USPAS Accelerator Physics 2015
Old Dominion University

Chapter 5 Review and Chapter 6:

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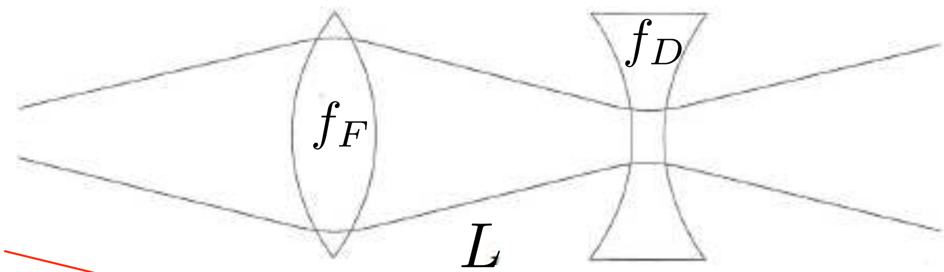
Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift L



Thin quadrupole matrices

$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

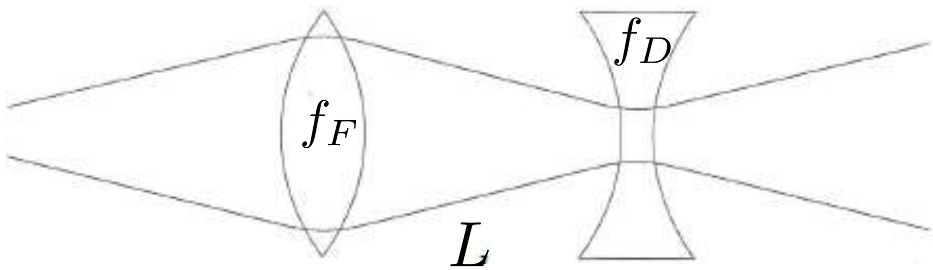
$$\frac{1}{f_{\text{doublet}}} = \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \qquad \text{(C\&M 5.1 with } f_F = -f_D\text{)}$$

$$f_D = f_F = f \qquad \Rightarrow \qquad \frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2}$$

There is **net focusing** given by this **alternating gradient** system

A fundamental point of optics, and of accelerator **strong focusing**

Strong Focusing: Another View



$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$$\text{incoming paraxial ray} \quad \begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \end{pmatrix} x_0$$

For this to be focusing, x' must have opposite sign of x where these are coordinates of transformation of incoming paraxial ray

$$\boxed{f_F = f_D \quad x' < 0 \quad \textbf{BUT} \quad x > 0 \text{ iff } f_F > L}$$

Equal strength doublet is net focusing under condition that each lens' focal length is greater than distance between them

More Math: Hill's Equation

- Let's go back to our quadrupole equations of motion for $R \to \infty$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)$$

What happens when we let the focusing K vary with s?

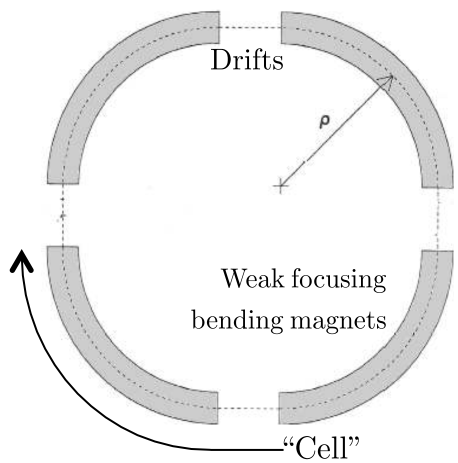
Also assume K is **periodic** in s with some periodicity C

$$x'' + K(s)x = 0 \qquad K(s) \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)(s) \qquad K(s+C) = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated "cell" of the layout

(Such as a FODO cell in the previous slide)

The simple harmonic oscillator equation with a **periodically** varying spring constant K(s) is known as **Hill's Equation**



Hill's Equation Solution Ansatz

$$x'' + K(s)x = 0 \qquad K \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)(s)$$

- Solution is a quasi-periodic harmonic oscillator

$$x(s) = A\, w(s)\, \cos[\Psi(s) + \Psi_0]$$

where w(s) is periodic in C but the phase Ψ(s) is not!!

Substitute this educated guess ("ansatz") to find

$$x' = Aw' \cos[\Psi + \Psi_0] - Aw\Psi' \sin[\Psi + \Psi_0]$$

$$x'' = A(w'' - w\Psi'^2) \cos[\Psi + \Psi_0] - A(2w'\Psi' + w\Psi'') \sin[\Psi + \Psi_0]$$

$$x'' + K(s)x = -A(2w'\Psi' + w\Psi'') \sin(\Psi + \Psi_0) + A(w'' - w\Psi'^2 + Kw) \cos(\Psi + \Psi_0) = 0$$

For $w(s)$ and $\Psi(s)$ to be independent of Ψ_0, coefficients of the sin and cos terms must vanish identically

Courant-Snyder Parameters

$$2ww'\Psi' + w^2\Psi'' = (w^2\Psi')' = 0 \quad \Rightarrow \quad \Psi' = \frac{k}{w(s)^2}$$

$$w'' - (k^2/w^3) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2$$

- Notice that in both equations $w^2 \propto k$ so we can scale this out and define a new set of functions, **Courant-Snyder Parameters** or **Twiss Parameters**

$$\boxed{\begin{aligned} \beta(s) &\equiv \frac{w^2(s)}{k} \\ \alpha(s) &\equiv -\frac{1}{2}\beta'(s) \\ \gamma(s) &\equiv \frac{1+\alpha(s)^2}{\beta(s)} \end{aligned}} \quad \Rightarrow$$

$$\Psi' = \frac{1}{\beta(s)} \qquad \boxed{\Psi(s) = \int \frac{ds}{\beta(s)}}$$

$$\boxed{K\beta = \gamma + \alpha'}$$

$\beta(s), \alpha(s), \gamma(s)$ are all periodic in C

$\Psi(s)$ is **not** periodic in C

Towards The Matrix Solution

- What is the matrix for this Hill's Equation solution?

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

Take a derivative with respect to s to get $x' \equiv \frac{dx}{ds}$

$$\Psi' = \frac{1}{\beta(s)} \qquad x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

Now we can solve for A and B in terms of initial conditions $(x(0), x'(0))$

$$x_0 \equiv x(0) = A\sqrt{\beta(0)} \qquad x'_0 \equiv x'(0) = \frac{1}{\sqrt{\beta(0)}}[B - \alpha(0)A]$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x'_0 + \alpha(0)x_0]$$

And take advantage of the periodicity of β, α to find $x(C), x'(C)$

Hill's Equation Matrix Solution

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

$$x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x'_0 + \alpha(0)x_0]$$

$$x(C) = [\cos\Psi(C) + \alpha(0)\sin\Psi(C)]x_0 + \beta(0)\sin\Psi(C)x'_0$$

$$x'(C) = -\gamma(0)\sin\Psi(C)x_0 + [\cos\Psi(C) - \alpha(0)\sin\Psi(C)]x'_0$$

We can write this down in a matrix form where $\mu = \Psi(C) - \Psi(0)$ is the betatron phase advance through one period C

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$ phase advance per cell

Interesting Observations

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

phase advance per cell

- μ is independent of s: this is the **betatron phase advance** of this periodic system
- Determinant of matrix M is still 1!
- Looks like a rotation and some scaling
- M can be written down in a **beautiful** and **deep** way

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

remember $x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$

Convenient Calculations

- If we know the transport matrix M, we can find the lattice parameters (periodic in C)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance per cell} \quad \cos\mu = \frac{1}{2}\,\mathrm{Tr}\,M$$

$$\beta(0) = \beta(C) = \frac{m_{12}}{\sin\mu}$$

$$\alpha(0) = \alpha(C) = \frac{m_{11} - \cos\mu}{\sin\mu}$$

$$\gamma(0) \equiv \frac{1 + \alpha^2(0)}{\beta(0)}$$

General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using lattice parameters and $\Delta\Psi = \Psi(s_2) - \Psi(s_1)$

$$M_{s_1 \to s_2} = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\Psi + \alpha(s_1)\sin\Delta\Psi] & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\Psi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\Psi + [1+\alpha(s_1)\alpha(s_2)]\sin\Delta\Psi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\Psi - \alpha(s_2)\sin\Delta\Psi] \end{pmatrix}$$

(C&M Eqn 5.52)

- This does not have a pretty form like the periodic matrix

However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

A common use of this matrix is the m_{12} term:

$$\boxed{\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\Psi)\, x'(s_1)}$$

Effect of angle kick on downstream position

(Deriving the Non-Periodic Transport Matrix)

$$x(s) = Aw(s)\cos\Psi(s) + Bw(s)\sin\Psi(s)$$

$$x'(s) = A\left(w'(s)\cos\Psi(s) - \frac{\sin\Psi(s)}{w(s)}\right) + B\left(w'(s)\sin\Psi(s) + \frac{\cos\Psi(s)}{w(s)}\right)$$

Calculate A, B in terms of initial conditions (x_0, x_0') and (w_0, Ψ_0)

$$A = \left(w_0'\sin\Psi_0 + \frac{\cos\Psi_0}{w_0}\right)x_0 - (w_0\sin\Psi_0)x_0'$$

$$B = -\left(w_0'\cos\Psi_0 - \frac{\sin\Psi_0}{w_0}\right)x_0 + (w_0\cos\Psi_0)x_0'$$

Substitute (A,B) and put into matrix form: $\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

$$m_{11}(s) = \frac{w(s)}{w_0}\cos\Delta\Psi - w(s)w_0'\sin\Delta\Psi$$

$$\Delta\Psi \equiv \Psi(s) - \Psi_0$$

$$w(s) = \sqrt{\beta(s)}$$

$$m_{12}(s) = w(s)w_0\sin\Delta\Psi$$

$$m_{21}(s) = -\frac{1 + w(s)w_0w'(s)w_0'}{w(s)w_0}\sin\Delta\Psi - \left[\frac{w_0'}{w(s)} - \frac{w'(s)}{w_0}\right]\cos\Delta\Psi$$

$$m_{22}(s) = \frac{w_0}{w(s)}\cos\Delta\Psi + w_0w'\sin\Delta\Psi$$

Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi}-\text{periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \Psi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance} \quad \mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \qquad \mathrm{Tr}\, M = 2\cos\mu$$

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

Transport Matrix Stability Criteria

- For long systems (rings) we want $M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ stable as $n \to \infty$
  - If 2x2 M has eigenvectors (V_1, V_2) and eigenvalues (λ_1, λ_2):

$$M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = A\lambda_1^n V_1 + B\lambda_2^n V_2$$

  - M is also unimodular (det M=1) so $\lambda_{1,2} = e^{\pm i\mu}$ with complex μ
  - For $\lambda_{1,2}^n$ to remain bounded, μ must be real

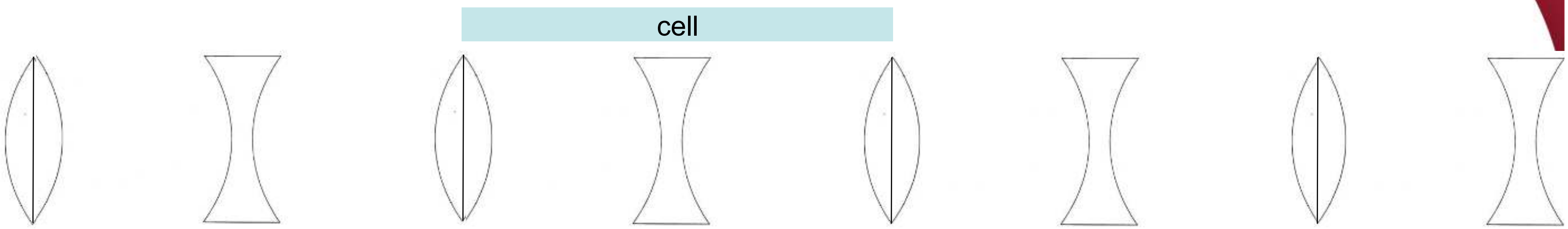
  - We can always transform M into diagonal form with the eigenvalues on the diagonal (since det M=1); this does not change the trace of the matrix

$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = \mathrm{Tr}\, M$$

  - The **stability requirement** for these types of matrices is then

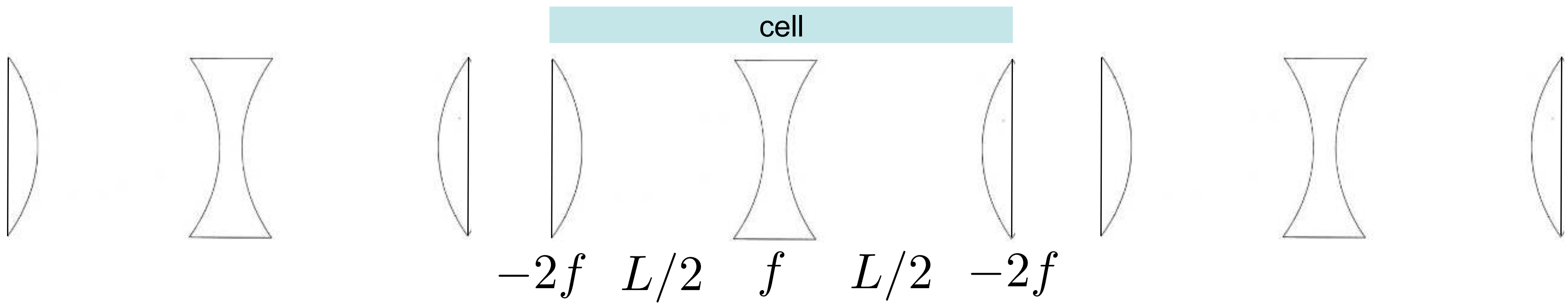
$$\mu \text{ real} \quad \Rightarrow \quad \boxed{-1 \leq \frac{1}{2}\mathrm{Tr}\, M \leq 1}$$

Lattice Optics: FODO Lives!



- Most accelerator lattices are designed in modular ways
  - Design and operational clarity, separation of functions
- One of the most common modules is a FODO module
  - Alternating focusing and defocusing “strong” quadrupoles
  - Spaces between are combinations of drifts and dipoles
  - Strong quadrupoles dominate the focusing
  - Periodicity is one FODO “cell” so we’ll investigate that motion
  - Horizontal beam size largest at centers of focusing quads
  - Vertical beam size largest at centers of defocusing quads

Periodic Example: FODO Cell Phase Advance



- Select periodicity between centers of **focusing** quads
  - A natural periodicity if we want to calculate maximum $\beta(s)$

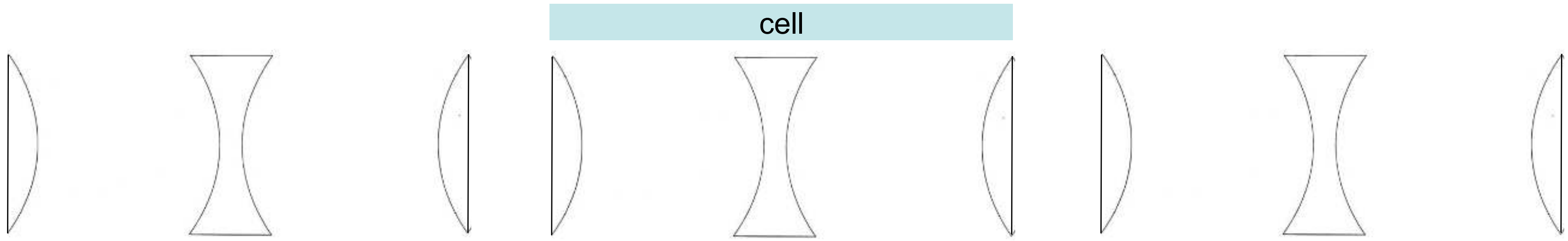
$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\mu = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\mu = 1 - 2\sin^2\frac{\mu}{2} \quad \Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f}$$

  - μ only has real solutions (stability) if $\frac{L}{4} < f$

Periodic Example: FODO Cell Beta Max/Min



$$-2f \quad L/2 \quad f \quad L/2 \quad -2f$$

- What is the maximum beta function, $\hat{\beta}$?
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad m_{12} = \beta \sin \mu$$

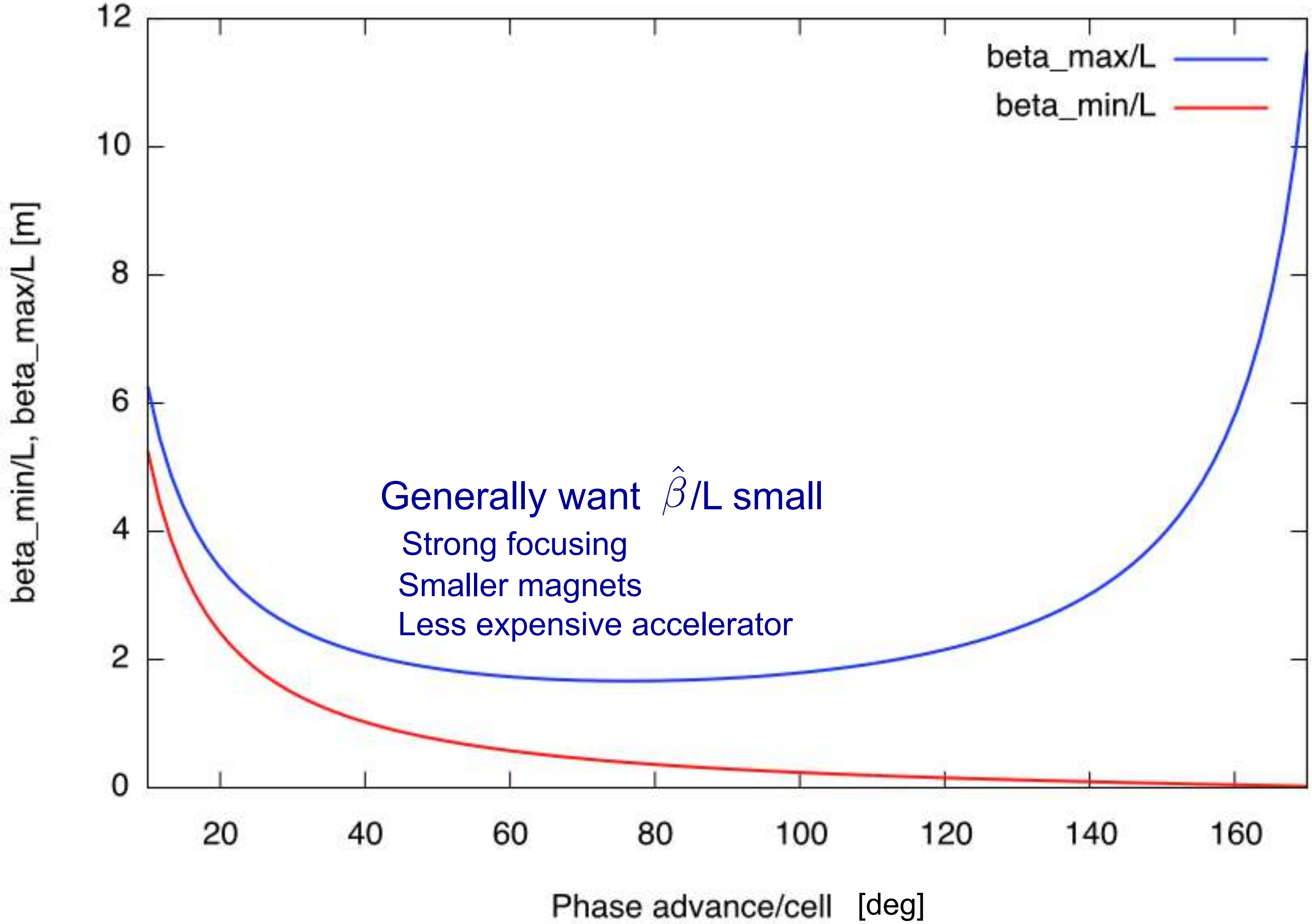
$$\hat{\beta} \sin \mu = \frac{L^2}{4f} + L = L\left(1 + \sin \frac{\mu}{2}\right)$$

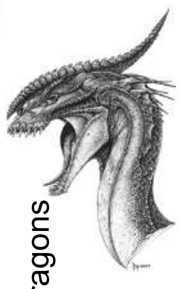
$$\boxed{\hat{\beta} = \frac{L}{\sin \mu}\left(1 + \sin \frac{\mu}{2}\right)}$$

  - Follow a similar strategy reversing F/D quadrupoles to find the minimum β(s) within a FODO cell (center of D quad)

$$\boxed{\check{\beta} = \frac{L}{\sin \mu}\left(1 - \sin \frac{\mu}{2}\right)}$$

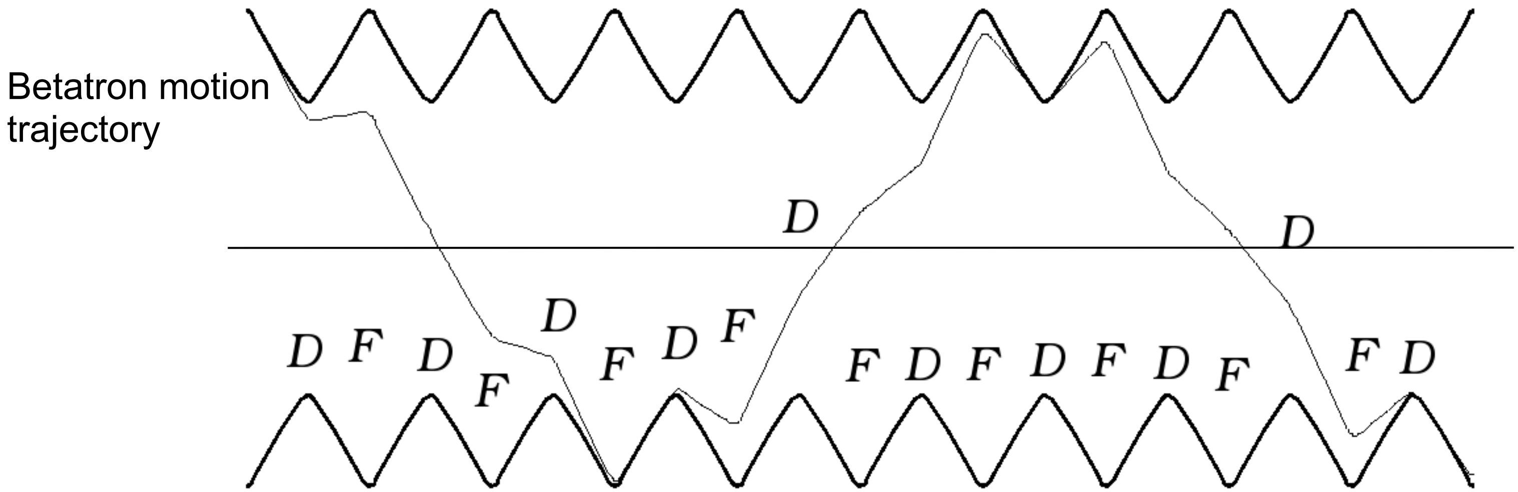
FODO Betatron Functions vs Phase Advance





Here there be dragons

FODO Beta Function, Betatron Motion

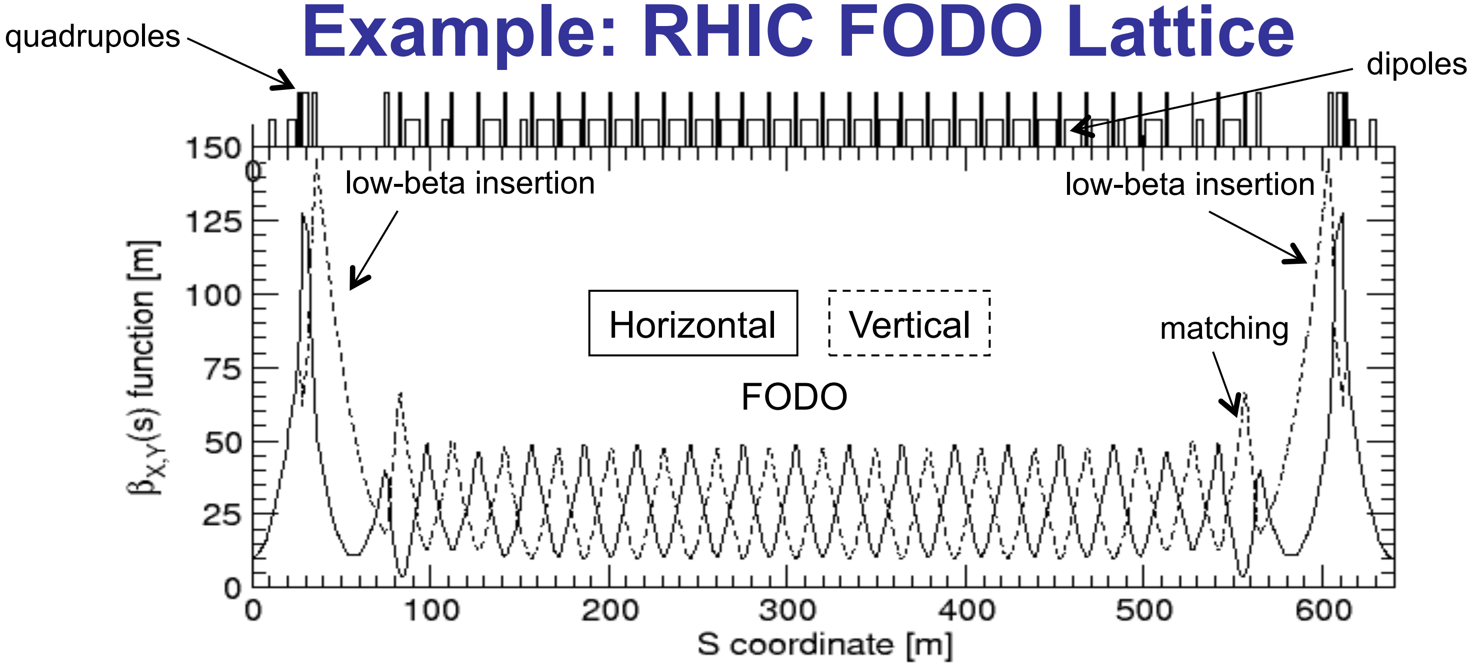


(C&M Fig. 5.3)

- This is a picture of a FODO lattice, showing contours of $\pm\sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$
  - This also shows a particle oscillating through the lattice
  - Note that $\sqrt{\beta(s)}$ provides an "envelope" for particle oscillations
    - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
  - Min beta is at defocusing quads, max beta is at focusing quads
  - 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \quad \mu = 360^\circ/6.5 \approx 55^\circ$$

Example: RHIC FODO Lattice



- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about L=30 m
  - Phase advance per FODO cell is about $\mu = 77^\circ = 1.344\ \mathrm{rad}$

$$\hat{\beta} = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right) \approx 53\ \mathrm{m}$$

$$\check{\beta} = \frac{L}{\sin\mu}\left(1 - \sin\frac{\mu}{2}\right) \approx 8.7\ \mathrm{m}$$

☺

Propagating Lattice Parameters

- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos\mu + J \sin\mu = \begin{pmatrix} \cos\mu + \alpha(s_1)\sin\mu & \beta(s_1)\sin\mu \\ -\gamma(s_1)\sin\mu & \cos\mu - \alpha(s_1)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Homework ☺

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$

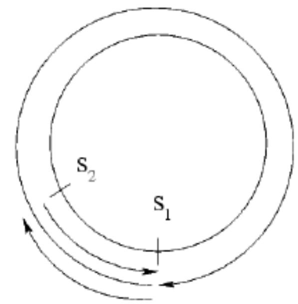
Propagating Lattice Parameters

- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos \mu + J \sin \mu = \begin{pmatrix} \cos \mu + \alpha(s_1) \sin \mu & \beta(s_1) \sin \mu \\ -\gamma(s_1) \sin \mu & \cos \mu - \alpha(s_1) \sin \mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The J(s) matrices at s_1, s_2 are related by

$$J(s_2) = M(s_1, s_2) J(s_1) M^{-1}(s_1, s_2)$$



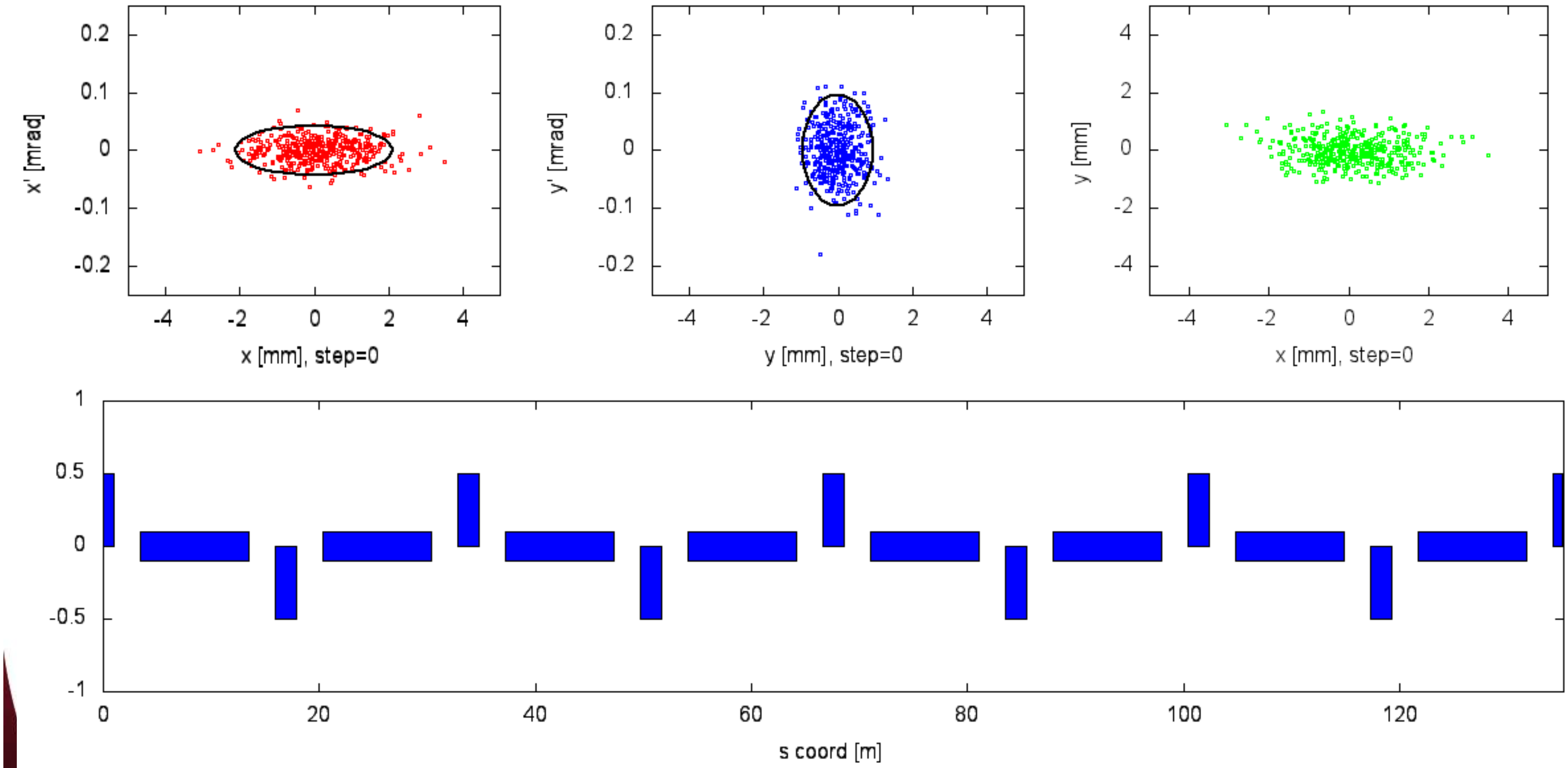
Then expand, using det M=1

$$J(s_2) = \begin{pmatrix} \alpha(s_2) & \beta(s_2) \\ -\gamma(s_2) & -\alpha(s_2) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \alpha(s_1) & \beta(s_1) \\ -\gamma(s_1) & -\alpha(s_1) \end{pmatrix} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}}$$

Quadratic: Lattice elements repeat themselves for $\mathbf{M} = \pm \mathbf{I}$

=========== What's the Ellipse? ===========



- Area of an ellipse that envelops a given percentage of the beam particles in phase space is related to the **emittance**

We can express this in terms of our lattice functions!

Invariants and Ellipses

$$x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

- We assumed A was constant, an **invariant of the motion**

  A can be expressed in terms of initial coordinates to find

$$\boxed{\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2}$$

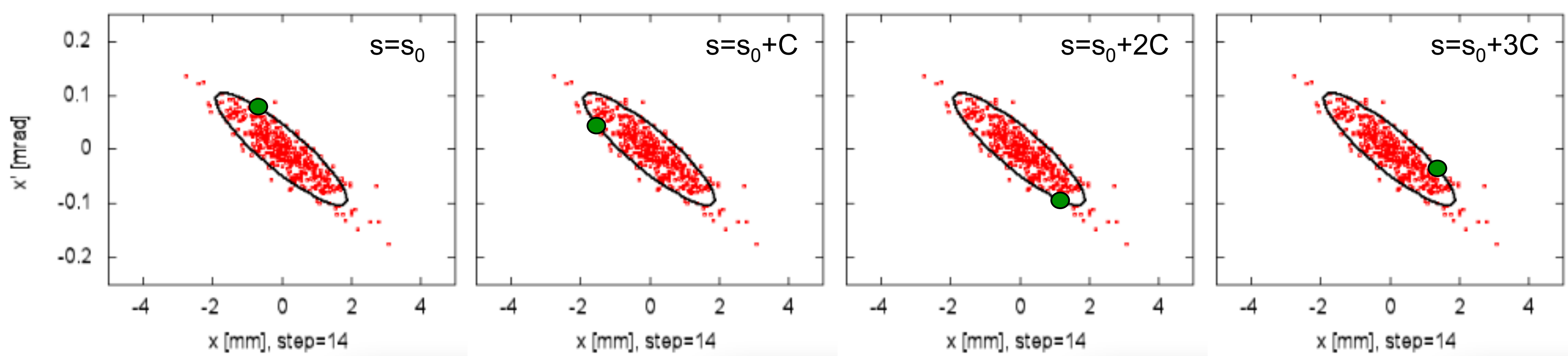
  This is known as the **Courant-Snyder invariant**: for all s,

$$\mathcal{W} = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

  Similar to total energy of a simple harmonic oscillator

  $\mathcal{W}$ looks like an elliptical area in (x, x') phase space

  Our matrices look like scaled rotations (ellipses) in phase space



Emittance

- The area of the ellipse inscribed by any given particle in phase space as it travels through our accelerator is called the **emittance** ϵ: it is "constant" and given by

$$\boxed{\epsilon = \pi \mathcal{W} = \pi[\gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2]}$$

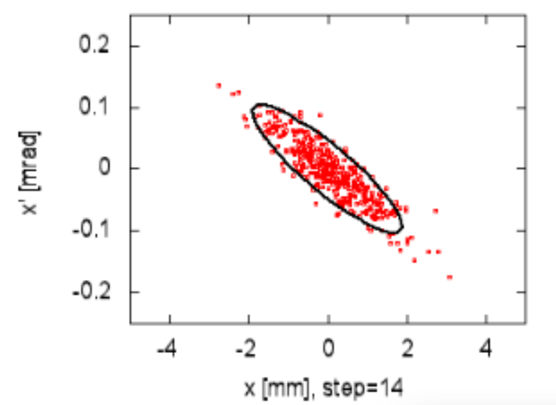
Emittance is often quoted as the area of the ellipse that would contain a certain fraction of all (Gaussian) beam particles

e.g. RMS emittance contains 39% of 2D beam particles

Related to RMS beam size σ_{RMS}

$$\sigma_{\mathrm{RMS}} = \sqrt{\epsilon\beta(s)}$$

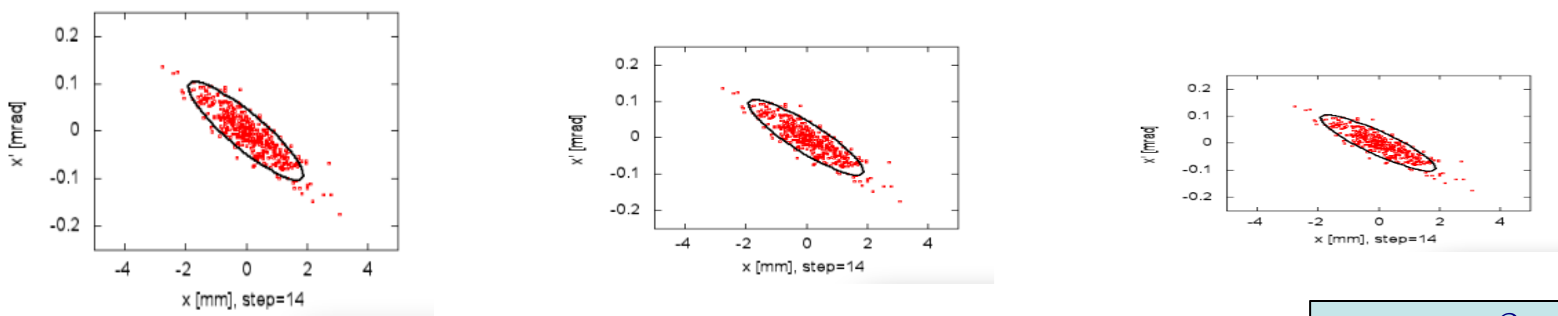
RMS beam size depends on s!



RMS emittance convention is fairly standard for electron rings, with units of mm-mrad

Adiabatic Damping and Normalized Emittance

- But we introduce electric fields when we accelerate
  - When we accelerate, invariant emittance is not invariant!
  - We are defining areas in (x, x') phase space
  - The definition of x doesn’t change as we accelerate
  - But $x' \equiv dx/ds = p_x/p_0$ **does** since p_0 changes!
  - p_0 scales with relativistic beta, gamma: $p_0 \propto \beta\gamma$
  - This has the effect of compressing x’ phase space by $\beta\gamma$

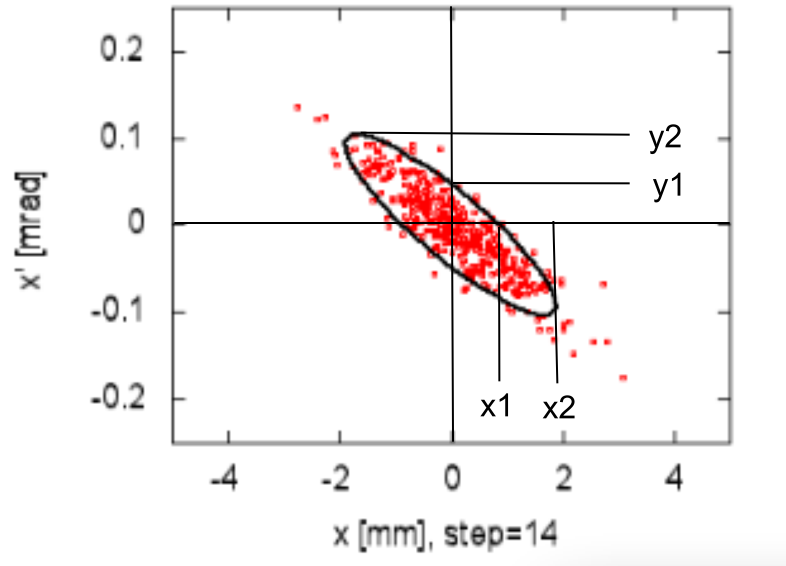


  - **Normalized emittance** is the invariant in this case $\boxed{\epsilon_{\mathrm{N}} \equiv \beta\gamma\epsilon}$

    unnormalized emittance goes down as we accelerate

    This is called **adiabatic damping**, important in, e.g., linacs

Phase Space Ellipse Geography



- Now we can figure out some things from a phase space ellipse at a given s coordinate:

$$x_1 = \sqrt{\mathcal{W}/\gamma(s)} \qquad x_2 = \sqrt{\mathcal{W}\beta(s)}$$

$$y_1 = \sqrt{\mathcal{W}/\beta(s)} \qquad y_2 = \sqrt{\mathcal{W}\gamma(s)}$$

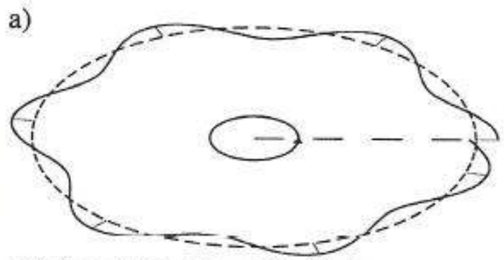
Rings and Tunes

- A synchrotron is by definition a periodic focusing system
  - It is very likely made up of many smaller periodic regions too
  - We can write down a periodic **one-turn matrix** as before

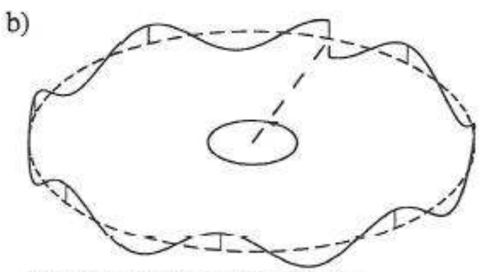
$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

  - We define **tune** as the total betatron phase advance in one revolution around a ring divided by the total angle 2π

$$Q_{x,y} = \frac{\Delta\mu_{x,y}}{\Delta\theta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$



Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation
with tune: $Q_v = 7.5$,
i.e., 7.5 oscillations per turn.

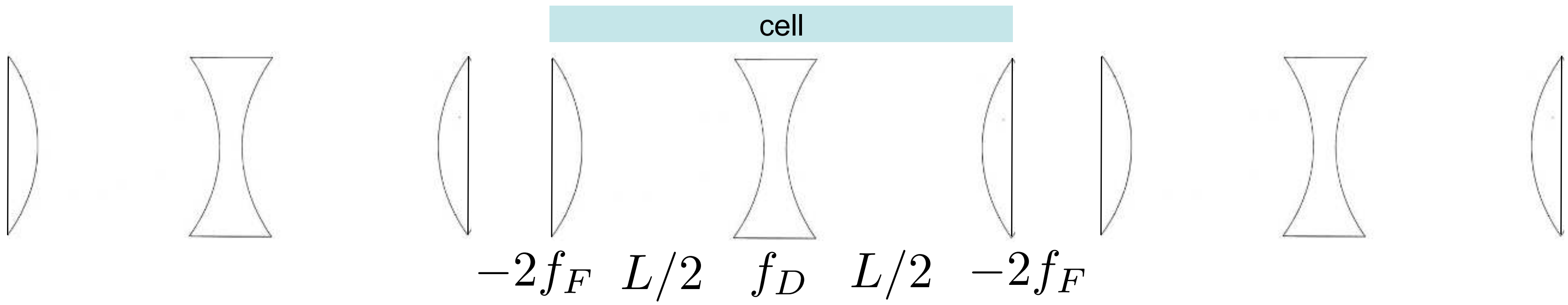
Tunes

- There are horizontal and vertical tunes
  - turn by turn oscillation frequency
- Tunes are a direct indication of the amount of focusing in an accelerator
  - Higher tune implies tighter focusing, lower $\langle \beta_{x,y}(s) \rangle$
- Tunes are a critical parameter for accelerator performance
  - Linear stability depends greatly on phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted by changing groups of quadrupoles

$$\boxed{M_{\text{one turn}} = I \cos(2\pi Q) + J \sin(2\pi Q)}$$

6.2: Stability Diagrams

- Designers often want or need to change the focusing of the two transverse planes in a FODO structure
  - What happens if the focusing/defocusing strengths differ?



  - Recalculate the M matrix and use dimensionless quantities

$$F \equiv \frac{L}{2f_F} \qquad D \equiv \frac{L}{2f_D}$$

then take the trace for stability conditions to find

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad\qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

Stability Diagrams II

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

- For stability, we must have $-1 < \cos\mu < 1$
- Using $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$, stability limits are where

$$\sin^2\frac{\mu}{2} = 0 \qquad \sin^2\frac{\mu}{2} = 1$$

- These translate to an a “necktie” stability diagram for FODO

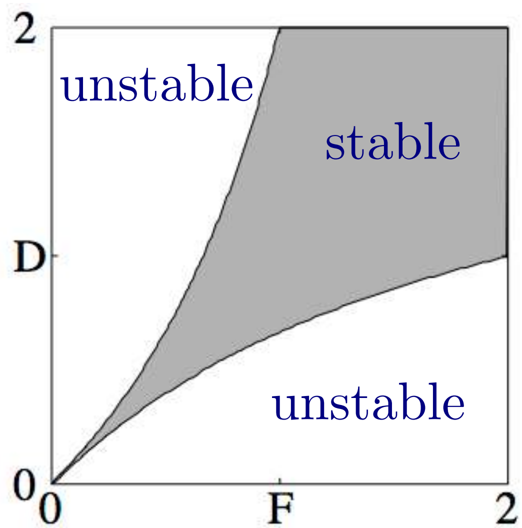


Figure. 6.1 Stability or “necktie” diagram for an alternate focusing lattice. The shaded area is the region of stability.

6.3: Dispersion

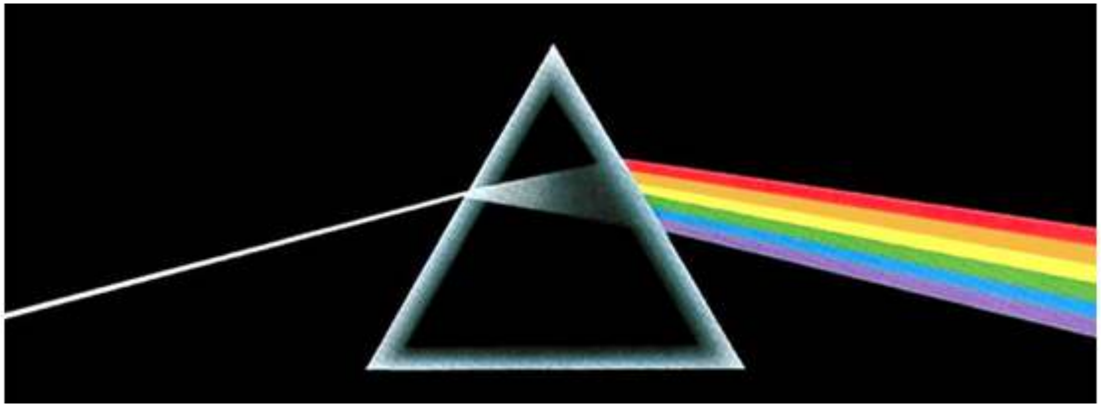
- There is one more important lattice parameter to discuss
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

$$x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$$

Dispersion originates from momentum dependence of dipole bends

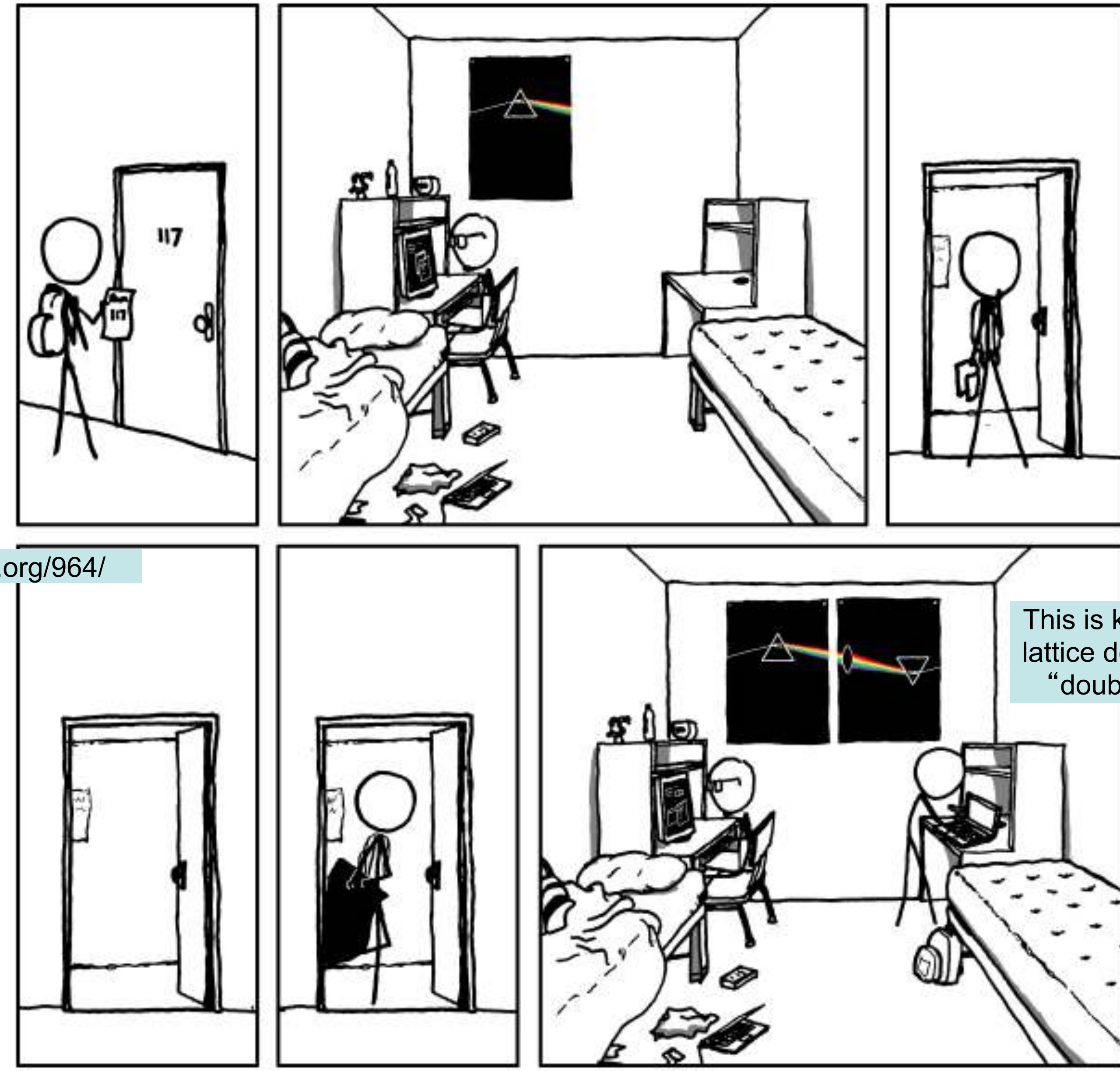
Equivalent to separation of optical wavelengths in prism

White light with many frequencies (momenta) enters, all with same initial trajectories (x,x')



Different positions due to different bend angles of different wavelengths (frequencies, momenta) of incoming light

(xkcd interlude)



http://www.xkcd.org/964/

This is known in accelerator lattice design language as a "double bend achromat"

Dispersion

- Add explicit momentum dependence to equation of motion again

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

Assume our ansatz solution and use initial conditions to find

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x_0' + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Particular solution of inhomogeneous differential equation with periodic ρ(s)

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}$$

The trajectory has two parts:

$$\boxed{x(s) = \mathrm{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}}$$

Dispersion Continued

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat} : D = D' = 0$$

  - If we take $\delta_0 = 1$ we can solve this in a clever way

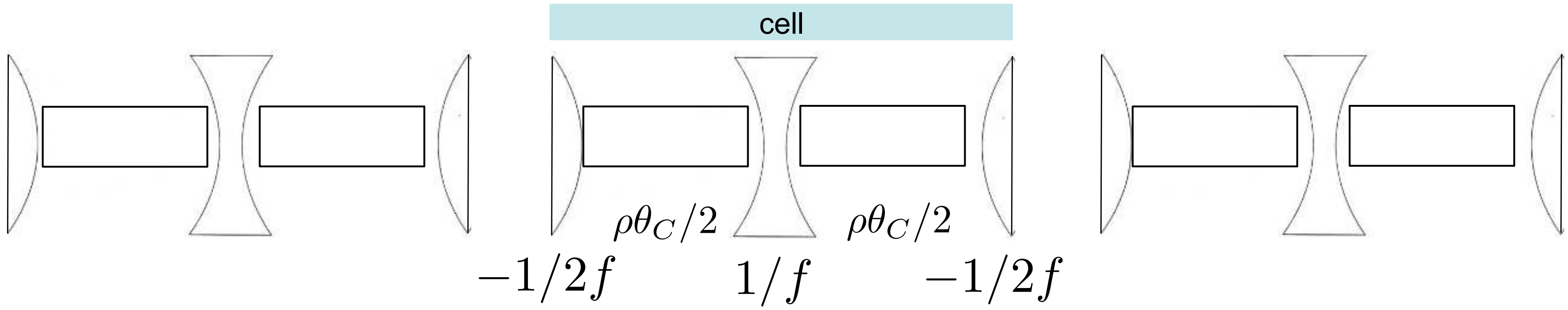
$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

  - Solving gives

$$\boxed{\begin{aligned} \eta(s) &= \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\mu)} \\ \eta'(s) &= \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\mu)} \end{aligned}}$$

FODO Cell Dispersion



- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\mathrm{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\mathrm{FODO}} = M_{-2f} M_{\mathrm{dipole}} M_f M_{\mathrm{dipole}} M_{-2f}$$

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

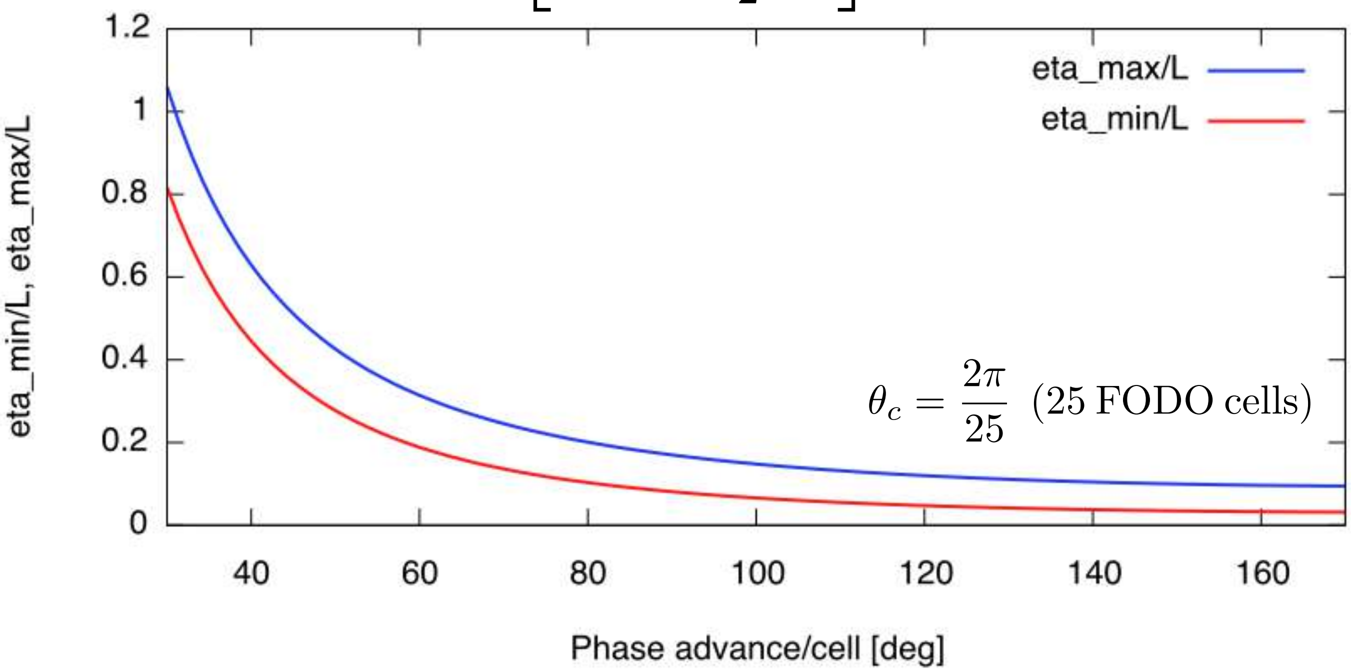
FODO Cell Dispersion

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

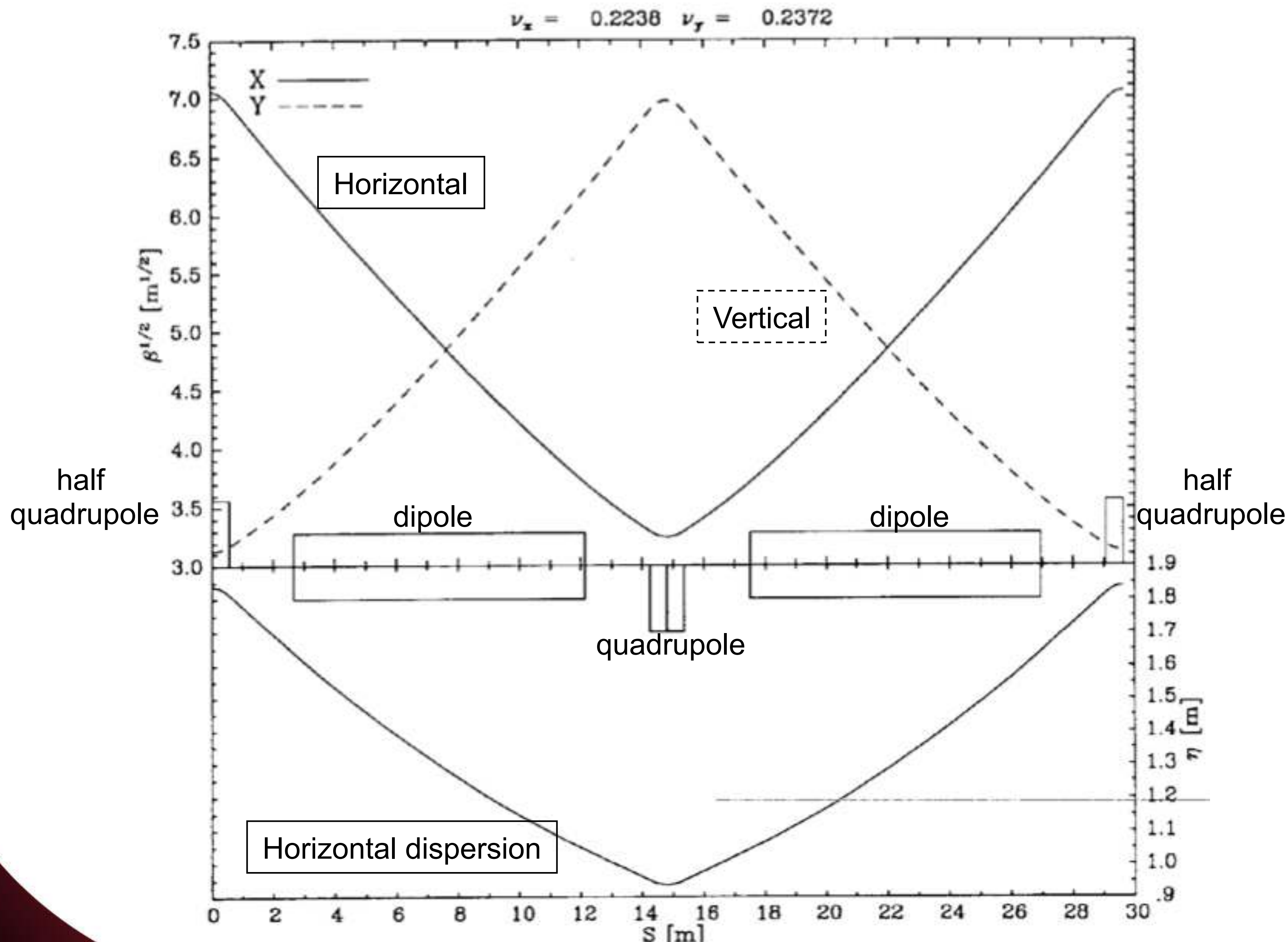
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at min}$$

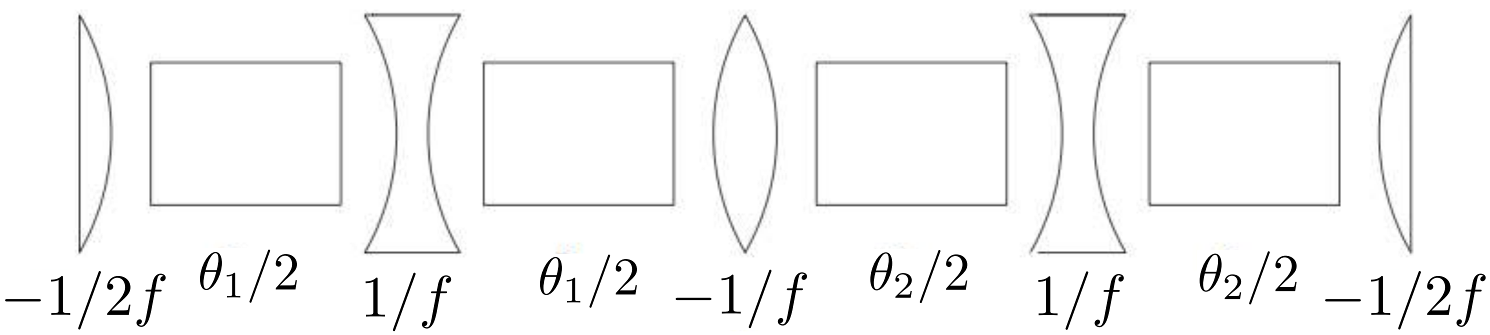


RHIC FODO Cell



6.6: Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO (η_x, η'_x) to zero.



  - Consider two FODO cells with different total bend angles θ_1, θ_2
    - Same quadrupole focusing to not disturb $\beta_x,\ \mu_x$ much
    - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
    - $\alpha_x = 0$ at ends to simplify periodic matrix

FODO Dispersion Suppressor

Zero dispersion area

slope η' =0

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\mu_x & \beta_x \sin 2\mu_x & D(s) \\ -\frac{\sin 2\mu_x}{\beta_x} & \cos 2\mu_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

FODO peak dispersion,

slope η' =0

$$\text{multiply matrices} \quad \Rightarrow \quad D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

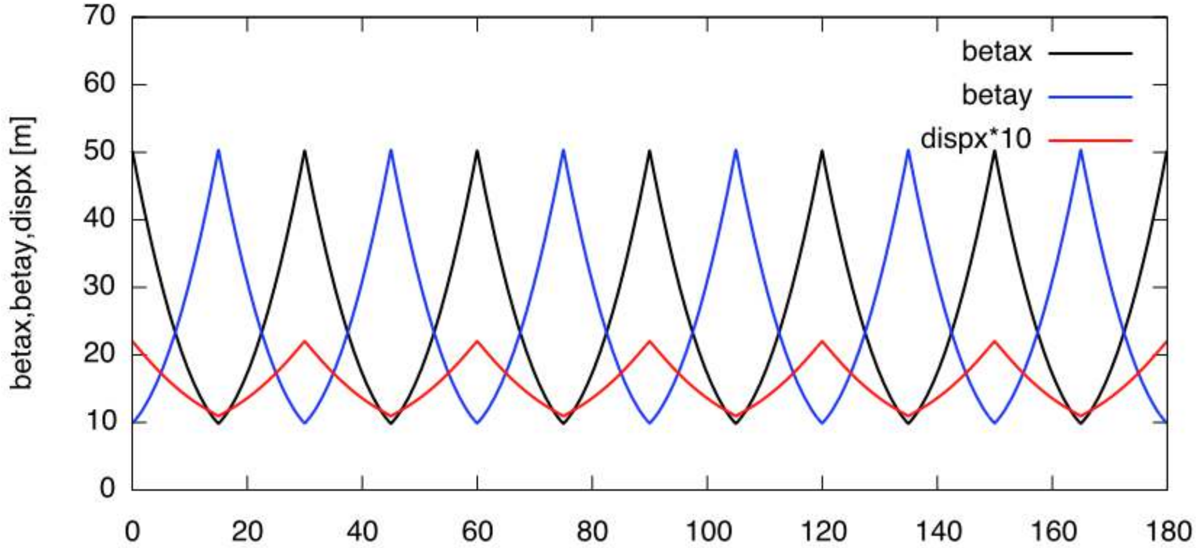
$$D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

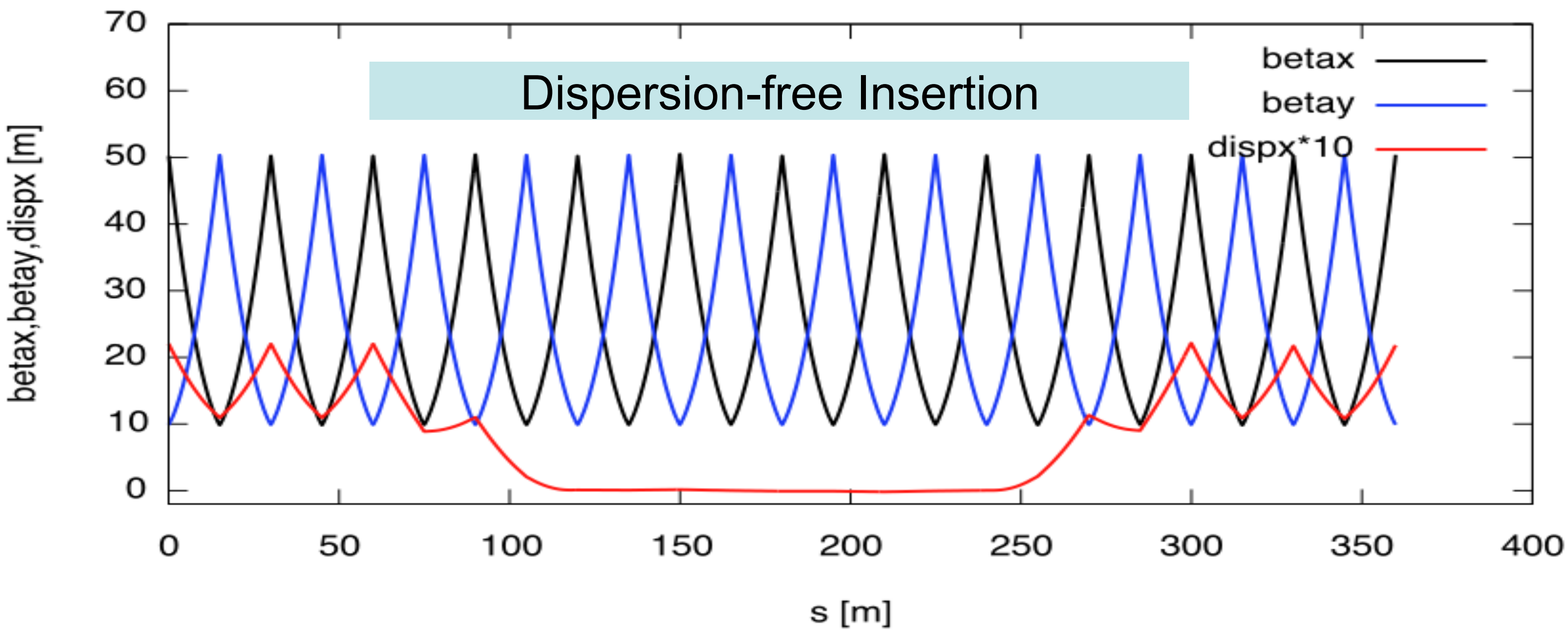
$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

$$\boxed{\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta}$$

$$\boxed{\theta = \theta_1 + \theta_2} \quad \text{two cells, one FODO bend angle} \quad \rightarrow \quad \text{reduced bending}$$

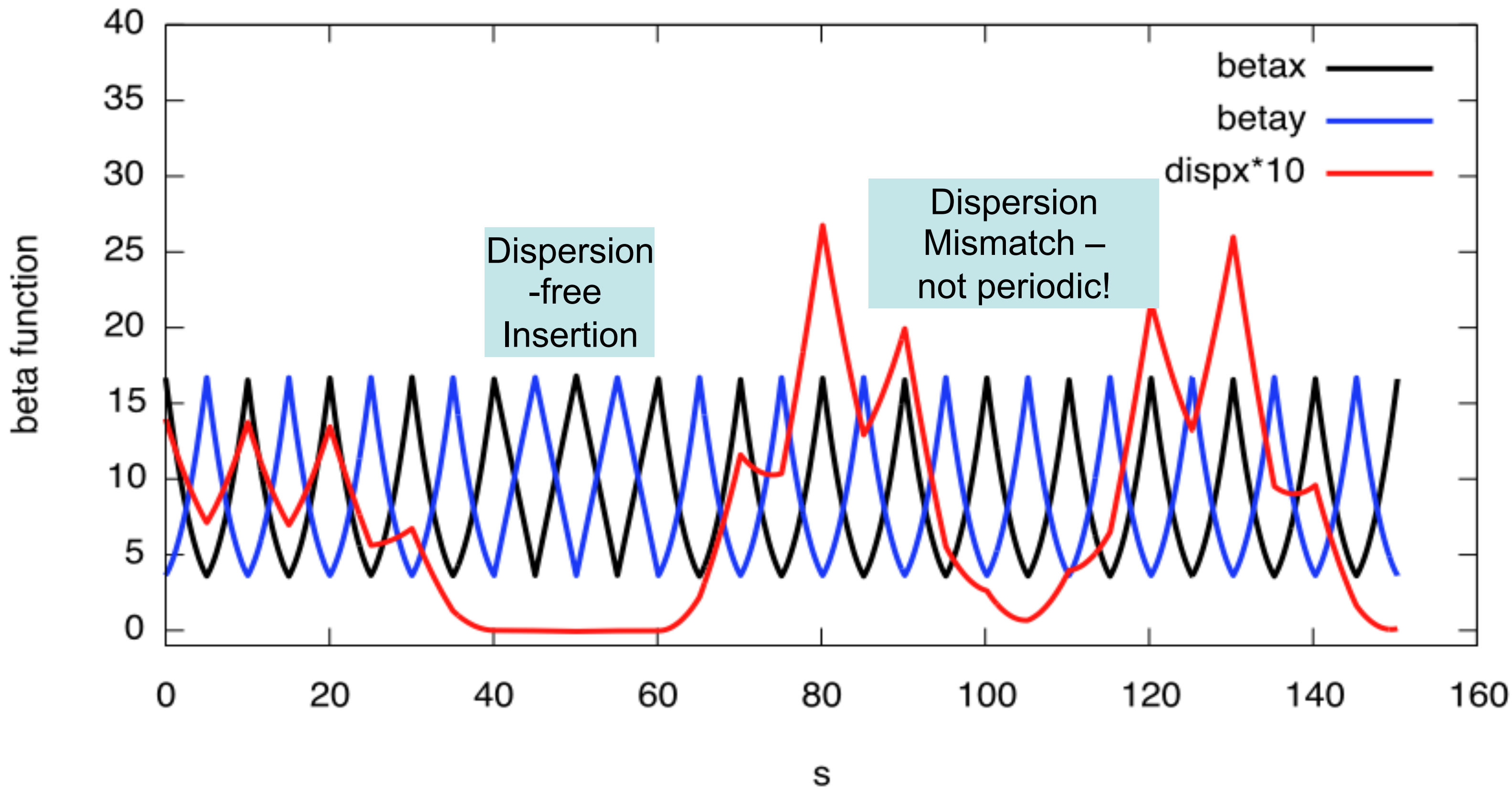
FODO Cell Dispersion and Suppressor





Mismatched Dispersion

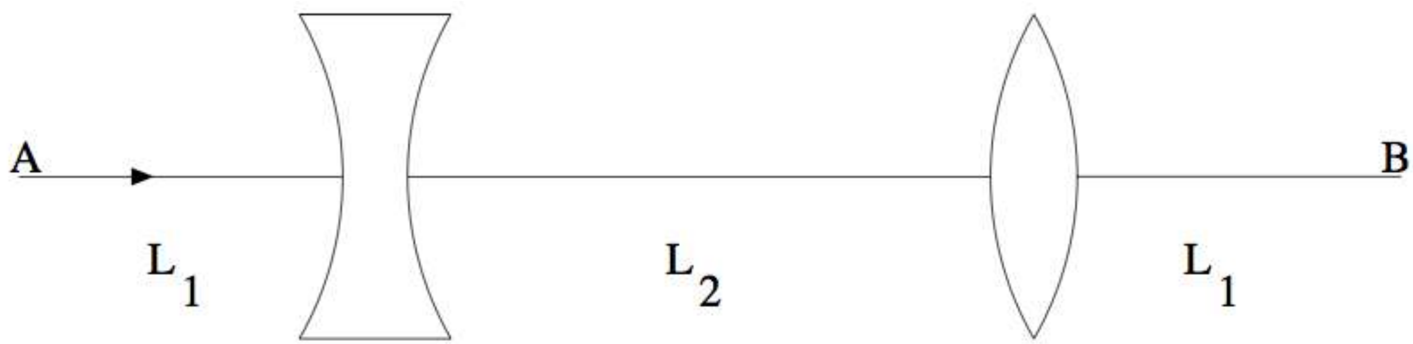
- Someone in class asked what mismatched dispersion looks like
  - For example, this is what happens when the second dispersion suppressor is eliminated and the dipole-free FODO cells run right up against the FODO cells with dipoles



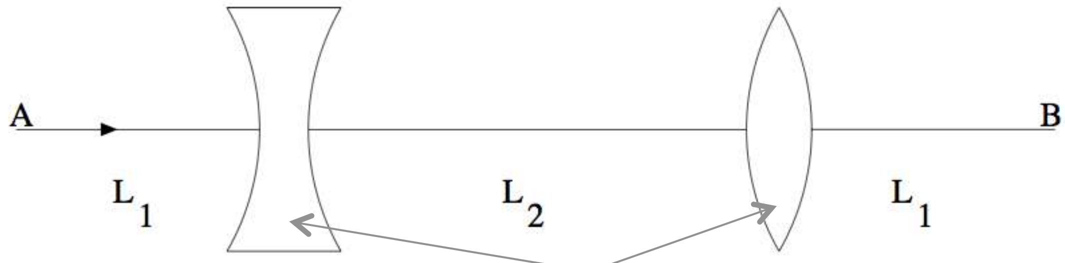
6.5: $\pi/2$ Insertion

- Insertions and matching: **modular** accelerator design
- FODO sections have very regular spacings of quads
  - Periodicity of quadrupoles => periodicity of focusing
- But we need some long quadrupole-free sections
  - RF, injections, extraction, experiments, long instruments

- Can we design a “module” that fits in a FODO lattice with a long straight section, and matches to FODO optics?
  - Yes: a minimal option is called the $\pi/2$ insertion
  - Matching lattice functions $(\beta,\alpha)_{x,y}$ at locations A,B



π/2 Insertion



$$\mathbf{M} = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}$$

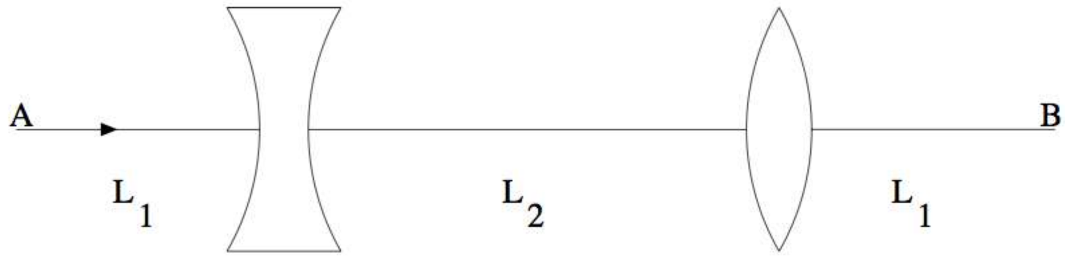
$$\mathbf{M} = \begin{pmatrix} 1 - \frac{l_1 l_2}{f^2} & 2l_1 + l_2 - \frac{l_1^2 l_2}{f^2} \\ -\frac{l_2}{f^2} & 1 - \frac{l_1 l_2}{f^2} - \frac{l_2}{f} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

$$\cos\mu = 1 - \frac{l_1 l_2}{f^2} \qquad \beta \sin\mu = \left(2 - \frac{l_1 l_2}{f^2}\right) l_1 + l_2 \qquad \gamma \sin\mu = \frac{l_2}{f^2}$$

$$m_{11} - m_{22} \text{ comparison :} \quad l_2 = \alpha f \sin\mu$$

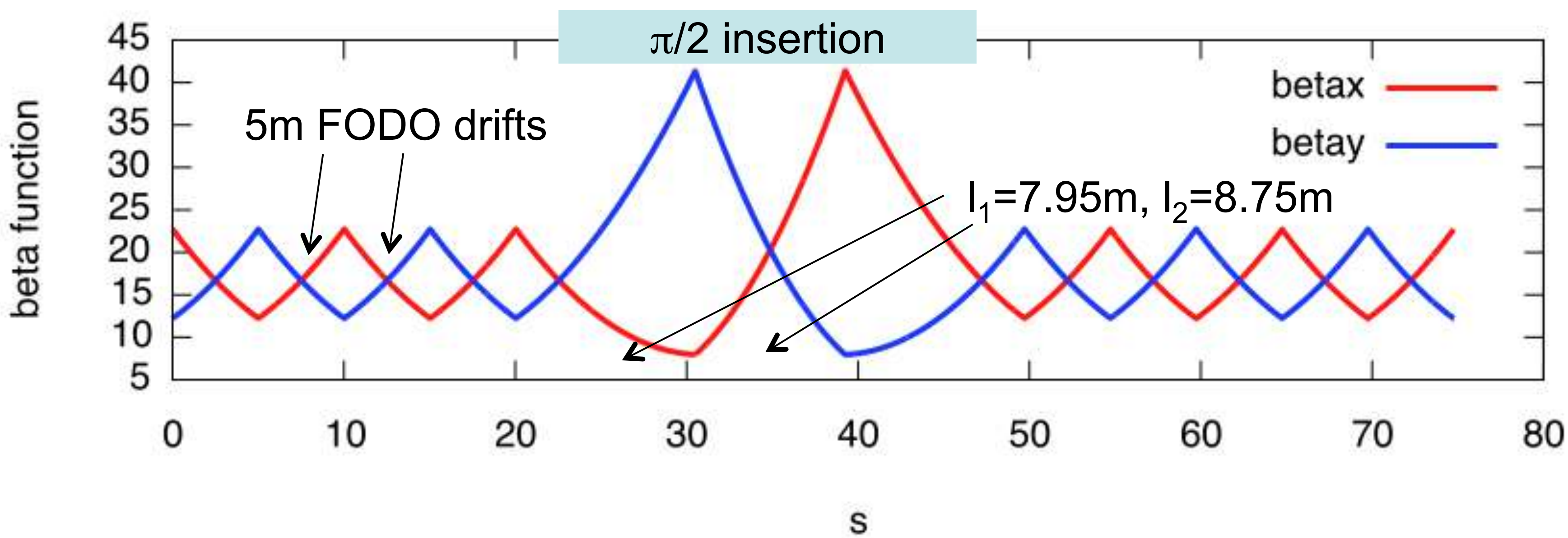
$$\text{Maximum } l_2 \text{ when } \sin\mu = 1, \ \mu = \frac{\pi}{2}, \ \cos\mu = 0$$

$\pi/2$ Insertion

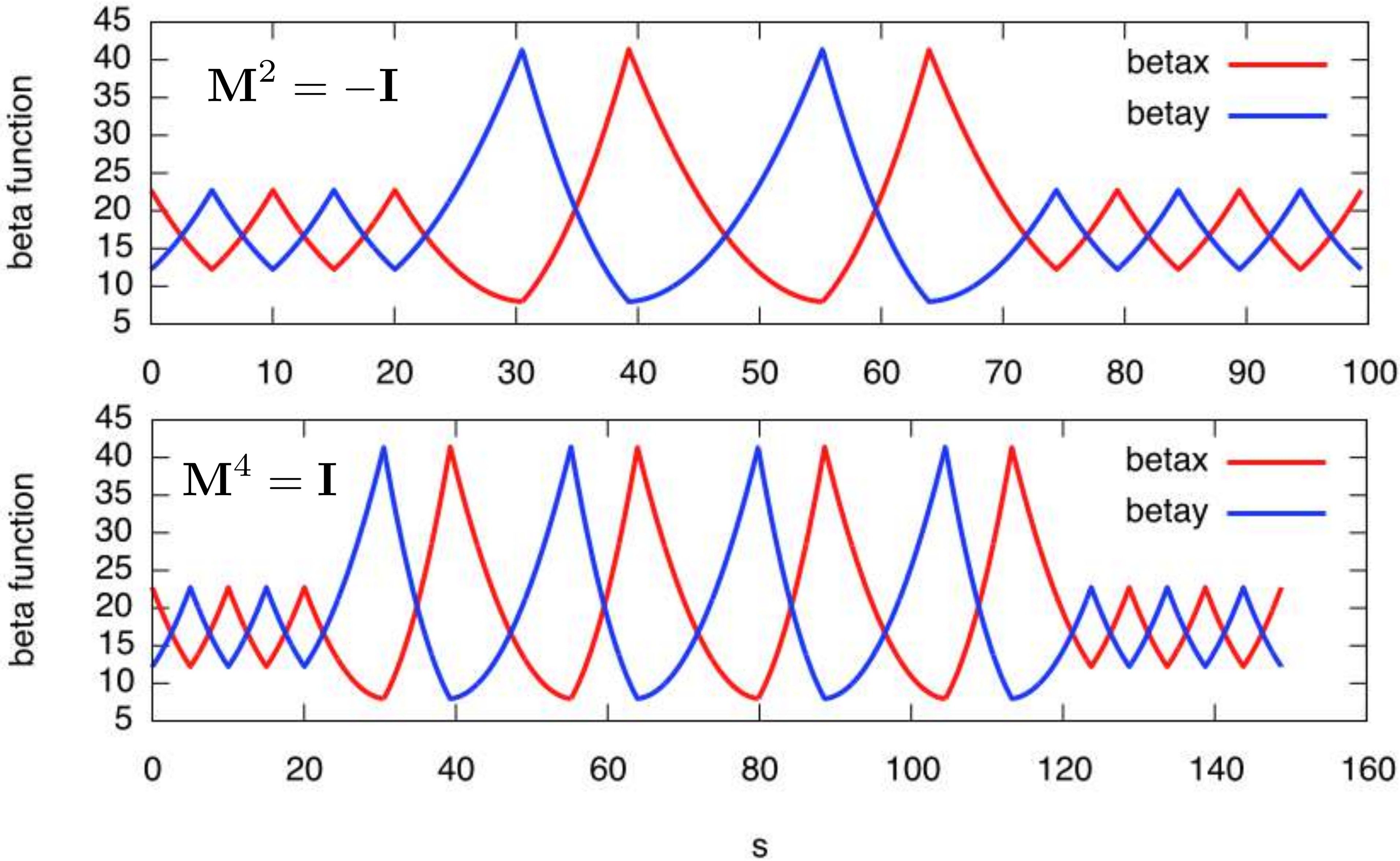


$$\text{Design constraints}: f = \frac{\alpha}{\gamma} \quad l_2 = \frac{\alpha^2}{\gamma} \quad l_1 = \beta - l_2$$

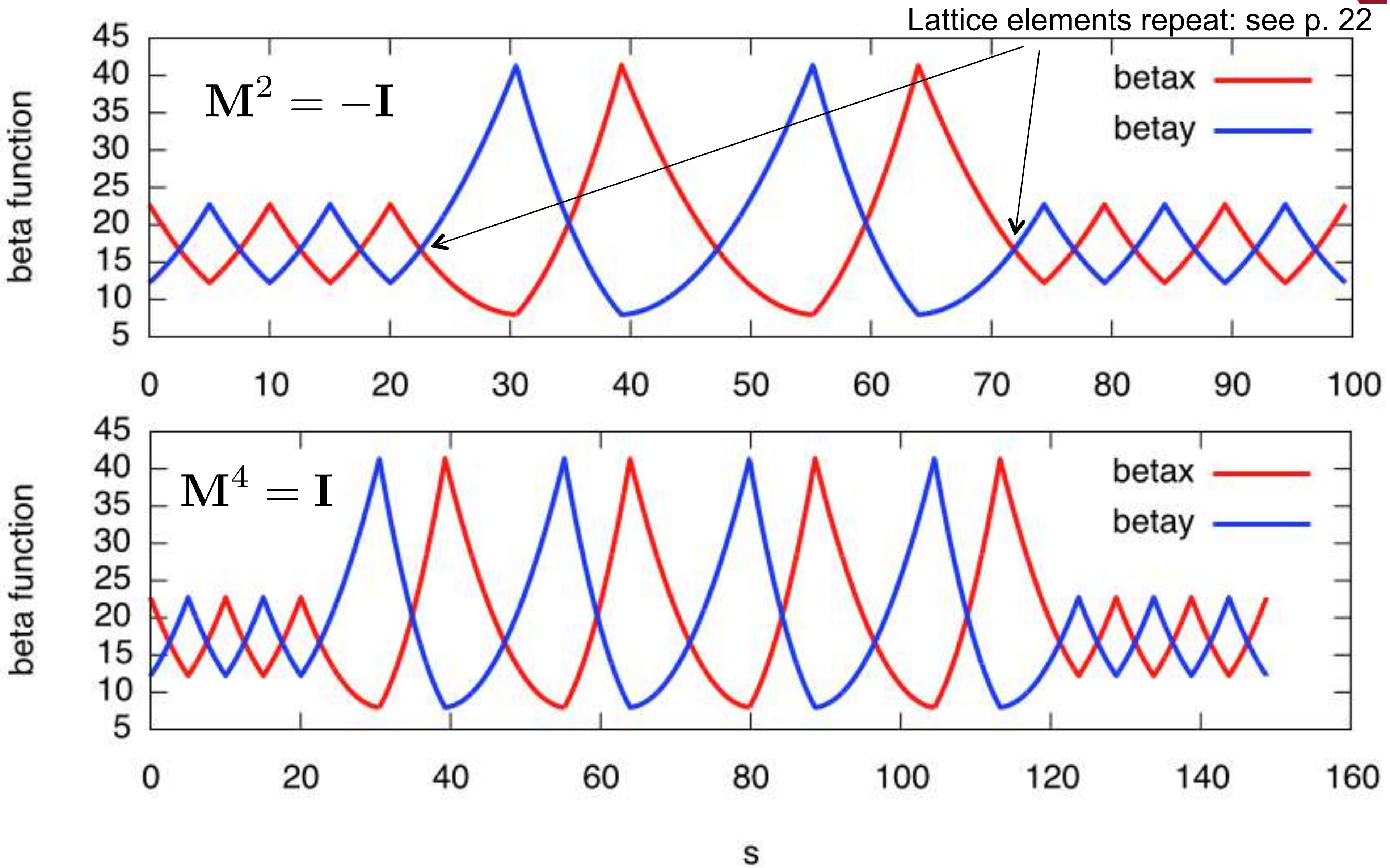
$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J \qquad (\text{recall } J^2 = -I)$$



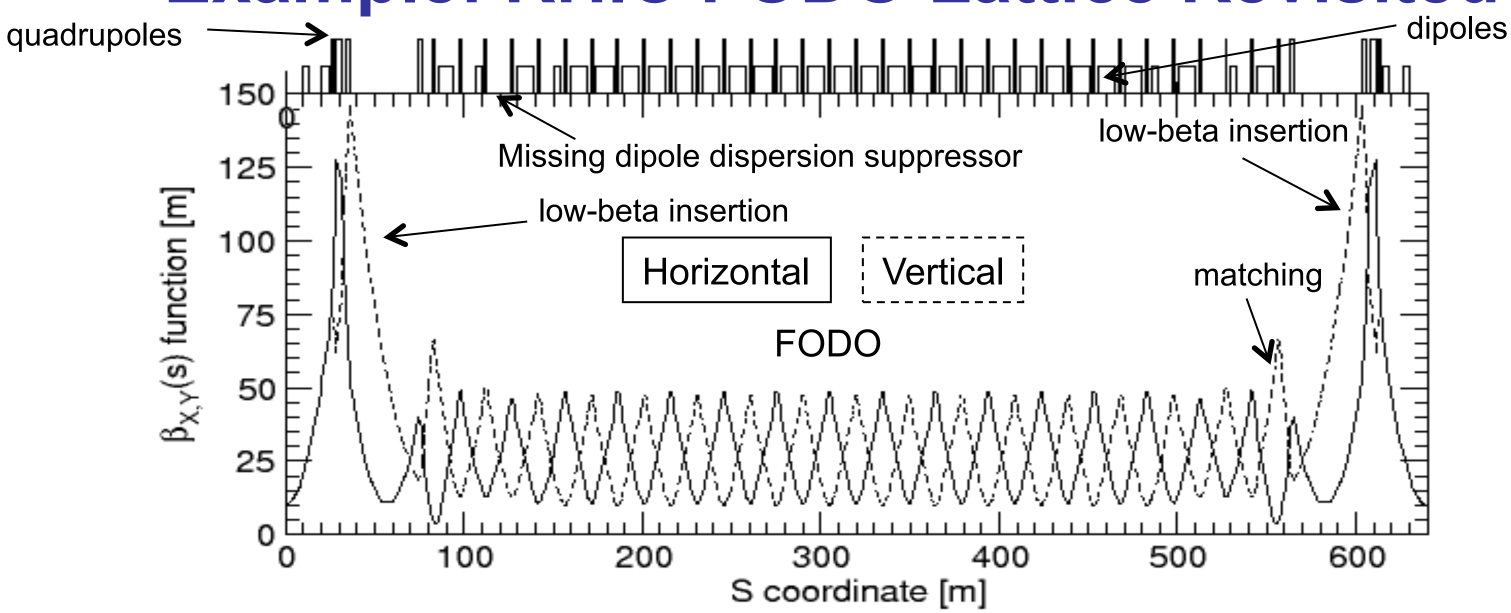
Multiple π/2 Insertions



Multiple π/2 Insertions



Example: RHIC FODO Lattice Revisited



- Note modular design, including low-beta insertions
  - Used for experimental collisions
  - Minimum beam size σ (with zero dispersion)
    - **maximize luminosity**
  - Large s, beam size in "low beta quadrupoles"
  - Other facilities also have longitudinal bunch compressors
    - Minimize longitudinal beam size (bunch length) for, e.g, FELs