

Supplementary Lecture Notes for Accelerator Physics

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E&M and Math Formulae Review

Appendix E of Conte & MacKay contains some other useful stuff.

Speed of light in vacuum: $c \equiv 299792458 \text{ m/s} \simeq 30 \text{ cm/ns}$

Fields (See Chapter 9 of Conte & MacKay):

$$\vec{D} = \epsilon \vec{E} \quad \vec{B} = \mu \vec{H} \quad \epsilon = \epsilon_r \epsilon_0 \quad \mu = \mu_r \mu_0$$

For free space (vacuum): $\epsilon_r = \mu_r = 1$

$$\mu_0 \equiv 4\pi \times 10^{-7} \text{ Tm/A}, \quad \epsilon_0 = \frac{1}{\mu_0 c^2} \simeq 8.85 \times 10^{-12} \text{ C/V/m}^2$$

Maxwell's Equations:

Gauss' Laws		Stoke's Laws	
$\nabla \cdot \vec{D} = \rho$	$\nabla \cdot \vec{B} = 0$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
$\iint_{\partial V} \vec{D} \cdot d\vec{S} = \iiint_V \rho dV$	$\iint_{\partial V} \vec{B} \cdot d\vec{S} = 0$	$\oint \vec{E} \cdot d\vec{\ell} = \frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{S}$	$\oint \vec{H} \cdot d\vec{\ell} = \iint \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \iint \vec{D} \cdot d\vec{S}$

Steady-state boundary conditions between two media:

$$\begin{aligned} (\vec{D}_2 - \vec{D}_1) \cdot \hat{n} &= \rho_{\text{surface}} \\ (\vec{B}_2 - \vec{B}_1) \cdot \hat{n} &= 0 \\ (\vec{E}_2 - \vec{E}_1) \times \hat{n} &= 0 \\ (\vec{H}_2 - \vec{H}_1) \times \hat{n} &= -\vec{J}_{\text{surface}} \end{aligned}$$

Fields in terms of potentials:

$$\begin{aligned} \vec{E} &= -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla \times \vec{A} \end{aligned}$$

Coulomb's law:

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{r}$$

Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Energy density:

$$u = \frac{1}{2}(\vec{D} \cdot \vec{E} + \vec{H} \cdot \vec{B})$$

Momentum density:

$$\vec{g} = \frac{\vec{E} \times \vec{H}}{c^2}$$

Poynting vector (energy flow):

$$\vec{S} = \vec{E} \times \vec{H}$$

Some trig identities and other useful stuff:

$$e^{ix} = \cos x + i \sin x$$

$$e^x = \cosh x + \sinh x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \quad \sinh(\theta + \phi) = \sinh \theta \cosh \phi + \cosh \theta \sinh \phi$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \quad \cosh(\theta + \phi) = \cosh \theta \cosh \phi + \sinh \theta \sinh \phi$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

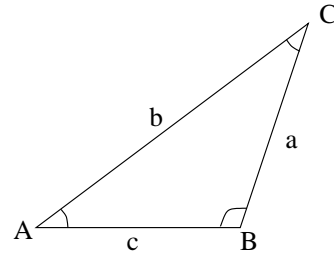
$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$



$$\int \frac{dx}{1+x^2} = \tan^{-1} x$$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1$$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2$$

Fourier transforms:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \hat{f}(\omega) d\omega$$

Poisson sum formula:

$$\sum_{n=-\infty}^{\infty} \delta(t - n\tau) = \frac{1}{\tau} \sum_{m=-\infty}^{\infty} e^{i\frac{m2\pi t}{\tau}}$$

Parseval's theorem:

$$h(t) = \int f(t')g(t-t')dt' \quad \Leftrightarrow \quad \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$$

Characteristic Equations of Symplectic Matrices

Given an $m \times m$ matrix \mathbf{M} , we can define a characteristic polynomial by

$$P(\lambda) = |\mathbf{M} - \mathbf{I}\lambda| = \sum_{j=0}^m A_j \lambda^j, \quad (2.1)$$

so that the characteristic equation for the eigenvalues λ_j is

$$P(\lambda) = \prod_{j=1}^m (\lambda - \lambda_j) = 0. \quad (2.2)$$

Clearly

$$A_0 = P(0) = |\mathbf{M}| = \prod_{j=1}^m \lambda_j, \quad (2.3)$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the m eigenvalues of \mathbf{M} . Expanding the product in Eq. (2.2), it is easy to show that

$$A_1 = -\text{tr}(\mathbf{M}) = -\sum_{j=1}^m M_{jj}, \quad \text{and} \quad A_m = (-1)^m. \quad (2.5)$$

It should also be noted that the coefficients A_j are all invariants under similarity transformations of the type $\mathbf{M} \rightarrow \mathbf{W}\mathbf{M}\mathbf{W}^{-1}$ for any nonsingular $m \times m$ matrix \mathbf{W} .

If we now restrict ourselves to nonsingular matrices, i. e. $|\mathbf{M}| \neq 0$, then \mathbf{M}^{-1} will exist. Note that none of the eigenvalues of \mathbf{M} will be zero in this case. Another characteristic polynomial for \mathbf{M}^{-1} is defined by

$$Q(\mu) = |\mathbf{M}^{-1} - \mathbf{I}\mu| = \sum_{j=0}^m B_j \mu^j, \quad (2.6)$$

with m eigenvalues μ_j found from the equation $Q(\mu_j) = 0$. Dividing $P(\lambda)$ by $|\mathbf{M}|$ produces

$$\frac{1}{\lambda^m |\mathbf{M}|} P(\lambda) = |\mathbf{M}^{-1} - \mathbf{I}\lambda^{-1}| = Q(\lambda^{-1}), \quad (2.7)$$

so the eigenvalues of the \mathbf{M}^{-1} are simply the inverse of the eigenvalues of the \mathbf{M} : $\mu_j = \lambda_j^{-1}$. Combining Eqs. (2.1, 2.6, and 2.7) yields

$$\sum_{j=0}^m A_j \lambda^j = |\mathbf{M}| \sum_{j=0}^m B_{m-j} \lambda^j, \quad (2.8)$$

which must hold for all values of λ , so we obtain the condition

$$A_j = |\mathbf{M}| B_{m-j}. \quad (2.9)$$

2.1 Symplectic restriction

Now if $\mathbf{M} \in \text{Sp}(2n, \mathbb{R}; \mathbf{S})$ is a symplectic matrix with $m = 2n$, then $|\mathbf{-M}| = (-1)^{2n}|\mathbf{M}| = 1$, and we get

$$A_j = B_{m-j}. \quad (2.10)$$

Since $\mathbf{M}^{-1} = \mathbf{S}^T \mathbf{M}^T \mathbf{S}$ and $\mathbf{S}^T \mathbf{S} = \mathbf{I}$, we find

$$Q(\mu) = |\mathbf{M}^{-1} - \mathbf{I}\mu| = |\mathbf{S}^T \mathbf{S}| |\mathbf{M}^{-1} - \mathbf{I}\mu| = |\mathbf{S}(\mathbf{S}^T \mathbf{M}^T \mathbf{S} - \mathbf{I}\mu)\mathbf{S}^T| = |\mathbf{M}^T - \mathbf{I}\mu|. \quad (2.11)$$

But \mathbf{M}^T has the same eigenvalues as \mathbf{M} , so we must have $Q(\mu) = P(\mu)$ for a symplectic matrix. This means that the reciprocal of an eigenvalue of a symplectic matrix is also an eigenvalue; they come in pairs: λ_j and $1/\lambda_j$. Additionally, we should note that since \mathbf{M} is real, all the A_j are real and therefore if λ_j is an eigenvalue of \mathbf{M} , then so is its complex conjugate λ_j^* .

Therefore the characteristic polynomial for a symplectic matrix has the symmetry with $A_j = A_{2n-j}$, so

$$P(\lambda) = 1 + A_1 \lambda^1 + A_2 \lambda^2 + \dots + A_{n-1} \lambda^{n-1} + A_n \lambda^n + A_{n-1} \lambda^{n+1} + \dots + A_2 \lambda^{2n-2} + A_1 \lambda^{2n-1} + \lambda^{2n}, \quad (2.12)$$

or slightly more succinctly

$$P(\lambda) = A_n \lambda^n + \sum_{j=0}^{n-1} A_j (\lambda^j + \lambda^{2n-j}). \quad (2.13)$$

Dividing the characteristic equation by λ^n (since none of the eigenvalues can be zero) gives

$$\lambda^{-n} P(\lambda) = A_n + \sum_{j=0}^{n-1} A_j \left(\lambda^{n-j} + \frac{1}{\lambda^{n-j}} \right) = 0. \quad (2.14)$$

Another way to arrive at the symmetry of the A_j is to expand $P(\lambda)$ in terms of the eigenvalues in symbolic terms even though we may not have solved for them. Let us order the m eigenvalues of \mathbf{M} so that the second half are the respective reciprocals of the first n eigenvalues:

$$\lambda_{n+j} = \frac{1}{\lambda_j}. \quad (2.15)$$

Now we can write

$$P(\lambda) = \sum_{j=0}^m A_j \lambda^j = \prod_{j=1}^m (\lambda - \lambda_j). \quad (2.16)$$

To simplify things, let us write the coefficients for the case $2n = m = 6$:

$$A_0 = \det(\mathbf{M}) = 1 = A_6, \quad (2.17a)$$

$$A_1 = A_5 = - \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{k=j+1}^4 \sum_{l=k+1}^5 \sum_{m=l+1}^6 \lambda_i \lambda_j \lambda_k \lambda_l \lambda_m = \sum_{i=1}^6 \lambda_j, \quad (2.17b)$$

$$A_2 = A_4 = \sum_{i=1}^3 \sum_{j=i+1}^4 \sum_{k=j+1}^5 \sum_{l=k+1}^6 \lambda_i \lambda_j \lambda_k \lambda_l = \sum_{i=1}^5 \sum_{j=i+1}^6 \lambda_i \lambda_j, \quad (2.17c)$$

$$A_3 = - \sum_{i=1}^4 \sum_{j=i+1}^5 \sum_{k=j+1}^6 \lambda_i \lambda_j \lambda_k. \quad (2.17d)$$

The eigenvalues of the matrix $\mathbf{K} = \mathbf{M} + \mathbf{M}^{-1}$ are

$$\kappa_j = \lambda_j + \lambda_j^{-1}, \quad (2.18)$$

and the characteristic polynomial of \mathbf{K} can be written as

$$R(\kappa) = |\mathbf{K} - \mathbf{I}\kappa| = \sum_{j=0}^{2n} C_j \kappa^j = \left(\prod_{j=1}^n (\kappa - \kappa_j) \right)^2. \quad (2.19)$$

So we must be able to write $R(\kappa)$ as

$$\sum_{i=0}^n C_i \kappa^i = \left(\sum_{i=0}^n D_i \kappa^i \right)^2 = \sum_{i=0}^n \sum_{j=0}^n D_i D_j \kappa^{i+j} = \sum_{i=0}^{2n} \left(\sum_{j=0}^i D_j D_{i-j} \right) \kappa^i. \quad (2.20)$$

The characteristic equation for \mathbf{K} is the square of an n^{th} order polynomial, and we need only factor $R(\kappa)$ and solve for the n roots of

$$\prod_{j=1}^n (\kappa - \kappa_j) = 0. \quad (2.21)$$

The eigenvalues of \mathbf{M} will then be

$$\lambda_j = \frac{\kappa_j}{2} + \sqrt{\left(\frac{\kappa_j}{2}\right)^2 - 1}, \quad \text{and} \quad \lambda_{n+j} = \frac{\kappa_j}{2} - \sqrt{\left(\frac{\kappa_j}{2}\right)^2 - 1}. \quad (2.22)$$

For $n = 3$, the D_j may be obtained from the C_j

$$D_0 = \sqrt{C_0} = \sqrt{|\mathbf{K}|}, \quad (2.23)$$

$$D_1 = \frac{C_1}{2D_0} = -\frac{\text{tr}(\mathbf{K})}{\sqrt{|\mathbf{K}|}}, \quad (2.24)$$

$$D_2 = \frac{C_2 - D_1^2}{2D_0}, \quad (2.25)$$

$$D_3 = \frac{C_3 - 2D_1 D_2}{2D_0}. \quad (2.26)$$

Expanding and comparing the right-hand products of Eqs. (2.16 and 2.19) with $\kappa_j = \lambda_j + \lambda_j^{-1}$ we find the coefficients of $R(\kappa)$ in terms of the coefficients in the characteristic equation for \mathbf{M} :

$$C_0 = A_3 + 2A_2, \quad (2.27)$$

$$C_1 = A_2 - 3, \quad (2.27)$$

$$C_2 = A_1, \quad (2.27)$$

$$C_3 = 1. \quad (2.27)$$

2.2 Solution of cubic equation

From §3.8.2 of Abramowitz and Stegun we have the polynomial

$$z^3 + a_2z^2 + a_1z + a_0 = 0.$$

To find the three solutions they give the following algorithm:

$$\begin{aligned} q &= \frac{a_1}{3} - \frac{a_2^2}{9}, \\ r &= \frac{a_1a_2 - 3a_0}{6} - \frac{a_2^3}{27}, \\ s_1 &= \left(r + \sqrt{q^3 + r^2} \right)^{1/3}, \\ s_2 &= \left(r - \sqrt{q^3 + r^2} \right)^{1/3}, \\ z_1 &= s_1 + s_2 - \frac{a_2}{3}, \\ z_2 &= -\frac{s_1 + s_2}{2} - \frac{a_2}{3} + i\frac{\sqrt{3}}{2}(s_1 - s_2), \\ z_3 &= -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - i\frac{\sqrt{3}}{2}(s_1 - s_2). \end{aligned}$$

Note that if $q^3 + r^2 \geq 0$, we should take the real cube roots in the equations for s_1 and s_2 . I find that the algorithm has problems in Octave if $q^3 + r^2 < 0$.

The cubic equation above could be thought of as characteristic equation for the diagonal 3×3 matrix

$$\mathbf{M} = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix},$$

i. e.,

$$z^3 - (-a_2)z^2 + a_1z - (-a_0) = 0.$$

In this case we must have the three coefficients

$$-a_0 = |\mathbf{M}| = z_1z_2z_3,$$

$$-a_2 = \text{tr}(\mathbf{M}) = z_1 + z_2 + z_3,$$

$$a_1 = z_1z_2 + z_2z_3 + z_3z_1,$$

which are invariant under similarity transformations

$$\mathbf{M}' = \mathbf{W}\mathbf{M}\mathbf{W}^{-1}$$

where \mathbf{W} is any complex 3×3 matrix with nonzero determinant.

Let's start with the equation:

$$z^3 - b_2z^2 + b_1z - b_0 = 0.$$

First make the substitution $z = x - k$ and find k to eliminate the x^2 term:

$$\begin{aligned} 0 &= (x^3 - 3kx^2 + 3k^2x - k^3) - b_2(x^2 - 2kx + k^2) + b_1(x - k) - b_0. \\ 0 &= x^3 - (3k + b_2)x^2 + (3k^2 + 2kb_2 + b_1)x - (k^3 + b_2k^2 + b_1k + b_0). \\ k &= -\frac{b_2}{3}, \quad \text{i.e.,} \quad z = x + \frac{b_2}{3}. \end{aligned}$$

The equation then becomes

$$\begin{aligned} 0 &= x^3 + (3k^2 + 2kb_2 + b_1)x - (k^3 + b_2k^2 + b_1k + b_0). \\ 0 &= x^3 + \left(\frac{b_2^2}{3} - \frac{2b_2^2}{3} + b_1\right)x - \left(-\frac{b_2^3}{27} + \frac{b_2^3}{9} - \frac{b_1b_2}{3} + b_0\right). \\ 0 &= x^3 + c_1x + c_0, \quad \text{where} \\ c_1 &= b_1 - \frac{b_2^2}{3}, \quad \text{and} \\ c_0 &= -\left(\frac{2b_2^3}{27} - \frac{b_1b_2}{3} + b_0\right). \end{aligned}$$

Now make the substitution $x = y - \frac{c_1}{3y}$, i. e. $z = y - \frac{c_1}{3y} + \frac{b_2}{3}$ to transform the equation to

$$\begin{aligned} 0 &= y^3 - 3\frac{c_1}{3y}y^2 + 3\frac{c_1^2}{9y^2}y - \frac{c_1^3}{27y^3} + c_1y - \frac{c_1^2}{3y} + c_0 \\ 0 &= y^3 - \frac{c_1^3}{27}y^{-3} + c_0 \\ 0 &= (y^3)^2 + c_0(y^3) - \frac{c_1^3}{27}, \quad \text{a quadratic equation in } y^3. \\ y^3 &= -\frac{c_0}{2} \pm \sqrt{\left(\frac{c_0}{2}\right)^2 + \frac{c_1^3}{27}}. \\ q &= \frac{c_1}{3} = \frac{b_1}{3} - \frac{b_2^2}{9}. \\ r &= -\frac{c_0}{2} = -\left(\frac{b_1b_2 - 3b_0^2}{6} - \frac{b_2^3}{27}\right), \\ Y_{\pm} &= y^3 = r \pm \sqrt{r^2 + q^3} = \rho_j e^{i(\theta_j + n2\pi)}, \\ &\quad \text{for } j \in \{1, 2\}, \quad \theta_j \in [0, 2\pi), \quad \text{and } n \in \{0, 1, 2\} \end{aligned}$$

It seems as though we can just take Y_+ to calculate

$$\begin{aligned} y_1 &= \sqrt[3]{r + \sqrt{r^2 + q^3}}, \quad y_2 = y_1 e^{i2\pi/3}, \quad \text{and} \quad y_3 = y_1 e^{i4\pi/3} \\ z_j &= y_j - \frac{q}{y_j} - k = y_j + \frac{3b_1 - b_2^2}{3y_j} + \frac{b_2}{3} \end{aligned}$$

With some tests, it seems that Y_+ or Y_- both generate the same three roots for $n \in \{0, 1, 2\}$. This algorithm works with complex coefficients as well.

References for Chapter 2

- [1] Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover (1972).
- [2] Richard Talman, *Geometric Mechanics*, John Wiley and Sons, New York (2000).
- [3] Robert A. Horn and Charles R. Johnson, *Matrix Analysis*, Cambridge (1985).

3

How to plot the Courant-Snyder ellipse

Given the usual Twiss parameterization of an ellipse,

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \mathcal{W}, \quad \text{with} \quad \gamma\beta - \alpha^2 = 1, \quad (1)$$

the obvious idea which comes to mind is to plot both

$$x' = f_+(x) = \frac{-\alpha + \sqrt{\mathcal{W}\beta - x^2}}{\beta x},$$

$$x' = f_-(x) = \frac{-\alpha - \sqrt{\mathcal{W}\beta - x^2}}{\beta x}.$$

In `gnuplot` this will almost work because `gnuplot` does not complain about the complex values, but only plots the real values; however the left and right ends of the ellipse near x_{\min} and x_{\max} will have little gaps. (The gaps can be lessened by increasing the number of samples via the “`set samples`” command, but this may not be satisfactory.)

How can we eliminate little gaps altogether? Use parametric plotting with the “`set parametric`” command. We can plot a circle of radius r centered at (x_0, y_0) with something like:

```
x0 = 0.1
y0 = 0.0
r = 10
set parametric
plot r*cos(t)+x0,r*sin(t)+y0
```

Since the positive y -intercept of the ellipse in Eq. (1) is just $(x_0, y_0) = (\sqrt{w/\gamma}, 0)$, our desired parameterization will be simply

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t + \alpha \sin t & \beta \sin t \\ -\gamma \sin t & \cos t - \alpha \sin t \end{pmatrix} \begin{pmatrix} \sqrt{w/\gamma} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t + \alpha \sin t \\ -\gamma \sin t \end{pmatrix} \sqrt{\frac{w}{\gamma}}.$$

So a simple `gnuplot` script could be:

```
#!/usr/bin/gnuplot
set parametric
alpha = -1.5; beta = 10.0; gamma = (1.0+alpha**2)/beta
w=1.0e-6
x0 = sqrt(w/gamma); y0 = 0
xoft(t) = (cos(t)+alpha*sin(t)) *x0
yoft(t) = -gamma*sin(t) *x0
plot xoft(t), yoft(t)
```

Matrix Elements: Synchrotron Coupling

For a charged particle of charge q in an external electromagnetic field, we may write the relativistic Hamiltonian as

$$\mathcal{H}(x, P_x, y, P_y, z, P_z; t) = U = \sqrt{(\vec{P} - \vec{A})^2 + m^2 c^4} + q\Phi, \quad (4.1)$$

with vector potential \vec{A} , and electric potential Φ , canonical momentum $\vec{P} = \vec{p} + q\vec{A}$, and total energy U . Here the kinetic momentum $\vec{p} = \gamma\beta mc$. In the usual cylindrical coordinates of accelerator physics with radius of curvature ρ , the Hamiltonian may be written as

$$\begin{aligned} \mathcal{H}(x, P_x, y, P_y, s, P_s; t) &= U \\ &= c \sqrt{(P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2} + m^2 c^2 + q\Phi. \end{aligned} \quad (4.2)$$

Recalling that a canonical transformation from variables (\vec{q}, \vec{p}) to variables (\vec{Q}, \vec{P}) preserves the Poincaré-Cartan integral invariant

$$\vec{p} \cdot d\vec{q} - H dt = \vec{P} \cdot d\vec{Q} - K dt, \quad (4.3)$$

we can interchange one canonical pair (q_j, p_j) with the time and energy pair $(t, -H)$ by writing the invariant as

$$\left(\sum_{i \neq j} p_i dq_i + (-H) dt \right) - (-p_j) dq_j. \quad (4.4)$$

This transformation gives the new Hamiltonian

$$\begin{aligned} \mathbb{H}(x, P_x, y, P_y, t, -U; s) &= -P_s \\ &= -qA_s \\ &\quad - \left(1 + \frac{x}{\rho}\right) \sqrt{\left(\frac{U - q\Phi}{c}\right)^2 - (mc)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2}. \end{aligned} \quad (4.5)$$

If there are no electrostatic fields then we may write $\Phi = 0$; the fields in rf cavities may be obtained from the time derivative of \vec{A} . Ignoring solenoids for now, with only transverse magnetic guide fields and the longitudinal electric fields of the cavities, then it is sufficient to have only A_s , so

$$A_x = 0, \quad A_y = 0, \quad \text{and} \quad \Phi = 0. \quad (4.6)$$

For dipoles, quadrupoles and cavities a vector potential of the form

$$\begin{aligned} qA_s &= q \left(1 + \frac{x}{\rho}\right) (\vec{A} \cdot \hat{s}) \\ &= -\frac{p_{sy}}{\rho} x - \frac{p_{sy}K}{2} (x^2 - y^2) + \dots \\ &\quad + \frac{qV}{\omega_{\text{rf}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos(\omega_{\text{rf}}t + \phi_0). \end{aligned} \quad (4.7)$$

is sufficient. Here the circumference is L , and the magnetic guide field parameter is

$$K = \frac{1}{\rho^2} + \frac{q}{p_{\text{sy}}} \left(\frac{\partial B}{\partial x} \right)_0, \quad (4.8)$$

and p_{sy} is the momentum of the synchronous design particle. The effective rf phase as the synchronous particle passes the cavities is ϕ_0 , to give a net energy gain per turn of $[qV \cos(\phi_0)]$. In Eq. (4.7) the effect of all rf cavities has been lumped at the location $s = 0$ in the ring.

The time coordinate may be broken up into the time for the synchronous particle to arrive at the location s plus a deviation Δt for the particular particle's arrival time:

$$t = t_{\text{sy}}(s) + \Delta t(s) = \frac{2\pi h}{\omega_{\text{rf}} L} s + \Delta t = \frac{s}{\beta c} + \Delta t. \quad (4.9)$$

If the beam is held at constant energy, then we may make a canonical transformation of the time coordinate Δt to rf phase φ given by

$$\varphi = \omega_{\text{rf}} \Delta t. \quad (4.10)$$

If acceleration is assumed to be adiabatically slow, so that ω_{rf} changes very slowly, and the magnetic guide fields track the momentum of the synchronous particle, keeping the synchronous particle on a fixed trajectory, we can allow for an adiabatic energy ramp according to

$$U_{\text{sy}} = U_0 + \frac{qV \sin \phi_0}{L} s, \quad (4.11)$$

where the energy gain per turn $[qV \sin \phi_0]$ is much less than the total energy U_s . In this case it might not unreasonable to use φ as the longitudinal coordinate, so long as we are prepared to allow for adiabatic damping of the phase space areas. To convert the time coordinate into an rf phase angle relative to the phase of the synchronous particle, we can use the generating function

$$F_2(x, p_x, t, W; s) = xp_x + \left[\omega_{\text{rf}} W - \left(U_0 + \frac{qV \sin \phi_0}{L} s \right) \right] t - \frac{2\pi h}{L} W s + \frac{qV \pi h \sin \phi_0}{\omega_{\text{rf}} L^2} s^2, \quad (4.12)$$

to find a new canonical momentum W corresponding to the phase coordinate. This is what was used to arrive at Eq. 7.61 of Ref. [1].*

Before proceeding down this path it will behoove us to examine the effect of ramping the energy. The deviation in energy of another particle of energy U from the synchronous particle may be defined as

$$\Delta U = U - U_{\text{sy}}. \quad (4.13)$$

For the synchronous particle the phase of the rf cavity should be

$$\begin{aligned} \phi_{\text{sy}} &= \phi_0 + \int_0^{t_{\text{sy}}} \omega_{\text{rf}} dt \\ &= \phi_0 + \int_0^{t_{\text{sy}}} \frac{2\pi h \beta c}{L} dt_{\text{sy}} \end{aligned} \quad (4.14)$$

* In writing § 7.6 of Ref. [1], I was following the formalism of Suzuki[2].

With changing energy and the velocity dependence of ω_{rf} , calculation of this integral becomes a problem and φ does not appear to be such an attractive candidate for a canonical coordinate. This is why Chris Iselin chose to take $\zeta = -c\Delta t$ as the longitudinal coordinate variable in the MAD program[3]. Of course there are other parameters which are not necessarily constants in real accelerators. It is quite common to vary the radial position of the closed orbit, as well as the synchronous phase of the rf – particularly during the phase jump at transition crossing. Pulsed quadrupoles are frequently used to cause a rapid change in the transition energy at transition during acceleration.

If we consider a ramp with a constant increase of energy per turn

$$\begin{aligned} U_{\text{sy}} &= U_0 + Rs \quad \text{with} \\ R &= \frac{qV}{L} \sin \phi_0, \end{aligned} \quad (4.15)$$

then the time evolution as a function of path length of the synchronous particles is given by

$$\begin{aligned} t_{\text{sy}}(s) &= \int_0^s \frac{ds}{\beta c} \\ &= \int_0^s \left[1 - \left(\frac{mc^2}{U_0 + Rs'} \right)^2 \right]^{1/2} ds' \\ &= \frac{mc^2}{R} \int_{\frac{mc^2}{U_0}}^{\frac{U_0 + Rs}{mc^2}} \sqrt{1 - \xi^{-2}} d\xi, & s' &= \frac{\xi mc^2 - U_0}{R} \\ &= -\frac{mc^2}{R} \int_{\frac{mc^2}{U_0}}^{\frac{mc^2}{U_0 + Rs}} \sqrt{1 - \eta^2} \frac{d\eta}{\eta^2}, & \xi &= \frac{1}{\eta} \\ &= \frac{mc^2}{R} \int_{\cos^{-1}\left(\frac{mc^2}{U_0}\right)}^{\cos^{-1}\left(\frac{mc^2}{U_0 + Rs}\right)} \tan^2 \theta d\theta, & \eta &= \cos \theta \\ &= \frac{mc^2}{R} [\tan \theta - \theta] \Big|_{\cos^{-1}\left(\frac{mc^2}{U_0}\right)}^{\cos^{-1}\left(\frac{mc^2}{U_0 + Rs}\right)} \\ &= \frac{mc^2}{2R} \left[\frac{U_0 + Rs}{mc^2} \sqrt{1 - \left(\frac{mc^2}{U_0 + Rs} \right)^2} - \frac{U_0}{mc^2} \sqrt{1 - \left(\frac{mc^2}{U_0} \right)^2} \right. \\ &\quad \left. + \cos^{-1} \left(\frac{mc^2}{U_0 + Rs} \right) - \cos^{-1} \left(\frac{mc^2}{U_0} \right) \right] \\ &= \frac{mc^2}{2R} \left[\beta\gamma - \beta_0\gamma_0 + \cos^{-1} \left(\frac{1}{\gamma} \right) - \cos^{-1} \left(\frac{1}{\gamma_0} \right) \right] \end{aligned} \quad (4.16)$$

Provided that the ramping is sufficiently slow, then acceleration may be treated adiabatically.

At least in the adiabatic case, then we can find the new canonical coordinate and Hamiltonian from Eq. (4.12):

$$-U = \frac{\partial F_2}{\partial t} = \omega_{\text{rf}} W - \left(U_0 + \frac{qV \sin \phi_0}{L} s \right) \quad (4.17a)$$

$$\varphi = \frac{\partial F_2}{\partial W} = \omega_{\text{rf}} t - \frac{2\pi h}{L} s \quad (4.17b)$$

$$\begin{aligned}
\frac{\partial F_2}{\partial s} &= -\frac{qV \sin \phi_0}{L} t - \frac{2\pi h}{L} s + \frac{2\pi h qV \sin \phi_0}{\omega_{\text{rf}} L^2} s \\
&= -\frac{qV \sin \phi_0}{L} \left[\frac{2\pi h}{\omega_{\text{rf}} L} s + \Delta t \right] - \frac{2\pi h}{L} s + \frac{2\pi h qV \sin \phi_0}{\omega_{\text{rf}} L^2} s \\
&= -\frac{qV \sin \phi_0}{L} \frac{\varphi}{\omega_{\text{rf}}} - \frac{2\pi h}{L} s \\
&= -\frac{qV \sin \phi_0}{L} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*},
\end{aligned} \tag{4.17c}$$

where we have written $\lambda_{\text{rf}}^* = L/2\pi h$ as a constant effective rf wavelength corrected for the velocity.* Ignoring the vertical coordinate and momentum the new Hamiltonian is

$$\begin{aligned}
H_1(x, p_x, \varphi, W; s) &= \mathbf{H} + \frac{\partial F_2}{\partial s} \\
&= \frac{p_{\text{sy}}}{\rho} x + \frac{p_{\text{sy}} K}{2} x^2 + \frac{qV}{\omega_{\text{rf}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos\left(\phi_0 + \varphi + \frac{s}{\lambda_{\text{rf}}^*}\right) \\
&\quad - \left(1 + \frac{x}{\rho}\right) \left[\frac{(U_{\text{sy}} - \omega_{\text{rf}} W)^2 - m^2 c^4}{c^2} - p_x^2 \right]^{1/2} - \frac{qV \sin \phi_0}{L} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*} \\
&= \frac{p_{\text{sy}}}{\rho} x + \frac{p_{\text{sy}} K}{2} x^2 + \frac{qV}{\omega_{\text{rf}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos\left(\phi_0 + \varphi + \frac{s}{\lambda_{\text{rf}}^*}\right) \\
&\quad - \left(1 + \frac{x}{\rho}\right) \left[p_{\text{sy}}^2 - \frac{2\omega_{\text{rf}} U_{\text{sy}}}{c^2} W + \frac{\omega_{\text{rf}}^2}{c^2} W^2 - p_x^2 \right]^{1/2} - \frac{qV \sin \phi_0}{L} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*} \\
&\simeq \frac{p_{\text{sy}}}{\rho} x + \frac{p_{\text{sy}} K}{2} x^2 + \frac{qV}{\omega_{\text{rf}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos\left(\phi_0 + \varphi + \frac{s}{\lambda_{\text{rf}}^*}\right) \\
&\quad - p_{\text{sy}} \left(1 + \frac{x}{\rho}\right) \left[1 - \frac{\omega_{\text{rf}} U_{\text{sy}}}{p_{\text{sy}}^2 c^2} W + \left(\frac{\omega_{\text{rf}}^2}{2p_{\text{sy}}^2 c^2} - \frac{1}{8} \frac{4U_{\text{sy}}^2 \omega_{\text{rf}}^2}{p_{\text{sy}}^4 c^4} \right) W^2 - \frac{1}{2} \frac{p_x^2}{p_{\text{sy}}^2} \right] \\
&\quad - \frac{qV \sin \phi_0}{L} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*} \\
&\simeq p_{\text{sy}} \left\{ -1 + \frac{K}{2} x^2 + \frac{qV}{\omega_{\text{rf}} p_{\text{sy}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos\left(\phi_0 + \varphi + \frac{s}{\lambda_{\text{rf}}^*}\right) \right. \\
&\quad \left. + \left(1 + \frac{x}{\rho}\right) \left[\frac{\omega_{\text{rf}} U_{\text{sy}}}{p_{\text{sy}}^2 c^2} W + \frac{m^2 \omega_{\text{rf}}^2}{2p_{\text{sy}}^4} W^2 + \frac{1}{2} \frac{p_x^2}{p_{\text{sy}}^2} \right] - \frac{qV \sin \phi_0}{L p_{\text{sy}}} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*} \right\} \\
&\simeq p_{\text{sy}} \left\{ -1 + \frac{K}{2} x^2 + \frac{qV}{\omega_{\text{rf}} p_{\text{sy}}} \sum_{j=-\infty}^{\infty} \delta(s - jL) \cos\left(\phi_0 + \varphi + \frac{s}{\lambda_{\text{rf}}^*}\right) \right. \\
&\quad \left. + \left(1 + \frac{x}{\rho}\right) \left[\frac{1}{\lambda_{\text{rf}}^*} \frac{W}{p_{\text{sy}}} + \frac{1}{\gamma^2 \lambda_{\text{rf}}^{*2}} \left(\frac{W}{p_{\text{sy}}}\right)^2 + \frac{1}{2} \left(\frac{p_x}{p_{\text{sy}}}\right)^2 \right] \right\}
\end{aligned}$$

* The true rf wavelength is actually $\lambda_{\text{rf}} = L/2\pi h\beta$ and is a function of the momentum of the particle, whereas λ_{rf}^* is a constant depending only on the design circumference and harmonic number.

$$\left. - \frac{qV \sin \phi_0}{L p_{\text{sy}}} \frac{\varphi}{\omega_{\text{rf}}} - \frac{s}{\lambda_{\text{rf}}^*} \right\} \quad (4.18)$$

This is essentially the same as Eq. (7.61) of Ref. [1] where the longitudinal variables are

$$\varphi = \omega_{\text{rf}} \Delta t, \quad \text{and} \quad (4.19a)$$

$$W = -\frac{\Delta U}{\omega_{\text{rf}}}, \quad (4.19b)$$

L is the circumference, and for magnets with transverse fields and no horizontal-vertical coupling

$$K = \frac{1}{\rho^2} + \frac{q}{p} \left(\frac{\partial B}{\partial x} \right)_0. \quad (4.20)$$

If we want to calculate matrices for the basic magnetic elements, i. e., normal quads and dipoles, then the summation drops out, since $\delta(s - jL) = 0$ and $V = 0$ away from the rf cavities. Then keeping only terms to second order in the canonical variables we have

$$\begin{aligned} H_1 &\simeq -p_s + \frac{p_s K}{2} x^2 + \frac{p_x^2}{2p_s} \\ &\quad + \left(\frac{U_{\text{sy}} \omega_{\text{rf}}}{p_{\text{sy}} c^2} - \frac{2\pi h}{L} \right) W + \frac{m^2 \omega_{\text{rf}}^2}{2p_{\text{sy}}^3} W^2 + \frac{U_{\text{sy}} \omega_{\text{rf}}}{\rho p_{\text{sy}} c^2} W x \\ &\simeq -p_{\text{sy}} + \frac{p_{\text{sy}} K}{2} x^2 + \frac{p_x^2}{2p_{\text{sy}}} + \frac{m^2 \omega_{\text{rf}}^2}{2p_{\text{sy}}^3} W^2 + \frac{U_{\text{sy}} \omega_{\text{rf}}}{\rho p_{\text{sy}} c^2} W x, \end{aligned} \quad (4.21)$$

since the two terms in the coefficient of W cancel. We may rescale the Hamiltonian by $1/p_{\text{sy}}$ getting

$$H_{1.5} \simeq -1 + \frac{K}{2} x^2 + \frac{1}{2} w_x^2 + \frac{1}{\gamma^2 \lambda_{\text{rf}}^{*2}} w_\phi^2 + \frac{1}{\rho \lambda_{\text{rf}}^*} w_\phi x, \quad (4.22)$$

with the new canonical momenta

$$w_x = \frac{p_x}{p_{\text{sy}}}, \quad \text{and} \quad (4.23a)$$

$$\begin{aligned} w_\phi &= \frac{W}{p_{\text{sy}}} = -\frac{\Delta U}{\omega_{\text{rf}} p_{\text{sy}}}. \\ &= -\frac{\beta \gamma m c^3}{\gamma m c^2} \frac{\Delta p}{\frac{2\pi h \beta c}{L} p_{\text{sy}}} \\ &= -\lambda_{\text{rf}}^* \frac{\Delta p}{p_{\text{sy}}}. \end{aligned} \quad (4.23b)$$

In this case with φ and w_ϕ as canonically conjugate the longitudinal emittance would have units of length (meters), just like the horizontal and vertical planes. (Of course this should be obvious since all three emittances would come from the common Hamiltonian $H_{1.5}$.) In the paraxial approximation, we obviously have $w_x \simeq x'$.

4.1 Quadrupole Matrix

For a normal quadrupole the last term in Eq. (4.22) vanishes, and we have

$$H_{1.5} \simeq -1 + \frac{k}{2}x^2 + \frac{1}{2}w_x^2 + \frac{1}{\gamma^2\lambda_{\text{rf}}^{*2}}w_\phi^2, \quad (4.24)$$

with

$$k = \frac{q}{p_{\text{sy}}} \left(\frac{\partial B}{\partial x} \right)_0. \quad (4.25)$$

Evaluating for the equations of motion, produces

$$\frac{dx}{ds} = \frac{\partial H_{1.5}}{\partial w_x} = w_x \quad (4.26a)$$

$$\frac{dw_x}{ds} = -\frac{\partial H_{1.5}}{\partial x} = -kx \quad (4.26b)$$

$$\frac{d\varphi}{ds} = \frac{\partial H_{1.5}}{\partial w_\phi} = \frac{1}{\gamma^2\lambda_{\text{rf}}^{*2}}w_\phi \quad (4.26c)$$

$$\frac{dw_\phi}{ds} = -\frac{\partial H_{1.5}}{\partial \varphi} = 0 \quad (4.26d)$$

So the infinitesimal matrix of integration should look like

$$\mathbf{I} + \mathbf{G} ds = \begin{pmatrix} 1 & ds & 0 & 0 \\ -k ds & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{ds}{\gamma^2\lambda_{\text{rf}}^{*2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.27)$$

which has the corresponding generator matrix

$$\mathbf{G} ds = \begin{pmatrix} 0 & \frac{1}{\sqrt{k}} & 0 & 0 \\ -\sqrt{k} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma^2\lambda_{\text{rf}}^{*2}\sqrt{k}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \sqrt{k} ds \quad (4.28)$$

Integration leads to the quadrupole transfer matrix

$$\mathbf{M} = \begin{pmatrix} \cos(\sqrt{k}l) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}l) & 0 & 0 \\ -\sqrt{k} \sin(\sqrt{k}l) & \cos(\sqrt{k}l) & 0 & 0 \\ 0 & 0 & 1 & \frac{l}{\gamma^2\lambda_{\text{rf}}^{*2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.29)$$

4.2 Drift Matrix

For a drift $k = 0$ and Eq. (4.27) becomes

$$\mathbf{I} + \mathbf{G} ds = \begin{pmatrix} 1 & ds & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{ds}{\gamma^2\lambda_{\text{rf}}^{*2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.30)$$

and leads to the full matrix

$$\mathbf{M} = \begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{l}{\gamma^2\lambda_{\text{rf}}^{*2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.31)$$

4.3 Sector Bend Matrix

$$K = \frac{1}{\rho^2} + \frac{1}{\rho^2} \frac{\rho}{B_0} \left(\frac{\partial B}{\partial x} \right)_0 = \frac{1-n}{\rho^2} \quad (4.32)$$

where n is the field index.

$$H_{1.5} \simeq -1 + \frac{1-n}{2\rho^2} x^2 + \frac{1}{2} w_x^2 + \frac{1}{2\gamma^2 \lambda_{rf}^{*2}} w_\phi^2 + \frac{1}{\rho \lambda_{rf}^*} w_\phi x, \quad (4.33)$$

$$\frac{dx}{ds} = \frac{\partial H_{1.5}}{\partial w_x} = w_x \quad (4.34a)$$

$$\frac{dw_x}{ds} = -\frac{\partial H_{1.5}}{\partial x} = -\frac{1-n}{\rho^2} x - \frac{1}{\rho \lambda_{rf}^*} w_\phi \quad (4.34b)$$

$$\frac{d\varphi}{ds} = \frac{\partial H_{1.5}}{\partial w_\phi} = \frac{1}{\gamma^2 \lambda_{rf}^{*2}} w_\phi + \frac{1}{\rho \lambda_{rf}^*} x \quad (4.34a)$$

$$\frac{dw_\phi}{ds} = -\frac{\partial H_{1.5}}{\partial \varphi} = 0 \quad (4.34a)$$

$$\mathbf{I} + \mathbf{G} ds = \begin{pmatrix} 1 & ds & 0 & 0 \\ -\frac{1-n}{\rho^2} ds & 1 & 0 & -\frac{1}{\rho \lambda_{rf}^*} ds \\ \frac{1}{\rho \lambda_{rf}^*} ds & 0 & 1 & \frac{1}{\gamma^2 \lambda_{rf}^{*2}} ds \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.35)$$

$$\mathbf{G} = \frac{1}{\rho} \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\frac{1-n}{\rho} & 0 & 0 & -\frac{1}{\lambda_{rf}^*} \\ \frac{1}{\lambda_{rf}^*} & 0 & 0 & \frac{\rho}{\gamma^2 \lambda_{rf}^{*2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.36a)$$

$$\mathbf{G}^2 = \frac{1}{\rho^2} \begin{pmatrix} -(1-n) & 0 & 0 & -\frac{\rho}{\lambda_{rf}^*} \\ 0 & -(1-n) & 0 & 0 \\ 0 & \frac{\rho}{\lambda_{rf}^*} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.36b)$$

$$\mathbf{G}^3 = \frac{1}{\rho^3} \begin{pmatrix} 0 & -\rho(1-n) & 0 & 0 \\ \frac{(1-n)^2}{\rho} & 0 & 0 & \frac{1-n}{\lambda_{rf}^*} \\ -\frac{1-n}{\lambda_{rf}^*} & 0 & 0 & -\frac{\rho}{\lambda_{rf}^{*2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.36c)$$

$$\mathbf{G}^4 = \frac{1}{\rho^4} \begin{pmatrix} (1-n)^2 & 0 & 0 & \frac{(1-n)\rho}{\lambda_{rf}^*} \\ 0 & (1-n)^2 & 0 & 0 \\ 0 & -\frac{(1-n)\rho}{\lambda_{rf}^*} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{1-n}{\rho^2} \mathbf{G}^2. \quad (4.36d)$$

$$\mathbf{M} = \begin{pmatrix} \cos[\sqrt{1-n}\theta] & \frac{\rho \sin[\sqrt{1-n}\theta]}{\sqrt{1-n}} & 0 & \frac{\rho(1-\cos[\sqrt{1-n}\theta])}{(1-n)\lambda_{rf}^*} \\ \frac{\sqrt{1-n} \sin[\sqrt{1-n}\theta]}{\rho} & \cos[\sqrt{1-n}\theta] & 0 & \frac{\sin[\sqrt{1-n}\theta]}{\sqrt{1-n}\lambda_{rf}^*} \\ -\frac{\sin[\sqrt{1-n}\theta]}{\sqrt{1-n}\lambda_{rf}^*} & -\frac{\rho(1-\cos[\sqrt{1-n}\theta])}{(1-n)\lambda_{rf}^*} & 1 & \frac{\rho}{\lambda_{rf}^{*2}} \left\{ \frac{\theta}{\gamma^2} - \frac{\sqrt{1-n}\theta - \sin[\sqrt{1-n}\theta]}{(1-n)^{-3/2}} \right\} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.37)$$

References for Chapter 4

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Comments on Canonical Coordinates

5.1 Nonparaxial considerations in the transverse plane

The paraxial approximation is generally obtained by dividing the Hamiltonian by the design momentum p_0 and making a small angle approximation. In the scale transformation obtained by dividing H by p_0 the real transverse momenta conjugate to the x and y coordinates are

$$s_x = \frac{p_x}{p_0} = \sin \theta_x \quad (5.1a)$$

$$s_y = \frac{p_y}{p_0} = \sin \theta_y, \quad (5.1b)$$

where θ_x and θ_y are the projections of the trajectory angle with respect to the design orbit. As stated earlier, the approximation is made for small angles by

$$\sin \theta_j \simeq \tan \theta_j.$$

Suppose we would still like to transform the transverse momenta to x' and y' without the paraxial approximation. What canonical coordinates could we expect to find. We need to construct a new F_2 function for the canonical transformation. Remember that $F_2(x, x', y, y'; s)$ is a function of old coordinates and new momenta with the partial derivatives:

$$\frac{\partial F_2}{\partial x} = \frac{p_x}{p_0} = x' \sqrt{1 - x'^2 - y'^2} \quad (5.2a)$$

$$\frac{\partial F_2}{\partial y} = \frac{p_y}{p_0} = y' \sqrt{1 - x'^2 - y'^2} \quad (5.2b)$$

$$\frac{\partial F_2}{\partial x'} = q_x \quad (5.2c)$$

$$\frac{\partial F_2}{\partial y'} = q_y. \quad (5.2d)$$

From the first pair of equations we find a good choice for F_2 to be

$$F_2(x, x', y, y'; s) = (xx' + yy') \sqrt{1 - x'^2 - y'^2}. \quad (5.3)$$

Evaluating the second pair of equations (5.2a&b) gives:

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = \frac{1}{\sqrt{1 - x'^2 - y'^2}} \begin{pmatrix} 1 - 2x'^2 - y'^2 & -x'y' \\ -x'y' & 1 - x'^2 - 2y'^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (5.4)$$

Inverting the matrix and solving for the old coordinates yields

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{1 - x'^2 - y'^2} \frac{1}{D} \begin{pmatrix} 1 - x'^2 - 2y'^2 & x'y' \\ x'y' & 1 - 2x'^2 - y'^2 \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}, \quad (5.5)$$

where the determinant

$$\begin{aligned} D &= (1 - 2x'^2 - y'^2)(1 - x'^2 - 2y'^2) - x'^2 y'^2 \\ &= [1 - (x'^2 + y'^2)][1 - 2(x'^2 + y'^2)]. \end{aligned} \quad (5.6)$$

So the old coordinates in terms of the new ones must be replaced by

$$x = \frac{(1 - x'^2 - 2y'^2)q_x + x'y'q_y}{(1 - x'^2 - y'^2)^{1/2}[1 - 2(x'^2 + y'^2)]} \quad (5.7a)$$

$$y = \frac{x'y'q_x + (1 - 2x'^2 - y'^2)q_y}{(1 - x'^2 - y'^2)^{1/2}[1 - 2(x'^2 + y'^2)]}. \quad (5.7b)$$

This will create one bloody-awful mess, won't it?

If we expand q_x and q_y in power series to 3rd order we obtain

$$q_x = x - \frac{1}{2}(3x'^2 + y'^2)x - x'y'y + \dots \quad (5.8a)$$

$$q_y = y - \frac{1}{2}(x'^2 + 3y'^2)y - x'y'x + \dots \quad (5.8b)$$

So if we are only interested in terms up to second order, the paraxial approximation will probably work, but if we want to keep terms to third order or higher, then we should use the canonical momenta p_x/p_0 and p_y/p_0 rather than x' and y' .

5.2 Longitudinal coordinate variations

There are a several different combinations for the longitudinal canonical variables, for example:

$$\left(z, \frac{\Delta p}{p_0} \right), \quad (5.9a)$$

$$\left(-c\Delta t, \frac{\Delta u}{p_0 c} \right), \quad (5.9b)$$

$$\left(-\frac{(t - t_0)v_0\gamma_0}{\gamma_0 + 1}, \frac{K - K_0}{K_0} \right), \quad (5.9e)$$

$$(\phi, W), \quad (5.9d)$$

$$(\phi, w_\phi), \quad (5.9e)$$

The last two pairs Eqs. (5.9d and e) were explained in the previous chapter Eqs. (4.19 and 4.23b). Differentiating the equation

$$U^2 = p^2 c^2 + m^2 c^4, \quad (5.10a)$$

leads to the relation

$$du = \beta c dp, \quad (5.10b)$$

or on converting to fractional deviations

$$\frac{dp}{p_0} = \frac{1}{\beta} \frac{du}{p_0 c} = \frac{1}{\beta^2} \frac{du}{U_0}. \quad (5.11)$$

Conversion from the pair Eq. (5.9a) to pair Eq. (5.9b) used in the MAD^{3,4} program may be accomplished by

$$\begin{pmatrix} z \\ \frac{\Delta p}{p_0} \end{pmatrix} = \begin{pmatrix} -\beta_0 c(t - t_0) \\ \frac{\Delta p}{p_0} \end{pmatrix} = \begin{pmatrix} -\beta_0 c \Delta t \\ \beta_0^{-1} \frac{\Delta u}{p_0 c} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_0^{-1} \end{pmatrix} \begin{pmatrix} -c \Delta t \\ \frac{\Delta u}{p_0 c} \end{pmatrix} \quad (5.12)$$

The usual definition of dispersion gives the particular solution to the inhomogeneous horizontal Hill's equation (Eq. 5.77 of Ref. 2) must be modified by a factor of β to agree with the value calculated by MAD.

The pair in Eq. (5.9e) is used in the program COSY Infinity⁵. If we start from the good canonical pair

$$(\Delta t, -\Delta U) \quad (5.13)$$

and rescale as usual by dividing by p_0 , we may write

$$\Delta t \begin{pmatrix} -\frac{\Delta U}{p_0} \end{pmatrix} = \frac{c \Delta t \Delta K}{p_0 c}, \quad (5.14)$$

where $\Delta K = \Delta U$ is the change in kinetic energy $K = (\gamma - 1)mc^2$. Writing $p_0 c$ in terms of the kinetic energy, we have

$$p_0 c = mc^2 \gamma_0 \beta_0 = mc^2 (\gamma_0 - 1) \sqrt{\frac{\gamma_0 + 1}{\gamma_0 - 1}} = K_0 \sqrt{\frac{\gamma_0 + 1}{\gamma_0 - 1}} = \frac{K_0 (\gamma_0 + 1)}{\sqrt{\gamma_0^2 - 1}} = \frac{K_0 (\gamma_0 + 1)}{\gamma_0 \beta_0}. \quad (5.15)$$

Substituting this into Eq. (5.14) we find

$$\Delta t \begin{pmatrix} -\frac{\Delta U}{p_0} \end{pmatrix} = -\frac{\Delta t \gamma_0 v_0}{\gamma_0 + 1} \frac{\Delta K}{K_0}, \quad (5.16)$$

and we see that Eq. (5.9e) must also be a good canonical pair which preserves the area elements of longitudinal phase space.

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Comparison of the TEAPOT and Yoshida Integrators

It is common knowledge that modeling of accelerators with drifts and thin kicks preserves the symplecticity as required by a conserved Hamiltonian. An excellent review of integration techniques as used for accelerators given by Étienne Forest in [1]. This informal note is meant to explain some of the difference between a few common symplectic integrators and does not present anything new and original.

Following the example in the TEAPOT manual[2], I consider a thick magnetic quadrupole lens of length l and strength

$$k = \frac{q}{p} \left. \frac{\partial B_y}{\partial x} \right|_0. \quad (6.1)$$

In the focusing plane the quadrupole matrix may be written for a thick lens as

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} \cos \varphi & \frac{\sin \varphi}{\sqrt{k}} \\ -\sqrt{k} \sin \varphi & \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{\varphi^2}{2} + \frac{\varphi^4}{24} - \frac{\varphi^6}{720} + \dots & \frac{\sqrt{k} \varphi}{k} - \frac{\sqrt{k} \varphi^3}{6k} + \frac{\sqrt{k} \varphi^5}{120k} + \dots \\ -\sqrt{k} \varphi + \frac{\sqrt{k} \varphi^3}{6} - \frac{\sqrt{k} \varphi^5}{120} + \dots & 1 - \frac{\varphi^2}{2} + \frac{\varphi^4}{24} - \frac{\varphi^6}{720} + \dots \end{pmatrix} + \mathcal{O}(\varphi^7), \end{aligned} \quad (6.2)$$

where the dimensionless angle $\varphi = \sqrt{k}l$, and the trig functions have been expanded up to 6th order for comparison (See Fig. 1.) with TEAPOT integration[2,3] and Yoshida integration[4] up to 4th order.

Before proceeding, it is perhaps informative to recall the the decomposition of a thick lens into a thin lens with drifts using principle planes[5,6]. The exact matrix in Eq. (6.1) may be factored into

$$\mathbf{M} = \begin{pmatrix} 1 & \frac{1}{\sqrt{k}} \tan \frac{\varphi}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sqrt{k} \sin \phi & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\sqrt{k}} \tan \frac{\varphi}{2} \\ 0 & 1 \end{pmatrix}. \quad (6.3)$$

While this seems to give a very simple drift-kick-drift representation for the thick lens, it may have two drawbacks:

1. The sum of the two drift lengths is not the same as the original magnet. In fact for the defocusing plane the circular tan and sin functions must be replaced by hyperbolic functions resulting in different lengths for the drifts of the focusing and defocusing planes.
2. In this case, the thin lens kick term is identical to M_{21} of the thick lens. For more general elements, this requires actually performing the integration of Hamilton's equations to determine the focusing element M_{21} of the thick-lens, so one might well ask "What have we gained?"

In passing, we can note that any symplectic 2×2-matrix with equal elements on the diagonal can be decomposed as follows:

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{1+a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{1+a} \\ 0 & 1 \end{pmatrix}, \quad (6.4)$$

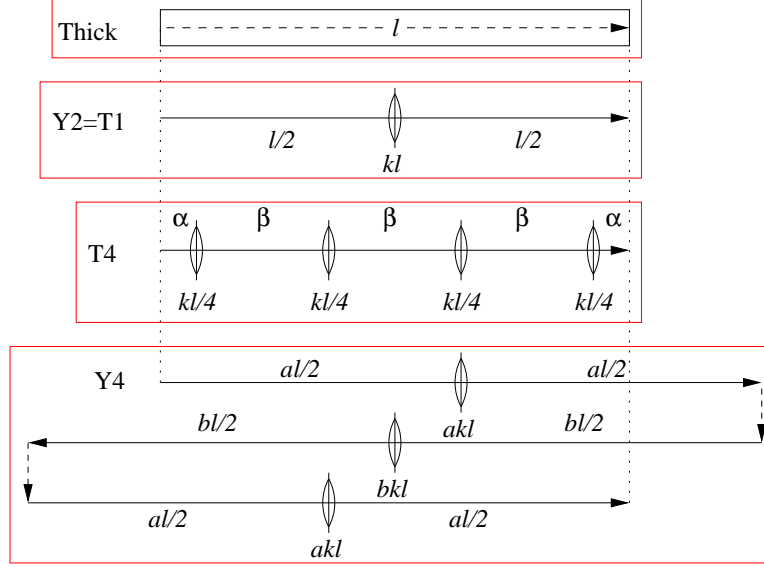


Figure 1. Comparison of the four integration models. The T1 integration method of TEAPOT is identical to the second order integrator Y2 of Yoshida. TEAPOT's T4 method uses positive drifts with four equal-strength thin kicks with two different drift lengths $\alpha = l/10$ and $\beta = 4l/15$. Yoshida's 4th order method Y4 uses only three thin kicks with backwards integration in the middle with $a = 1/(2 - \sqrt[3]{2})$ and $b = -2a$.

since the determinant $a^2 - bc = 1$. If the diagonal elements are unequal, then a similar decomposition may be performed with unequal drifts on either side of the thin kick.

The single drift-kick-drift algorithm of TEAPOT (T4) is identical to the 2nd order integrator of Yoshida (Y2):

$$\mathbf{Y}_2(\phi, k) = \mathbf{T}_1(\varphi, k) = \begin{pmatrix} 1 - \frac{\varphi^2}{2} & \frac{\sqrt{k}\varphi}{k} - \frac{\sqrt{k}\varphi^3}{4k} \\ -\sqrt{k}\varphi & 1 - \frac{\varphi^2}{2} \end{pmatrix}. \quad (6.5)$$

Taking with difference with the expansion in Eq. (6.2):

$$\mathbf{M}_{\text{thick}} - \mathbf{T}_1 = \begin{pmatrix} \frac{\varphi^4}{24} - \frac{\varphi^6}{720} + \dots & \frac{\sqrt{k}\varphi^3}{12k} + \frac{\sqrt{k}\varphi^5}{120k} + \dots \\ \frac{\sqrt{k}\varphi^3}{6} - \frac{\sqrt{k}\varphi^5}{120} + \dots & \frac{\varphi^4}{24} - \frac{\varphi^6}{720} + \dots \end{pmatrix} = \mathcal{O}(\varphi^3), \quad (6.6)$$

we see that this single kick algorithm is accurate to order φ^2 .

A similar expansion of the difference $\mathbf{M} - \mathbf{T}_4$

$$\mathbf{M}_{\text{thick}} - \mathbf{T}_4 = \begin{pmatrix} \frac{\varphi^4}{360} - \frac{23\varphi^6}{54000} + \dots & -\frac{\sqrt{k}\varphi^3}{180k} + \frac{7\sqrt{k}\varphi^5}{6750k} + \dots \\ -\frac{\sqrt{k}\varphi^3}{600} + \dots & \frac{\varphi^4}{360} - \frac{23\varphi^6}{54000} + \dots \end{pmatrix} = \mathcal{O}(\varphi^3), \quad (6.7)$$

is still only accurate to order φ^2 , while the Yoshida fourth order integrator differs by

$$\begin{aligned} \mathbf{M}_{\text{thick}} - \mathbf{Y}_4 &= \begin{pmatrix} -\frac{20 \times 2^{2/3} + 25^{1/3} + 32}{1440} \varphi^6 + \dots & \frac{5 \times 2^{2/3} + 5 \times 2^{1/3} + 6}{720\sqrt{k}} \varphi^5 + \dots \\ -\frac{15 \times 2^{2/3} + 20 \times 2^{1/3} + 26}{720} \sqrt{k} \varphi^5 + \dots & -\frac{20 \times 2^{2/3} + 25 \times 2^{1/3} + 32}{1440} \varphi^6 + \dots \end{pmatrix} \\ &= \begin{pmatrix} -0.066143 \varphi^6 + \dots & \frac{0.028106}{\sqrt{k}} \varphi^5 + \dots \\ -0.104180 \sqrt{k} \varphi^5 + \dots & 0.066143 \varphi^6 + \dots \end{pmatrix} = \mathcal{O}(\varphi^5). \end{aligned} \quad (6.8)$$

For single particle tracking there is no particular reason why backwards tracking in an integrator must be avoided; however when one considers collective effects such as space charge, then it may be advantageous to use an integrator which propagates only in the forward direction[7]. For most spin tracking, space-charge effects are ignored and fourth or higher order Yoshida should work quite well.

References for Chapter 6

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Example of Coupled Bunch Instability

During one ramp of polarized protons in Yellow ring of RHIC, only one of the two 28 MHz cavities for acceleration was being powered. The tuner for the other cavity was detuned to a fixed frequency away from the proper frequency. As the beam was accelerated from 24.3 GeV at injection to 100 GeV at storage the revolution frequency shifts from $f_{rf,i} = 28.1297$ MHz to $f_{rf,f} = 28.1494$ MHz. The normal harmonic number for the 28 MHz cavities is $h = 360$. At one point in the ramp, the 358th harmonic of the revolution frequency crossed the resonant frequency of the unpowered cavity initiating the multibunch instability shown in Fig. (7.1).

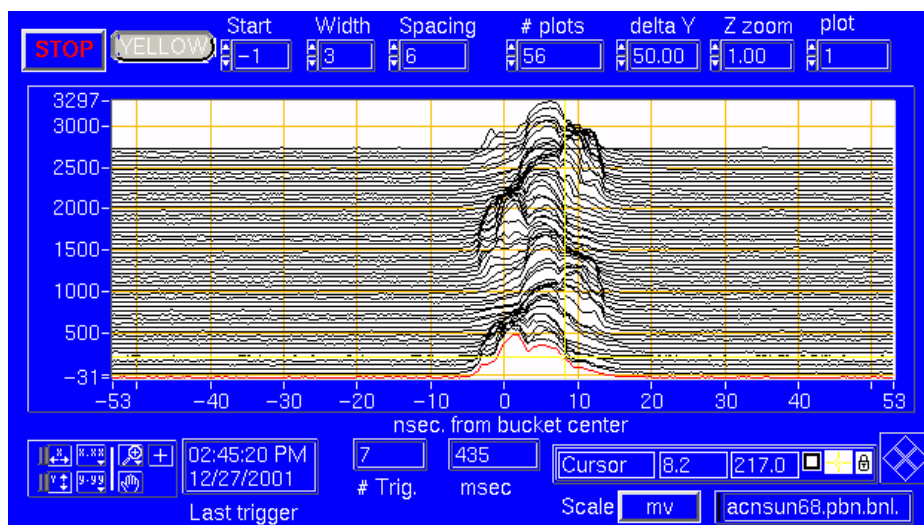


Figure. 7.1 Coupled bunch instability in the RHIC Yellow ring during acceleration of polarized protons. The 56 traces are the 55 bunches (plus one empty bucket) taken on one turn during acceleration. The populate every sixth rf bucket starting from bucket 1 up to bucket 331. There is a gap of 5 bunches (buckets 332-360) to leave room for the rise time of the abort kickers.

The bunched beam drives TM_{010} oscillations in the unpowered cavity. Each bunch will see the wake of previous bunches and gain (lose) a little energy from (to) the cavity depending on the relative phase of the wake oscillation when the bunch crosses the gap. As a result there is a slight beating of frequencies of the two cavities as indicated in Fig. (7.1).

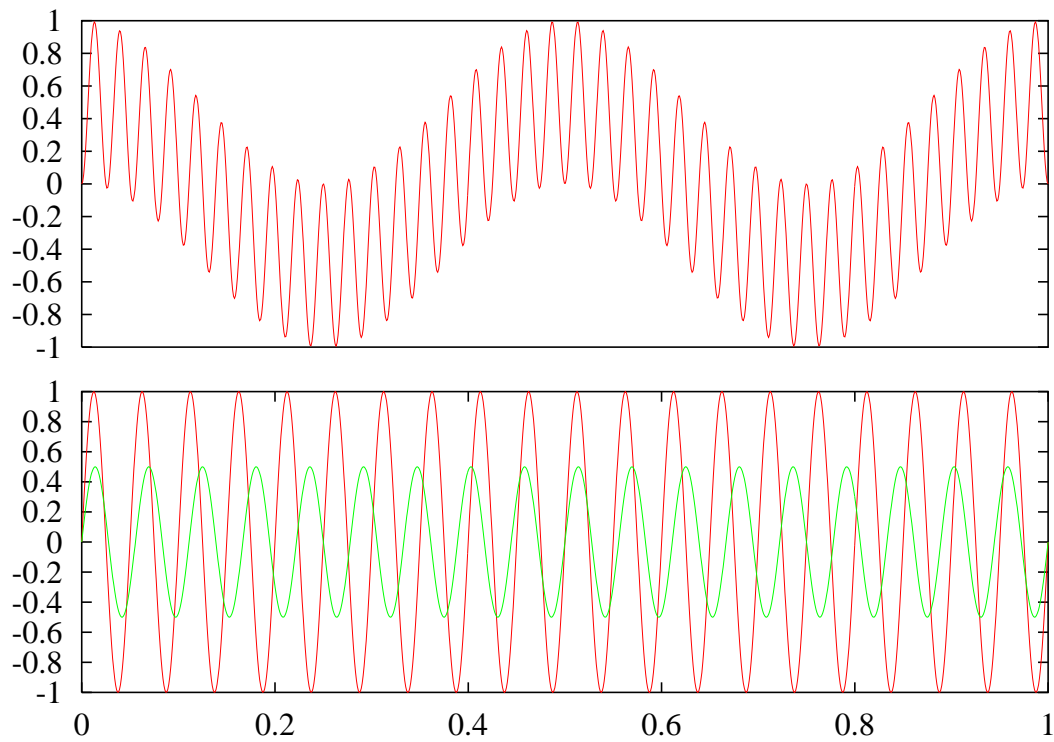
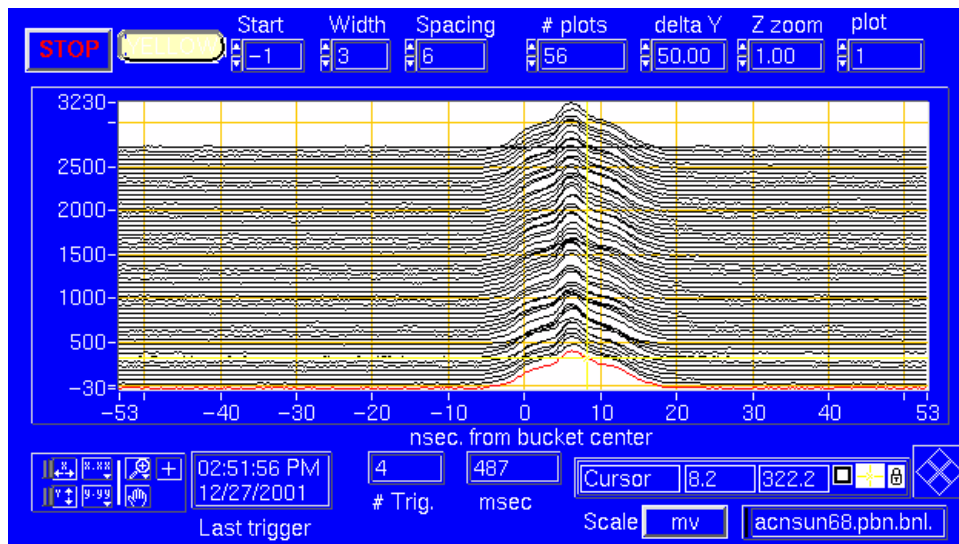


Figure. 7.2 Conceptual beating of the frequencies of the two cavities: $[\sin(2\pi hx) \sin(2\pi h'x)]$. Here the harmonic numbers $h = 20$ $[\sin(2\pi hx)]$ and $h' = 18$ $[0.5 \sin(2\pi h'x)]$ were used rather than 360 and 358, so that the individual cycles could be seen for the individual cavities.



This shows the 55 bunches later in the acceleration ramp after the oscillations have Landau damped.