x'' = -\frac{\partial \tilde{H}}{\partial x} = \frac{q}{p_0}\frac{\partial A_s}{\partial x} + \frac{1+\delta}{\rho}$$

$$B_x = \frac{1}{1+x/\rho}\frac{\partial A_s}{\partial y}, \quad B_y = -\frac{1}{1+x/\rho}\frac{\partial A_s}{\partial x}$$

$$x'' = -\frac{q}{p_0}\left(1+\frac{x}{\rho}\right)\left(B_{y0}+\frac{\partial B_y}{\partial x}x\right)+\frac{1+\delta}{\rho} = -\left(\frac{1}{\rho^2}+\frac{q}{p_0}\frac{\partial B_y}{\partial x}\right)x+\frac{\delta}{\rho} = -k_x(s)x+\frac{\delta}{\rho(s)}$$

Analytic Approach

- Equations of motion

$$x'' + k_x(s)x = \frac{\delta}{\rho(s)}$$

$$y'' + k_y(s)y = 0$$

where

$$k_x(s) = \frac{1}{\rho^2} + \frac{q}{p_0}\frac{\partial B_y}{\partial x}$$

$$k_y(s) = -\frac{q}{p_0}\frac{\partial B_y}{\partial x}$$

Hill's Equation

- Let us first consider the general homogeneous equation

$$z'' + k(s)z = 0$$

with

$$k(s+L) = k(s)$$

- According to Floquet's theorem, solutions have the form

$$z(s) = z_A(s) + z_B(s)$$

$$z_A(s) = Aw(s)\cos\Psi(s)$$

$$z_B(s) = Bw(s)\sin\Psi(s)$$

where $w(s)$ is the amplitude function periodic with the same period L and $\Psi(s)$ is a non-periodic phase of the oscillations.

Hill's Equation

- Substituting one of the solutions into the Hill's equation

$$z_A'(s) = Aw'\cos\Psi - Aw\Psi'\sin\Psi$$

$$z_A''(s) = Aw''\cos\Psi - 2Aw'\Psi'\sin\Psi - Aw\Psi''\sin\Psi - Aw\Psi'^2\cos\Psi$$

$$A[(w'' - w\Psi'^2 + kw)\cos\Psi - (2w'\Psi' + w\Psi'')\sin\Psi] = 0$$

- This gives

$$w'' - w\Psi'^2 + kw = 0$$

$$2w'\Psi' + w\Psi'' = 0 \quad \Rightarrow \quad \Psi' = 1/w^2$$

$$w'' + kw - \frac{1}{w^3} = 0$$

General Solution

- The general solution is

$$z(s) = Aw\cos\Psi + Bw\sin\Psi$$

$$z'(s) = A(w'\cos\Psi - \frac{\sin\Psi}{w}) + B(w'\sin\Psi + \frac{\cos\Psi}{w})$$

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - \frac{\sin\Psi}{w} & w'\sin\Psi + \frac{\cos\Psi}{w} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = F(w,\Psi)\begin{pmatrix} A \\ B \end{pmatrix}$$

- Initial conditions at $s = s_0$

$$\begin{pmatrix} z_0 \\ z_0' \end{pmatrix} = F(w_0,\Psi_0)\begin{pmatrix} A \\ B \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A \\ B \end{pmatrix} = F^{-1}(w_0,\Psi_0)\begin{pmatrix} z_0 \\ z_0' \end{pmatrix}$$

$$\begin{pmatrix} z \\ z' \end{pmatrix} = F(w,\Psi)\begin{pmatrix} A \\ B \end{pmatrix} = F(w,\Psi)F^{-1}(w_0,\Psi_0)\begin{pmatrix} z_0 \\ z_0' \end{pmatrix}$$

$$M(s) = F(w,\Psi)F^{-1}(w_0,\Psi_0)$$

General Solution

$$F(w,\Psi) = \begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - \dfrac{\sin\Psi}{w} & w'\sin\Psi + \dfrac{\cos\Psi}{w} \end{pmatrix}$$

$$\det(F) = 1$$

$$F^{-1}(w_0,\Psi_0) = \begin{pmatrix} w_0'\sin\Psi_0 + \dfrac{\cos\Psi_0}{w_0} & -w_0\sin\Psi_0 \\ -w_0'\cos\Psi_0 + \dfrac{\sin\Psi_0}{w_0} & w_0\cos\Psi_0 \end{pmatrix}$$

$$M = F(w,\Psi)F^{-1}(w_0,\Psi_0)$$

$$= \begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - \dfrac{\sin\Psi}{w} & w'\sin\Psi + \dfrac{\cos\Psi}{w} \end{pmatrix} \begin{pmatrix} w_0'\sin\Psi_0 + \dfrac{\cos\Psi_0}{w_0} & -w_0\sin\Psi_0 \\ -w_0'\cos\Psi_0 + \dfrac{\sin\Psi_0}{w_0} & w_0\cos\Psi_0 \end{pmatrix}$$

General Solution

$$M = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}$$

$$a(s) = \frac{w(s)}{w_0}\cos[\Psi(s)-\Psi_0] - w(s)w_0'\sin[\Psi(s)-\Psi_0]$$

$$b(s) = w(s)w_0\sin[\Psi(s)-\Psi_0]$$

$$c(s) = -\frac{1+w(s)w_0w'(s)w_0'}{w(s)w_0}\sin[\Psi(s)-\Psi_0] - \left[\frac{w_0'}{w(s)} - \frac{w'(s)_0}{w_0}\right]\cos[\Psi(s)-\Psi_0]$$

$$d(s) = \frac{w_0}{w(s)}\cos[\Psi(s)-\Psi_0] + w_0w'(s)\sin[\Psi(s)-\Psi_0]$$

General Solution

- Defining

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -\frac{1}{2}\beta' = -w(s)w'(s)$$

$$\mu(s) = \Psi(s) - \Psi_0$$

- $\beta(s)$ is called the betatron function, $\mu(s)$ is called the betatron phase, and $\alpha(s)$ is sometimes called a correlation function. Since $w(s)$ is periodic, both $\beta(s)$ and $\alpha(s)$ are also periodic with the same period.
- We note

$$-ww_0' = -\frac{w}{w_0}w_0w_0' = \sqrt{\frac{\beta}{\beta_0}}\alpha_0$$

$$\frac{w_0'}{w} - \frac{w'}{w_0} = \frac{w_0w_0'}{w_0w} - \frac{ww'}{ww_0} = \frac{\alpha - \alpha_0}{\sqrt{\beta_0\beta}}$$

$$\frac{w}{w_0} = \sqrt{\frac{\beta}{\beta_0}}$$

General Solution

$$M(s)=\begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}[\cos\mu(s)+\alpha_0\sin\mu(s)] & \sqrt{\beta_0\beta(s)}\sin\mu(s) \\ -\dfrac{[\alpha(s)-\alpha_0]\cos\mu(s)+[1+\alpha_0\alpha(s)]\sin\mu(s)}{\sqrt{\beta_0\beta(s)}} & \sqrt{\dfrac{\beta_0}{\beta(s)}}[\cos\mu(s)-\alpha(s)\sin\mu(s)] \end{pmatrix}$$

- Note

$$\vec{z}(s)=M(s)\vec{z}_0$$

$$z(s)=a(s)z_0+b(s)z_0'$$

$$z'(s)=a'(s)z_0+b'(s)z_0'=c(s)z_0+d(s)z_0' \quad \Rightarrow \quad c(s)=a'(s), \quad d(s)=b'(s)$$

$$M(s)=\begin{pmatrix} a(s) & b(s) \\ a'(s) & b'(s) \end{pmatrix}$$

- Matrix for one period

$$M(L)=\begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\dfrac{1+\alpha^2}{\beta}\sin\mu(s) & \cos\mu-\alpha\sin\mu \end{pmatrix}$$

Betatron Tunes

- Betatron phase advance over one period

$$\mu = \int_s^{s+L} \Psi' ds = \int_s^{s+L} \frac{1}{w^2} ds = \int_s^{s+L} \frac{1}{\beta(s)} ds$$

- The number of oscillations per turn is called a betatron tune

$$Q_{x,y} = \frac{\mu_{x,y}}{2\pi} = \frac{1}{2\pi} \oint \frac{1}{\beta_{x,y}} ds$$

- The betatron tunes are global parameters of the ring, i.e. the phase advance does not depend on the starting point of the integration.
- In an uncoupled case, the fraction parts of the tunes can be readily obtained from the diagonal 2×2 blocks of the transfer matrix:

$$q_{x,y} = \frac{1}{2\pi} \cos^{-1}\left(\frac{\mathrm{tr}(M_{x,y}^{2\times2})}{2} \right)$$

Courant-Snyder Invariant

- The particle trajectory can be written as

$$z(s)=\sqrt{\frac{\beta(s)}{\beta_0}}[\cos\mu(s)+\alpha_0\sin\mu(s)]z_0+\sqrt{\beta_0\beta(s)}\sin\mu(s)z_0'$$

$$=\sqrt{\beta(s)}\left[\frac{z_0}{\sqrt{\beta_0}}\cos\mu(s)+\left(\frac{z_0}{\sqrt{\beta_0}}\alpha_0+\sqrt{\beta_0}z_0'\right)\sin\mu(s)\right]$$

$$=\sqrt{W\beta(s)}[\cos\mu_0\cos\mu(s)-\sin\mu_0\sin\mu(s)]=\sqrt{W\beta(s)}\cos[\mu(s)+\mu_0]$$

where

$$\cos\mu_0=\frac{z_0/\sqrt{\beta_0}}{\sqrt{W}},\quad \sin\mu_0=-\frac{z_0\alpha_0/\sqrt{\beta_0}+\sqrt{\beta_0}z_0'}{\sqrt{W}}$$

$$W\equiv\left(\frac{z_0}{\sqrt{\beta_0}}\right)^2+\left(\frac{z_0}{\sqrt{\beta_0}}\alpha_0+\sqrt{\beta_0}z_0'\right)^2$$

$$=\frac{1+\alpha_0^2}{\beta_0}z_0^2+2\alpha_0z_0z_0'+\beta_0z_0'^2=\gamma_0z_0^2+2\alpha_0z_0z_0'+\beta_0z_0'^2$$

Courant-Snyder Invariant

- We can show that this quantity is conserved along the orbit

$$z = \sqrt{W\beta}\cos\Psi$$

$$z^2 = W\beta\cos^2\Psi$$

$$z' = \sqrt{W}\left(\frac{1}{2}\frac{\beta'}{\sqrt{\beta}}\cos\Psi - \sqrt{\beta}\Psi'\sin\Psi\right) = \sqrt{W}\left(-\frac{\alpha}{\sqrt{\beta}}\cos\Psi - \frac{1}{\sqrt{\beta}}\sin\Psi\right)$$

$$= -\sqrt{\frac{W}{\beta}}\left(\alpha\cos\Psi + \sin\Psi\right)$$

$$z'^2 = \frac{W}{\beta}\left(\alpha\cos\Psi + \sin\Psi\right)^2$$

$$zz' = -W\cos\Psi\left(\alpha\cos\Psi + \sin\Psi\right)$$

$$\gamma z^2 + 2\alpha zz' + \beta z'^2$$

$$= \gamma W\beta\cos^2\Psi - 2\alpha W\cos\Psi\left(\alpha\cos\Psi + \sin\Psi\right) + \beta\frac{W}{\beta}\left(\alpha\cos\Psi + \sin\Psi\right)^2$$

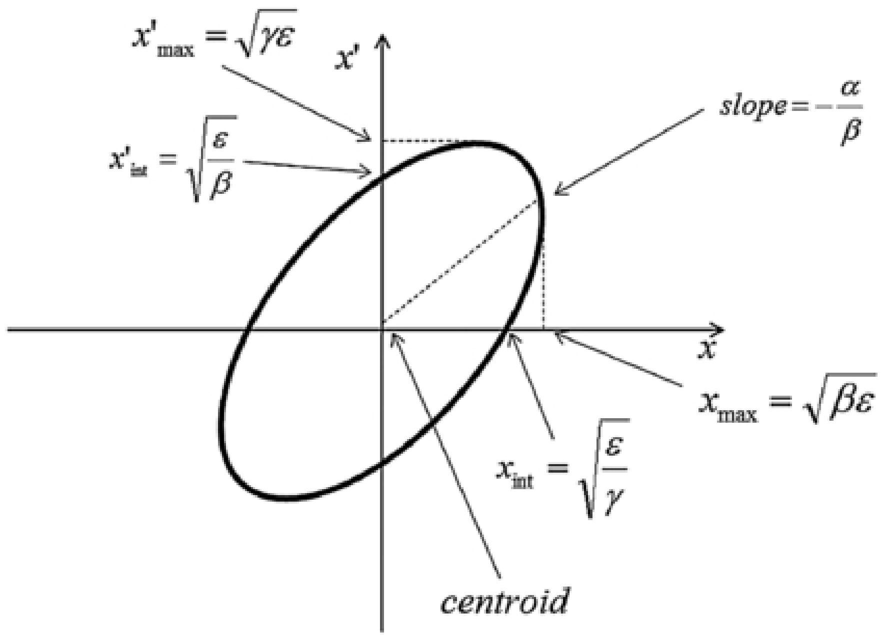
$$= W[(\gamma\beta - 2\alpha^2 + \alpha^2)\cos^2\Psi + (-2\alpha + 2\alpha)\cos\Psi\sin\Psi + \sin^2\Psi] = W$$

Courant-Snyder Invariant

- An invariant of motion called Courant-Snyder invariant

$$W = \gamma z^2 + 2\alpha z z' + \beta z'^2 = \gamma_0 z_0^2 + 2\alpha_0 z_0 z_0' + \beta_0 z_0'^2$$

- This equation describes a skew ellipse in the phase space with an area of πW



- For a periodic lattice, the particle lies on an ellipse determined by its initial conditions.
- Since the Twiss parameters are independent of initial conditions, there is a family of similar inscribed ellipses corresponding to different values of W.

Emittance

- A particle inside a certain ellipse has a phase space trajectory that stays inside the ellipse on successive turns since it is also an inscribed non-intersecting ellipse.
- Similarly, any particle outside an ellipse stays outside.
- The phase space region contained within the largest ellipse than an accelerator is able to transport is called the acceptance or admittance.
- It is convenient to choose an ellipse

$$\gamma z^2 + 2\alpha z z' + \beta z'^2 = \varepsilon$$

  containing a certain fraction of the beam, e.g. 39% (1σ or rms), 90% (4.6σ), or 95% (6σ). The ellipse will always contain the same fraction of the beam. This is another manifestation of Liouville's theorem.

- The area of the selected ellipse $\pi\varepsilon$ is called the emittance. It is quoted with the factor of π explicitly written out.

Beam Envelope

- The trajectory of a particle is given by

$$z(s) = \sqrt{W\beta(s)}\cos(\mu(s)+\mu_0)$$

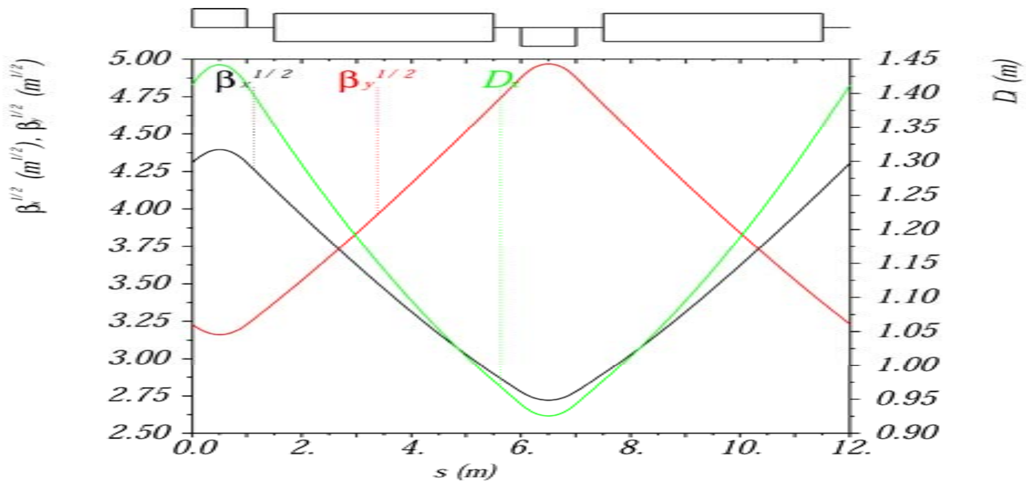
- The transverse amplitude is

$$z_{max} = \sqrt{W\beta(s)}$$

- The rms envelope of the beam is

$$z_{rms} = \sqrt{\varepsilon_{rms}\beta(s)}$$

Thus, $\sqrt{\beta(s)}$ can be interpreted as the beam envelope function specifying the transverse size of the beam as a function of s. Of course, the horizontal and vertical beam sizes are determined by their respective betatron functions and emittances.



Adiabatic Invariants

- Let us find an invariant of motion preserved during acceleration. The rate of energy change during acceleration is slow on the time scale of betatron oscillations. An adiabatic approximation applies, i.e. we can assume a slow approximately uniform acceleration of particles with no heating of the beam in its rest frame.
- The adiabatic approximation allows use of the Poincaré-Cartan integral invariant:

$$I = \oint p dq \approx \text{const.}$$

where p and q are canonically conjugate coordinates and the integration is over one period of the betatron oscillations.

- For horizontal motion:

$$q = x$$

$$p_x = \gamma m \dot{x} = \gamma m \frac{ds}{dt}\frac{dx}{ds} = (\gamma m \upsilon) x' = p x'$$

Adiabatic Invariants

- The horizontal invariant becomes

$$I_x = p\oint x'dx = p\pi\varepsilon_x = \pi mc\beta\gamma\varepsilon_x = \pi mc\varepsilon_x^*$$

where we found a quantity invariant under acceleration called the normalized emittance:

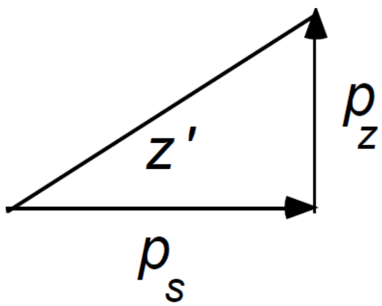
$$\pi\varepsilon_x^* = \beta\gamma\pi\varepsilon_x$$

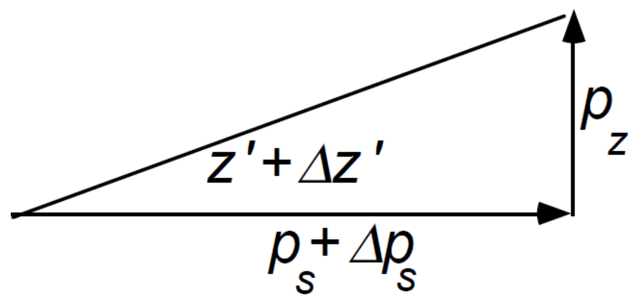
- Similarly, for the vertical motion:

$$I_y = p\oint y'dy = p\pi\varepsilon_y = \pi mc\beta\gamma\varepsilon_y = \pi mc\varepsilon_y^*$$

$$\pi\varepsilon_y^* = \beta\gamma\pi\varepsilon_y$$

- If the energy is adiabatically increased the area of the (x, p_x) ellipse remains constant while the area of the (x, x') ellipse shrinks. This effect is called adiabatic damping.





Dispersion

- To account for momentum deviation, consider the horizontal equation of motion with the inhomogeneous term:

$$x'' + k(s)x = \delta / \rho(s)$$

$$\delta = \Delta p / p$$

- The solution may be written as

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta_0$$

where $C(s)$ and $S(s)$ are solutions of the homogeneous equation and $D(s)$ is a particular solution of the inhomogeneous equation with $\delta = 1$ having the initial conditions

$$C(0) = S'(0) = 1 \quad \text{(cosine-line component)}$$

$$C'(0) = S(0) = 0 \quad \text{(sine-like component)}$$

$$D(0) = D'(0) = 0$$

- The slope is given by

$$x'(s) = C'(s)x_0 + S'(s)x_0' + D'(s)\delta_0$$

- The constant-energy trajectory equation in the matrix form is then

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}$$

Dispersion

- Consider the eigenvalues of a 1-turn matrix. The characteristic equation is

$$P_{3\times3}(\lambda) = P_{2\times2}(\lambda)(1-\lambda) = 0$$

The eigenvalues of the 2×2 matrix were obtained before

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(M_{2\times2}) \pm \sqrt{\left(\frac{1}{2}\mathrm{tr}(M_{2\times2})\right)^2 - 1} = e^{\pm i\mu}$$

and

$$\lambda_3 = +1$$

- Let us write the eigenvector corresponding to the 3rd eigenvalue as

$$\begin{pmatrix} \eta\delta \\ \eta'\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} \delta$$

- The periodic function η is called the dispersion function

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

where the matrix elements are evaluated for the periodic cell starting at position s.

Dispersion

- Solving for η and η'

$$\begin{pmatrix}\eta\\ \eta'\end{pmatrix}=\begin{pmatrix}C(s) & S(s)\\ C'(s) & S'(s)\end{pmatrix}\begin{pmatrix}\eta\\ \eta'\end{pmatrix}+\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}$$

$$\begin{pmatrix}1-C(s) & -S(s)\\ -C'(s) & 1-S'(s)\end{pmatrix}\begin{pmatrix}\eta\\ \eta'\end{pmatrix}=\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}$$

$$\det\begin{pmatrix}1-C(s) & -S(s)\\ -C'(s) & 1-S'(s)\end{pmatrix}=1-C(S)-S'(s)+\underbrace{C(S)S'(s)-S(s)C'(s)}_{1}$$

$$=2-\mathrm{tr}(M_{2\times 2})=2(1-\cos\mu)$$

$$\begin{pmatrix}\eta\\ \eta'\end{pmatrix}=\begin{pmatrix}1-C(s) & -S(s)\\ -C'(s) & 1-S'(s)\end{pmatrix}\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}=\frac{1}{2(1-\cos\mu)}\begin{pmatrix}1-S'(s) & S(s)\\ C'(s) & 1-C(s)\end{pmatrix}\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}$$

$$=\frac{1}{2(1-\cos\mu)}\begin{pmatrix}[1-S'(s)]D(s)+S(s)D'(s)\\ C'(s)D(s)+[1-C(s)]D'(s)\end{pmatrix}$$

- The dispersion, nominally, is the closed orbit for a particle with $\delta=1$ in linear approximation. Of course, the approximation and therefore the dispersion are only applicable when $\delta << 1$.

Dispersion

- Note that the periodic dispersion function $\eta(s) = dx / d\delta$ is not the same as the matrix element $D(s) = \partial x / \partial\delta$.
- $x_\delta(s) = \eta(s)\delta$ is the inhomogeneous part of the trajectory due to the dispersion.
- The total horizontal trajectory component consists of a part due to the betatron oscillations and the part due to the dispersion:

$$x(s) = x_\beta(s) + x_\delta(s) = x_\beta(s) + \eta(s)\delta$$

- Since the two components are independent, their contributions to the total beam size must be added in quadrature:

$$\sigma_{tot}(s) = \sqrt{\sigma_\beta^2(s) + [\eta(s)\sigma_\delta]^2} = \sqrt{\varepsilon_{rms}\beta(s) + [\eta(s)\sigma_\delta]^2}$$

using rms values of the involved quantities.

Momentum Compaction

- The momentum compaction: relative change of circumference per unit change in relative momentum offset

$$\alpha_p = \frac{\Delta L / L}{\Delta p / p}$$

- The circumference

$$L_{\delta=0} = L = \oint ds, \quad L_{\delta\neq 0} = L + \Delta L = \oint d\sigma$$

$$\oint d\sigma = \oint \rho_\delta d\theta = \oint (\rho + x_\delta) d\theta = \oint \left(1 + \frac{x_\delta}{\rho}\right) ds = L + \oint \frac{x_\delta}{\rho} ds \quad \Rightarrow$$

$$\Delta L = \oint \frac{x_\delta}{\rho} ds = \oint \frac{\eta(s)\delta}{\rho} ds = \delta \oint \frac{\eta(s)}{\rho(s)} ds$$

- The momentum compaction then becomes

$$\alpha_p = \frac{1}{L} \oint \frac{\eta(s)}{\rho(s)} ds$$