USPAS Accelerator Physics 2017
University of California, Davis

Chapter 3: Trajectory Mechanics, Hamiltonian Formalism

Todd Satogata (Jefferson Lab and ODU) / satogata@jlab.org
Cedric Hernalsteens (IBA) / cedric.hernalsteens@gmail.com
Randika Gamage (ODU) / bgama002@odu.edu

http://www.toddsatogata.net/2017-USPAS

Overview

- Hamiltonian "refresher"
- Section 3.1: Liouville's theorem
- Section 3.2-4: Hamiltonian refresher; Canonical momentum, potentials, coords
- Section 3.5: Symplecticity and its consequences
- Section 3.6: Canonical coordinates revisited
- Section 3.7: Symplectic generators

- This section can get mathematically tedious at times

Classical Dynamics

- Particle motion is described by classical dynamics
  - For now, assume conservative (non-dissipative) forces are only acting on the beam
  - Forces and potentials only depend on position, not momentum: $F(x,y,z,p_x,p_y,p_z) \Rightarrow F(x,y,z)$
    - E.g. external magnetic fields, internal self-fields
  - Relativity holds: $\vec{p} = \vec{\beta}\gamma mc \quad \vec{v} = (\vec{p}c)/\sqrt{p^2+m^2c^2}$
- Under these conditions we can leverage much of 19th century classical mechanics
  - Phase space evaluation and conservation
  - Hamiltonian approaches (conservative fields)
  - Implications of Hamilton's equations (symplecticity)
  - Canonical coordinates and transformations

3.1: Liouville's Theorem

- Consider particle phase space density function

$$f(x, y, z, p_x, p_y, p_z; t)$$

$$N(t) = \int f(x, y, z, p_x, p_y, p_z; t)\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z$$

- We **define** a 6-dimensional current, where $\dot{} \equiv \frac{d}{dt}$

$$\vec{J}_6 \equiv (f\dot{x}, f\dot{y}, f\dot{z}, f\dot{p}_x, f\dot{p}_y, f\dot{p}_z) = (f\vec{v}, f\vec{F})$$

- In any region of phase space, we expect continuity:

$$\frac{\partial f}{\partial t} + \vec{\nabla}_6 \cdot \vec{J}_6 = 0$$

- Liouville's Theorem: f is time-constant, or $\frac{df}{dt} = 0$

Liouville's Theorem: Proof

- Consider an arbitrary spatial volume V
- Rate of change of particles N(t) inside the volume must be equal to the negative of the total flux **leaving** the volume

$$\frac{\partial N(t)}{\partial t} = \frac{\partial}{\partial t}\int_V f\, dV = \int_V \frac{\partial f}{\partial t}\, dV$$

Rate of change of N(t) inside V

$$-\int_S \vec{J}_6 \cdot d\vec{S} = -\int_V (\vec{\nabla}_6 \cdot \vec{J}_6)\, dV$$

Negative of flux leaving volume V

6D Stokes Theorem or Continuity: $\frac{\partial f}{\partial t} + \vec{\nabla}_6 \cdot \vec{J}_6 = 0$

- Now evaluate $\vec{\nabla}_6 \cdot \vec{J}_6$

Liouville's Theorem: Proof

- Partition $\vec{\nabla}_6$ and recall $\vec{J}_6 = (f\vec{v}, f\vec{F})$:

$$\vec{\nabla}_6 = (\vec{\nabla}, \vec{\nabla}_p) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right)$$

$$\begin{aligned} \vec{\nabla}_6 \cdot \vec{J}_6 &= \vec{\nabla} \cdot (f\vec{v}) + \vec{\nabla}_p (f\vec{F}) \\ &= (\vec{\nabla} f) \cdot \vec{v} + f\underbrace{(\vec{\nabla} \cdot \vec{v})} + (\vec{\nabla}_p f) \cdot \vec{F} + f\underbrace{(\vec{\nabla}_p \cdot \vec{F})} \end{aligned}$$

Vanishes since $\vec{v} = (\vec{p}c)/\sqrt{p^2 + m^2c^2}$
does not depend on any position coordinates

- Be careful here: potential does not depend on *p* but the magnetic part of the Lorentz force does:

$$\vec{\nabla}_p \cdot \vec{F} = q\vec{\nabla}_p \cdot (\vec{v} \times \vec{B}) = q\vec{B} \cdot (\vec{\nabla}_p \times \vec{v}) - q\vec{v} \cdot \cancel{(\vec{\nabla}_p \times \vec{B})}$$

Liouville's Theorem: Proof

- So now let's evaluate $(\vec{\nabla}_p \times \vec{v})$ where $\vec{v} = (\vec{p}c)/\sqrt{p^2 + m^2c^2}$
- Just look at z component and use cyclicity:

$$\left[\vec{\nabla}_p \times \left(\frac{\vec{p}c}{\sqrt{p^2+m^2c^2}}\right)\right]_z = \frac{\partial}{\partial p_x}\left(\frac{p_y}{\sqrt{p^2+m^2c^2}}\right) - \frac{\partial}{\partial p_y}\left(\frac{p_x}{\sqrt{p^2+m^2c^2}}\right)$$

$$= \frac{-\frac{1}{2}(2p_xp_y)}{(p^2+m^2c^2)^{3/2}} - \frac{-\frac{1}{2}(2p_yp_x)}{(p^2+m^2c^2)^{3/2}}$$

$$= 0$$

- Using cyclicity we can conclude $(\vec{\nabla}_p \times \vec{v}) = 0$
- So the 6-divergence of the 6-current is

$$\vec{\nabla}_6 \cdot \vec{J}_6 = (\vec{\nabla} f) \cdot \vec{v} + (\vec{\nabla}_p f) \cdot \vec{F}$$

Liouville's Theorem: Proof (last page!)

- The continuity equation was $\frac{\partial f}{\partial t} + \vec{\nabla}_6 \cdot \vec{J}_6 = 0$
- We have shown that $\vec{\nabla}_6 \cdot \vec{J}_6 = (\vec{\nabla} f) \cdot \vec{v} + (\vec{\nabla}_p f) \cdot \vec{F}$
- We want to show that $\frac{df}{dt} = 0$

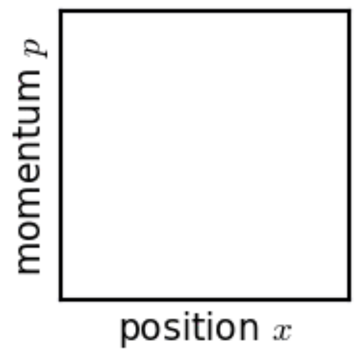
(total derivative)

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}_{(\vec{\nabla} f) \cdot \vec{v}} + \underbrace{\frac{\partial f}{\partial p_x}\frac{dp_x}{dt} + \frac{\partial f}{\partial p_y}\frac{dp_y}{dt} + \frac{\partial f}{\partial p_z}\frac{dp_z}{dt}}_{(\vec{\nabla}_p f) \cdot \vec{F}} + \frac{\partial f}{\partial t}$$

$$\begin{aligned} \frac{df}{dt} &= (\vec{\nabla} f) \cdot \vec{v} + (\vec{\nabla}_p f) \cdot \vec{F} + \frac{\partial f}{\partial t} \\ &= 0 \qquad \text{QED} \end{aligned}$$

Liouville's Theorem: Discussion

- Liouville's theorem here essentially states that the phase space density of particles *f* acts like an incompressible fluid

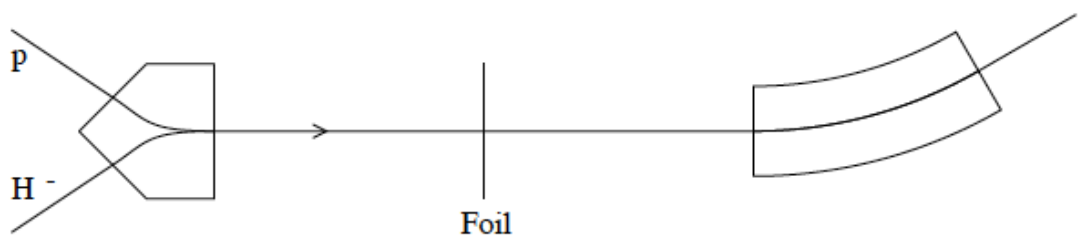


From Wikipedia

- Assumptions:
  - Only velocity-dependent force is from Lorentz/Maxwell
    - Works because this is a linear cross product force
  - All other forces are from purely spatial potentials
    - Potentials not assumed to be linear in position coordinates
    - Liouville even applies for **nonlinear** magnetic fields
    - Nonlinearities filament *f* to the point of **apparent growth**
  - Fully special relativistic treatment (valid in all frames)

Liouville's Theorem: Discussion

- These assumptions are valid for a lot of particle distribution motion in accelerators
  - Including conservative internal forces like space charge, intra-beam scattering, and Touschek scattering
- They are not appropriate for non-conservative or velocity-dependent forces in accelerators
  - Synchrotron radiation and radiation damping
  - Beam-material interactions (foils, targets, beam-gas)
  - Two-beam interactions (electron cooling, beam-beam)
  - Charge-changing interactions
    - E.g. Tandem Van de Graaffs, H- injection



3.2-3: Canonical Momentum and Potentials

- We want to use a full canonical formalism for our mechanics, including **Hamiltonian dynamics**
  - Under conservative forces $\vec{\nabla} \times \vec{F} = 0$
  - All forces depending only on position (and not momentum or velocity) are conservative
  - Also perform coord transformations: **generating functions**
- Lorentz force is **not** generally conservative

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

but we can define a new momentum in terms of the fields that makes a new "canonical force" that **is** conservative and obeys Hamiltonian dynamics

Canonical Momentum

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \frac{d\vec{p}}{dt} = q(\vec{\nabla} \times \vec{E} + \vec{\nabla} \times (\vec{v} \times \vec{B}))$$

$$= -q\frac{\partial \vec{B}}{\partial t} + q\left[\cancel{(\vec{B} \cdot \vec{\nabla})\vec{v}} - (\vec{v} \cdot \vec{\nabla})\vec{B} + \cancel{(\vec{\nabla} \cdot \vec{B})\vec{v}} - \cancel{(\vec{\nabla} \cdot \vec{v})\vec{B}}\right]$$

$$= -q\frac{\partial \vec{B}}{\partial t} - q\left[(\vec{v} \cdot \vec{\nabla})\vec{B}\right]$$

Position derivatives of v are zero

$$= -q\left[\frac{\partial \vec{B}}{\partial t} + \frac{\partial \vec{B}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{B}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{B}}{\partial z}\frac{dz}{dt}\right]$$

$$= -q\frac{d\vec{B}}{dt} = -\frac{d}{dt}\left[\vec{\nabla} \times (q\vec{A})\right]$$

- Here $\vec{A}$ is a vector potential such that $\vec{B} = \vec{\nabla} \times \vec{A}$
- Rearranging and swapping differentiations, we have

$$\vec{\nabla} \times \left[\frac{d}{dt}\left(\vec{p} + q\vec{A}\right)\right] = \vec{\nabla} \times \left[\frac{d\vec{P}}{dt}\right] = 0 \qquad \text{where} \qquad \vec{P} \equiv \vec{p} + q\vec{A}$$

Canonical Momentum and Force

- We can thus define a **canonical momentum** for charged particle motion under the influence of the Lorentz force:

$$\vec{P} = \vec{p} + q\vec{A}$$

  - This momentum has implicit dependence on position through the vector potential
- The corresponding **canonical force**

$$\vec{F}_{\mathrm{can}} \equiv \frac{d\vec{P}}{dt}$$

is conservative, in that $\vec{\nabla} \times \vec{F}_{\mathrm{can}} = 0$

3.4: Hamiltonian Refresher

- Recall the Hamiltonian represents the *total energy* of a system in terms of 2N variables: positions and canonical (conjugate) momenta:

$$H(\vec{x}, \vec{p}; t) = \mathrm{KE}(\vec{x}, \vec{p}) + \mathrm{PE}(\vec{x}, \vec{p}; t)$$

- Then equations of motion of the system are given by Hamilton's equations:

1D $$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

General $$\dot{\vec{x}} = \vec{\nabla}_p H \qquad \dot{\vec{p}} = -\vec{\nabla} H$$

- Example: 1D simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \Rightarrow \quad p = m\dot{x} = mv \qquad\qquad \dot{p} = F = -\frac{\partial H}{\partial x} = -kx$$

Hamiltonian Dynamics

- Can be derived from Lagrangian dynamics and principle of least action
  - Too long to go into here! See, e.g., Goldstein for details

- Central relevant points
  - Hamiltonian is **time-dependent** function of **canonical pairs of coordinates**, each a position and **canonical momentum**
$$H(x, p_x, y, p_y, \ldots; t)$$
  - The Hamiltonian has a physical interpretation as **the total energy of the system**
  - Dynamics then follow **Hamilton's equations** for each pair of canonical coordinates, e.g.

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\text{generally} \quad \begin{aligned} \dot{\vec{x}} &= \nabla_p H \\ \dot{\vec{p}} &= -\nabla H \end{aligned}$$

Hamilton Example 1: 1D Newtonian Dynamics

$$H(x, p_x; t) = \frac{p_x^2}{2m} + V(x)$$

$V(x)$ is space – dependent potential

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{x} = v_x = \frac{p_x}{m} \qquad \dot{p}_x = F_x = -\frac{\partial V(x)}{\partial x}$$

- Hamiltonian of kinetic energy plus potential energy gives "definition" of Newtonian momentum and Newtonian force from potential gradient

Hamiltonian Example 2: 1D Relativistic Free Particle

$$H(x, p_x; t) = \gamma m c^2 = \sqrt{\gamma^2 m^2 c^4} \qquad \gamma^2 = \frac{1}{1-\beta^2} = \frac{1-\beta^2}{1-\beta^2} + \frac{\beta^2}{1-\beta^2} = 1 + \gamma^2\beta^2$$

$$\boxed{H(x, p_x; t) = \sqrt{(1+\gamma^2\beta^2) m^2 c^4} = \sqrt{p_x^2 c^2 + m^2 c^4}}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{x} = v_x = \beta c = \frac{p_x c^2}{\sqrt{p^2 c^2 + m^2 c^4}}$$

$$= \frac{p_x c^2}{\gamma m c^2} = \frac{p_x}{\gamma m}$$

$$\Rightarrow p_x = \gamma \beta m c$$

$$\dot{p}_x = 0$$

p_x is a constant of the motion

Relativistic Hamiltonian

- So a free particle has relativistic Hamiltonian

$$H = \sqrt{p^2c^2 + m^2c^4} \qquad \dot{\vec{p}} = -\vec{\nabla} H \qquad \dot{\vec{x}} = \vec{\nabla}_p H$$

- Including fields generated by vector potential $\vec{A}(\vec{x}, t)$ and scalar potential $\phi(\vec{x}, t)$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

then the Hamiltonian is given in terms of the potentials and canonical momentum as

$$H = \underbrace{\sqrt{(\vec{P} - q\vec{A})^2c^2 + m^2c^4}}_{\text{Kinetic energy}} + \underbrace{q\phi}_{\text{Potential energy}}$$

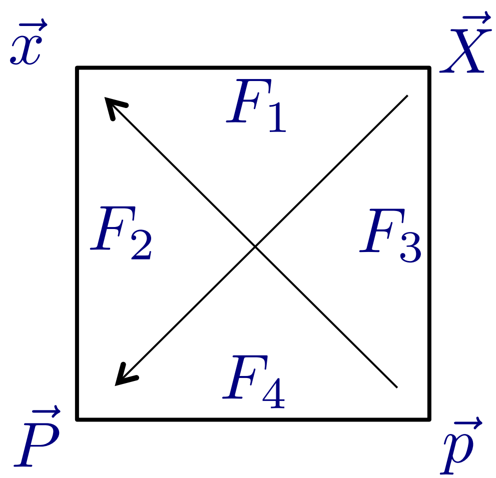
Generating Function Refresher

- Any coordinate system transformation we apply must be **canonical** (respecting Hamilton's equations)
  - These ensure that coordinate and momenta remain canonically conjugate
  - Have to go into cylindrical coordinates like yesterday

- Canonical transformations are best represented with **generating functions**
  - Say transformation is from $(\vec{x}, \vec{p}) \Rightarrow (\vec{X}, \vec{P})$
  - The transformation may also be time-dependent
  - There are four types of generating functions, $F_{1\text{-}4}$
  - Each is a function of one set of old coordinates or momenta, and one set of new coordinates or momenta

Generating Function Refresher

- Generating functions and mnemonic square:



$$
\begin{array}{lll}
F_1(\vec{x},\vec{X}): & \vec{p} = \partial F_1/\partial \vec{x} & \vec{P} = -\partial F_1/\partial \vec{X} \\
F_2(\vec{x},\vec{P}): & \vec{p} = \partial F_2/\partial \vec{x} & \vec{X} = \partial F_2/\partial \vec{P} \\
F_3(\vec{p},\vec{X}): & \vec{x} = -\partial F_3/\partial \vec{p} & \vec{P} = -\partial F_3/\partial \vec{X} \\
F_4(\vec{p},\vec{P}): & \vec{x} = -\partial F_4/\partial \vec{p} & \vec{X} = \partial F_4/\partial \vec{P}
\end{array}
$$

- Time-dependent transformations also add a total time derivative of the generating function to the Hamiltonian:

$$
H(\vec{x},\vec{p};t) \;\rightarrow\; H(\vec{X},\vec{P};t) + \frac{dF}{dt}
$$

Generating Function Examples

- Changes of coordinate or Hamiltonian scale are trivially canonical transformations
  - Essentially a change of units

- Coordinates and momenta are interchangeable

$$H(x,p;t) \qquad\qquad F_1(x,X) = \alpha x X$$

$$p = \frac{\partial F_1}{\partial x} = \alpha X \quad P = -\frac{\partial F_1}{\partial X} = -\alpha x$$

Generating Function Examples

- Time-dependent simple harmonic oscillator
  - For simplicity set scales so m=1, k=1

$$H(x,p) = \frac{p^2}{2} + \frac{x^2}{2}$$

  - p is **not** a constant of the motion, but we can find one!
  - Transform to **action-angle coordinates:** (ϕ, J)

$$F_1(x,\phi) = -\frac{x^2}{2}\tan\phi$$

$$p = \frac{\partial F_1}{\partial x} = -x\tan\phi \qquad J = -\frac{\partial F_1}{\partial \phi} = -\frac{x^2}{2}(-\sec^2\phi)$$

$$p = -\sqrt{2J}\sin\phi \qquad \Rightarrow x = \sqrt{2J}\cos\phi$$

$$H = J \qquad \dot{\phi} = \frac{\partial H}{\partial J} = 1 \qquad \dot{J} = 0$$

Coordinate Systems: Frenet-Serret

- We generally describe particle motion in a coordinate system relative to a **design trajectory**
  - Coordinates and momenta are then perturbatively "small" displacements from this design trajectory
  - Design trajectories can be arbitrarily complicated!
    - But they are always differentiable and usually simple
  - A general differential geometry treatment uses **Frenet-Serret formulas** since the design trajectory is an arbitrary curve in 3D Euclidian space
    - *s* is the **pathlength** along the design trajectory
    - $\hat{T}$ is a unit vector tangent to the curve in the direction of *s*
    - $\hat{N}$ is a normal vector defined by $d\hat{T}/ds$
    - $\hat{B}$ is a binormal unit vector defined by $\hat{T} \times \hat{N}$

https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

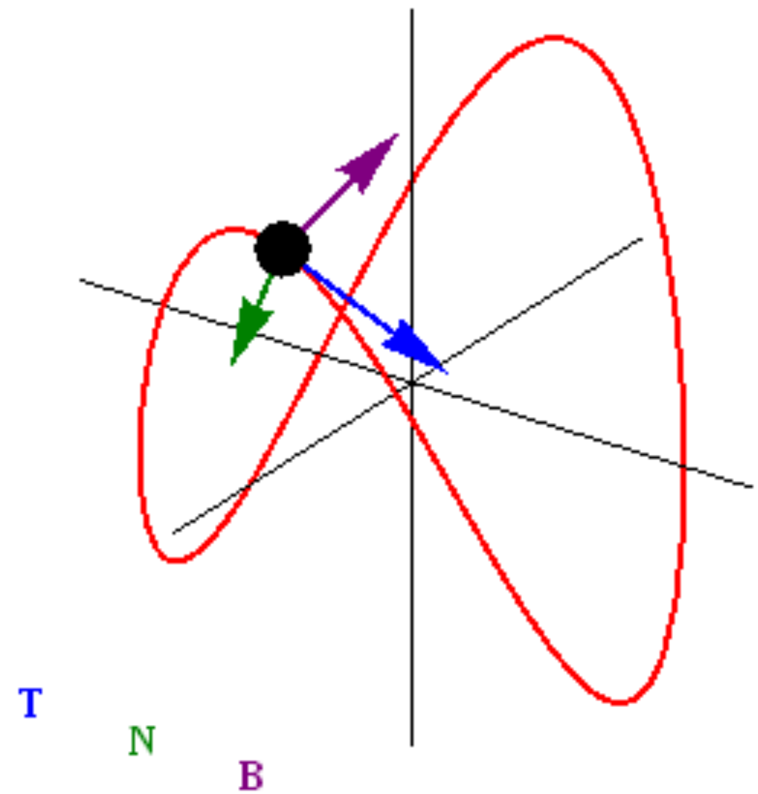
Coordinate Systems: Frenet-Serret

$$' \equiv \frac{d}{ds} \qquad \begin{pmatrix} \hat{T}'(s) \\ \hat{N}'(s) \\ \hat{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \hat{T}(s) \\ \hat{N}(s) \\ \hat{B}(s) \end{pmatrix}$$

$$\kappa(s) : \text{curvature} = \frac{1}{\rho(s)}$$

$$\tau(s) : \text{torsion}$$

Curvature can be thought of as "non-linearity" while torsion can be thought of as "non-planarity"



An excellent (long, but very general) approach with somewhat confusing notation is given by V. Litvinenko here

We instead follow the text and simplify a bit, in particular ignoring torsion…

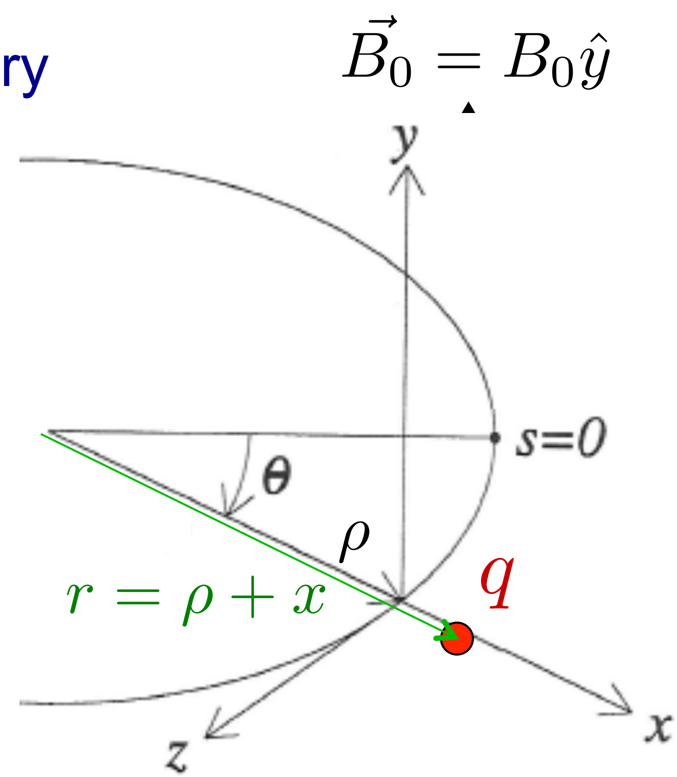
Review: Parameterizing Particle Motion

- Derive local Hamiltonian (excluding torsion)
- We need a **local coordinate system** $(\hat{x}, \hat{y}, \hat{z})$ **relative to the design trajectory**
  - s is the direction of the design trajectory
  - y is position in the B field direction
  - x is position in the radial direction
  - ρ is not a coordinate, but the **design** bending radius in magnetic field B_0
- Can express total radius r as

$$\boxed{r = \rho + x} \qquad \theta = \frac{s}{\rho} = \frac{\beta c t}{\rho}$$

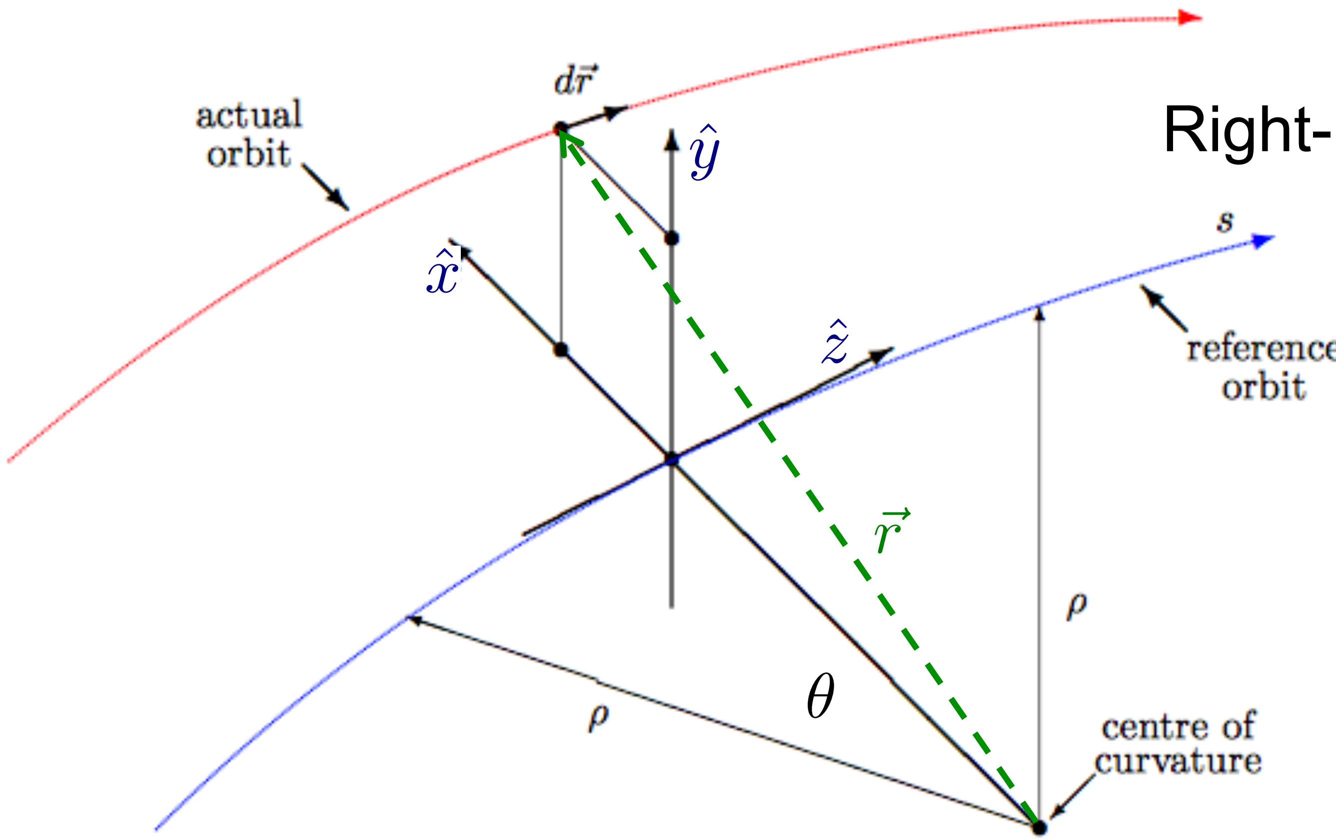
- Also define local trajectory angle

$$\boxed{x' = \frac{dx}{ds} = \frac{1}{\rho}\frac{dx}{d\theta} \approx \frac{p_x}{p_0}}$$



Review: MAD/madx Coordinate System

http://madx.web.cern.ch/madx/madX/doc/latexuguide/madxuguide.pdf p.11

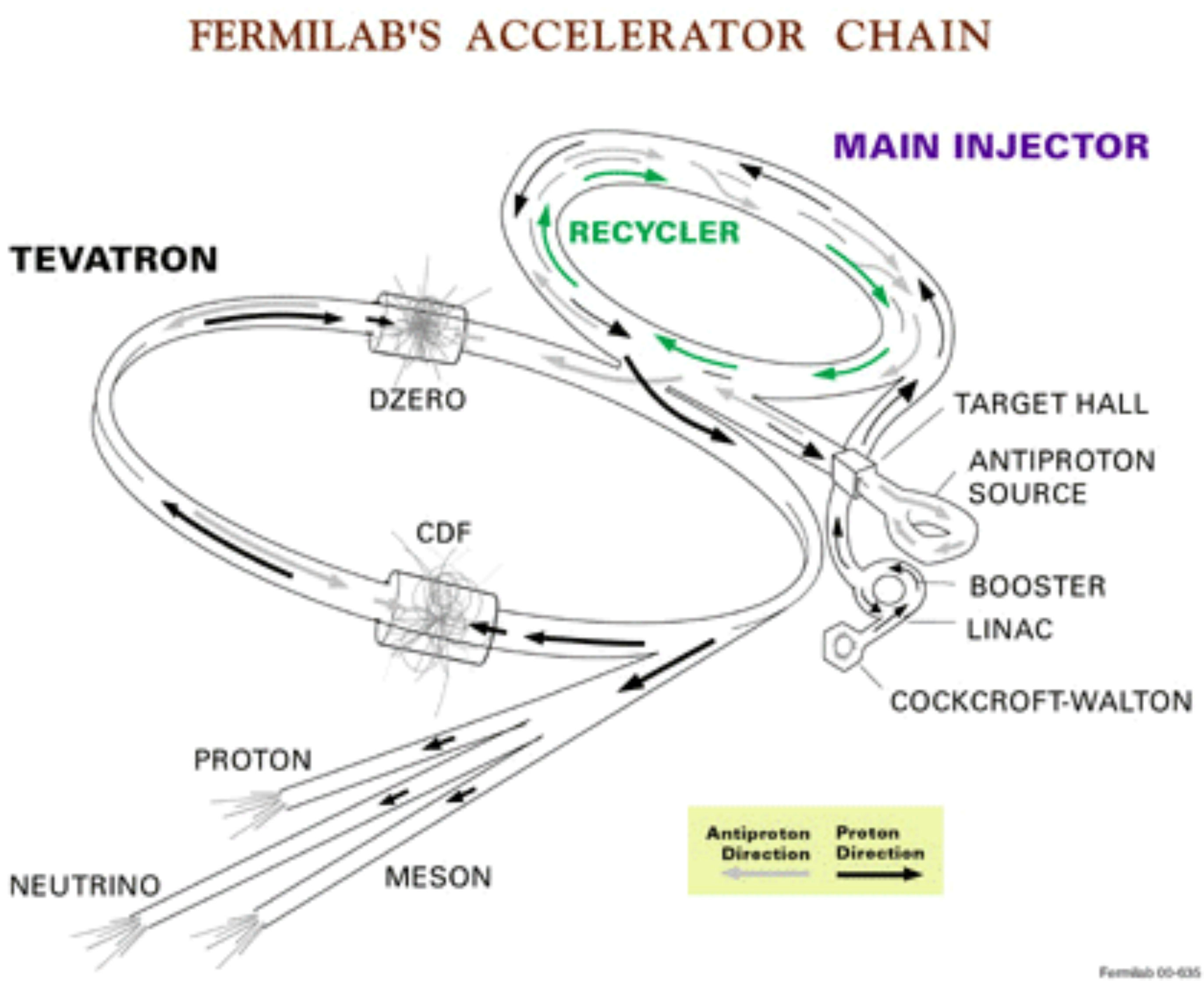


Right-handed

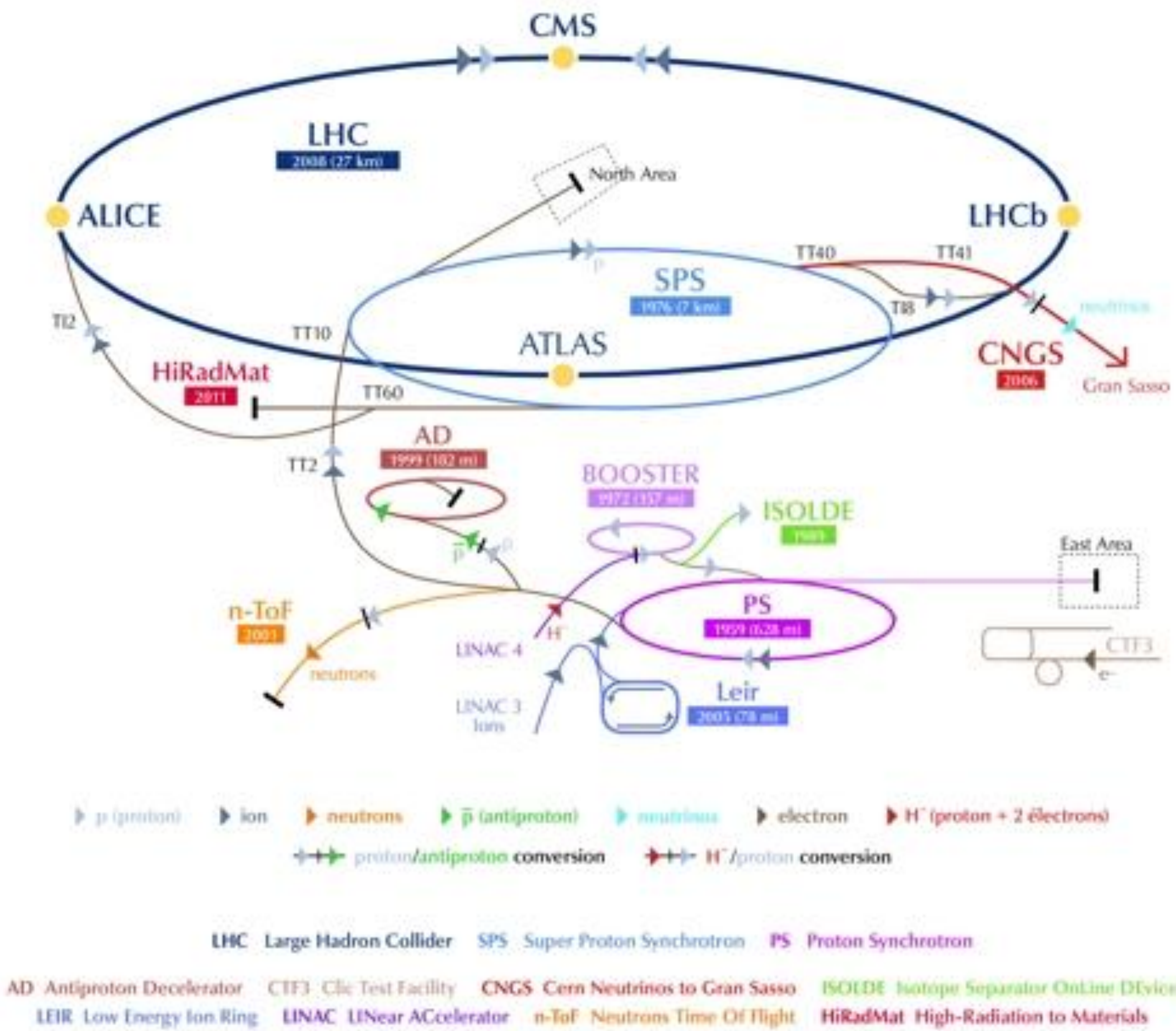
$$\vec{r} = (\rho + x)\hat{x} + y\hat{y} + z\hat{z}$$

$$R = \rho + x \qquad \frac{R}{\rho} = \left(1 + \frac{x}{\rho}\right)$$

Example: FNAL Coordinate Systems



Example: CERN Coordinate Systems]



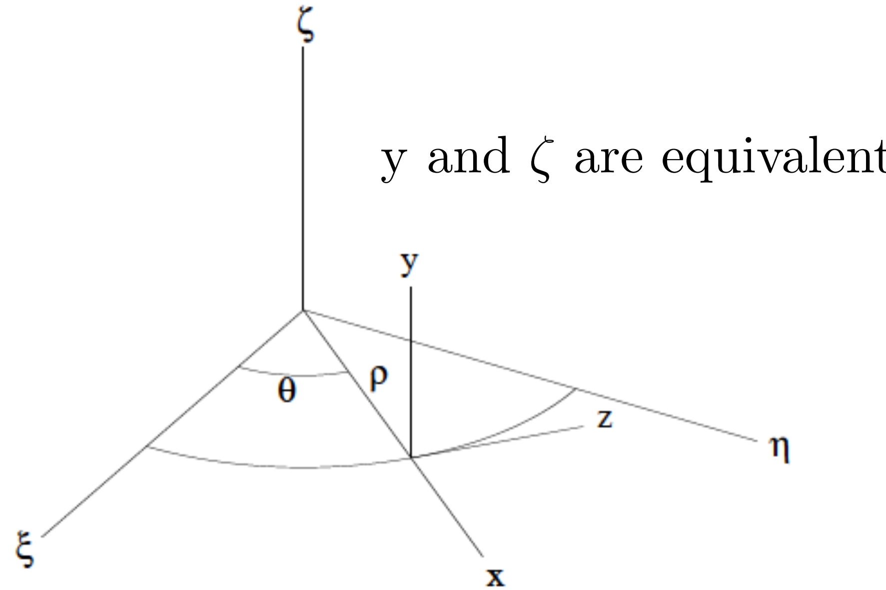
Coordinate Transformation

- Cylindrical and cartesian coordinate transformation:

$$\theta = \frac{s}{\rho}$$

$$\xi = (\rho + x)\cos\theta$$

$$\eta = (\rho + x)\sin\theta$$



We can then construct a generating function F_3 to find the corresponding canonical momenta in the cartesian system – we are transforming $(\xi, \eta, p_\xi, p_\eta) \to (x, s, p_x, p_s)$

$$\xi = (\rho + x)\cos(s/\rho) = -\frac{\partial F_3}{\partial p_\xi} \qquad \eta = (\rho + x)\sin(s/\rho) = -\frac{\partial F_3}{\partial p_\eta}$$

$$\to F_3(p_\xi, p_\eta, x, s) = -(\rho + x)\left(p_\xi \cos\frac{s}{\rho} + p_\eta \sin\frac{s}{\rho}\right)$$

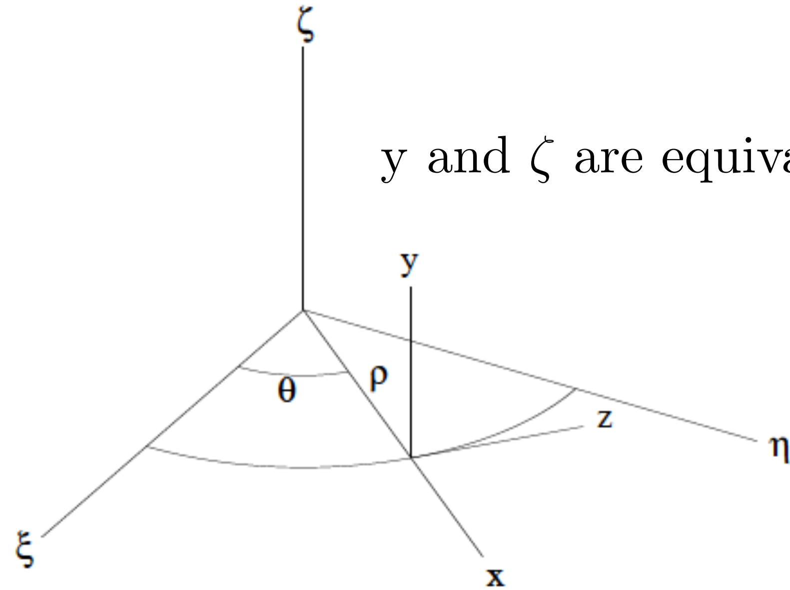
Coordinate Transformation

- Cylindrical and cartesian coordinate transformation:

$$\theta = \frac{s}{\rho}$$

$$\xi = (\rho + x)\cos\theta$$

$$\eta = (\rho + x)\sin\theta$$



$$\rightarrow F_3(p_\xi, p_\eta, x, s) = -(\rho + x)\left(p_\xi \cos\frac{s}{\rho} + p_\eta \sin\frac{s}{\rho}\right)$$

$$P_\xi = -\frac{\partial F_3}{\partial x} = p_\xi \cos\theta + p_\eta \sin\theta = p_r = \vec{p}\cdot\hat{x}$$

$$P_s = -\frac{\partial F_3}{\partial s} = \left(1 + \frac{x}{\rho}\right)(-p_\xi \sin\theta + p_\eta \cos\theta) = \left(1 + \frac{x}{\rho}\right)\vec{p}\cdot\hat{s}$$

Curvilinear Hamiltonian

- In the curvilinear system then

$$H = \sqrt{p^2c^2 + m^2c^4} \qquad \dot{\vec{p}} = -\vec{\nabla} H \qquad \dot{\vec{x}} = \vec{\nabla}_p H$$

$$\Rightarrow H = \sqrt{m^2c^4 + c^2\left[p_x^2 + p_y^2 + \left(\frac{p_s}{1+x/\rho}\right)^2\right]}$$

- In terms of canonical momenta:

$$H = c\sqrt{(P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1+x/\rho}\right)^2 + m^2c^2} + q\phi$$

where we have also defined $A_s \equiv \left(1 + \frac{x}{\rho}\right)\vec{A}\cdot\hat{\theta}$

Conjugate Hamiltonian

$$H = c\sqrt{(P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2 + m^2c^2} + q\phi$$

- We have one more little trick up our sleeves: it is natural to think not of time as the independent coordinate, but the pathlength *s*
- The conjugate momentum to *t* is –*H*, or the negative of the total energy *U*. Then

$$\begin{aligned} H &= -P_s(x, P_x, y, P_y, t, -U; s) \\ &= -qA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{U - q\phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2} \end{aligned}$$

- This is why longitudinal coordinates are usually naturally expressed in terms of time and energy

3.5: Symplecticity and its Consequences

- We have written a Hamiltonian for relativistic particle motion in general EM fields
  - *s* is "time-like" coordinate along design trajectory
  - $P_{x,y}$ are true canonical momenta

$$H = H(x, P_x, y, P_y, t, -U; s)$$

  - The coordinates are a six dimensional **phase space** column vector

$$\vec{X} = (x, P_x, y, P_y, t, -U)^{\mathrm{T}}$$

  with Hamilton's equations

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial P_i} \qquad \frac{dP_i}{ds} = -\frac{\partial H}{\partial x_i}$$

Symplectic Form

- We can write Hamilton's equations in a nice form:

$$\frac{dX_i}{ds} = \sum_j S_{ij} \frac{\partial H}{\partial X_j}$$

where

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Note a few things about *S*:

$$\det S = 1 \qquad S^2 = -I \qquad S^{\mathrm{T}} = -S \qquad S^{-1} = -S$$

S is like a matrix form of $i = \sqrt{-1}$

Matrix Coordinate Transformation

- Instead of writing down equations of motion from the Hamiltonian, we can write a **map *M***, or coordinate transformation, from location s_0 to s_1

$$M_{ij} \equiv \frac{\partial X_i(s_1)}{\partial X_j(s_0)}$$

- The total derivative of the coordinates is then

$$\frac{dX_i(s_1)}{ds} = \sum_j \frac{\partial X_i(s_1)}{\partial X_j(s_0)} \frac{dX_j(s_0)}{ds} = \sum_j M_{ij} \frac{dX_j(s_0)}{ds}$$

- Hamilton's equations at s_1:

$$\frac{dX_i(s_1)}{ds} = \sum_j S_{ij} \frac{\partial H}{\partial X_j(s_1)} = \sum_{jk} S_{ij} \frac{\partial H}{\partial X_k(s_0)} \underbrace{\frac{\partial X_k(s_0)}{\partial X_j(s_1)}}_{\left(\left(M^{-1}\right)^{\mathrm{T}}\right)_{jk}}$$

Definition of *M*

Matrix Coordinate Transformation

$$\frac{dX_i(s_1)}{ds} = \sum_{jk} S_{ij} \left(\left(M^{-1}\right)^{\mathrm{T}} \right)_{jk} \frac{\partial H}{\partial X_k(s_0)}$$

$$\frac{dX_l(s_0)}{ds} = \sum_{k} S_{lk} \frac{\partial H}{\partial X_k(s_0)} \quad \Rightarrow \quad \frac{\partial H}{\partial X_k(s_0)} = -\sum_{l} S_{kl} \frac{dX_l(s_0)}{ds}$$

$$\frac{dX_i(s_1)}{ds} = -\sum_{jkl} S_{ij} \left(\left(M^{-1}\right)^{\mathrm{T}} \right)_{jk} S_{kl} \frac{dX_l(s_0)}{ds}$$

$$= \sum_{j} M_{ij} \frac{dX_j(s_0)}{ds}$$

From previous page

Transpose,inverse commute

$$M = -S(M^{\mathrm{T}})^{-1} S \quad \Rightarrow \quad \boxed{M^{\mathrm{T}} S M = S}$$

Matrix *M* is **symplectic**

Symplecticity

$$\boxed{M^{\mathrm{T}} S M = S}$$

- *S* is also symplectic since $S^{\mathrm{T}} S S = (-S)(-I) = S$
- Symplecticity implies that *M* is unimodular

$$1 = \det S = \det(M^{\mathrm{T}} S M) = (\det M)^2 \quad \Rightarrow \quad \boxed{\det M = \pm 1}$$

- Inverse of *M*

$$S = M^{\mathrm{T}} S M \quad \Rightarrow \quad S S M^{-1} = S M^{\mathrm{T}} S \quad \Rightarrow \quad \boxed{M^{-1} = -S M^{\mathrm{T}} S = S^{\mathrm{T}} M^{\mathrm{T}} S}$$

- General symplectic conjugate of a 2N-d matrix *N*

$$\tilde{N} \equiv S^{\mathrm{T}} N^{\mathrm{T}} S$$

  - This is just N^{-1} if *N* is symplectic
- You will derive some useful identities for *M* in the homework

Symplecticity and Nonlinearity

- We have treated everything here using first-order differentiation
  - This is equivalent to linearizing the transport
  - Hence matrices (homogeneous linear transformations)
- However, symplecticity holds even for nonlinear transport
  - *M* becomes Jacobian matrix of the map
  - Linear map: independent of particle coordinates
    - We can write general transport matrices for linear magnet transport (drifts, dipoles, quadrupoles, linear RF, etc)
  - Nonlinear maps depend on particle coordinates
    - We will discuss more on nonlinear dynamics next week

3.6: Hamiltonian Simplifications

- Most accelerators are dominated by regions where there are only static transverse magnetic fields
  - We'll discuss field expansions and these magnets more this afternoon while discussing Chapter 4
- We have the freedom to choose vector and scalar potential gauges such that

$$\phi = 0 \qquad A_x = 0 \qquad A_y = 0$$

which gives

$$P_x = p_x = \gamma\beta_x mc \qquad P_y = p_y = \gamma\beta_y mc$$

$$H = -qA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{U}{c}\right)^2 - m^2c^2 - p_x^2 - p_y^2}$$

Standard Canonical Coordinates

- Rescale Hamiltonian by design momentum *p₀*

$$\frac{H}{p_0} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{U}{p_0 c}\right)^2 - \left(\frac{mc}{p_0}\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2}$$

$$\text{Paraxial approximation}: \frac{p_x}{p_0} \approx \frac{dx}{ds} = x' \quad \frac{p_y}{p_0} \approx \frac{dy}{ds} = y'$$

- And apply one last canonical coordinate transformation

$$(t, -U/p_0) \to (z \equiv s - v_0 t, \delta \equiv (p - p_0)/p_0)$$

$$H(x, x', y, y', z, \delta; s) = H(x, x', y, y', t, -U/p_0; s) + \frac{\partial F_2(t, \delta; s)}{\partial s}$$

$$z = \frac{\partial F_2}{\partial \delta} \qquad -U/p_0 = \frac{\partial F_2}{\partial t}$$

Standard Canonical Coordinates

$$F_2 = F_2(t,\delta;s) \qquad z = \frac{\partial F_2}{\partial \delta} \qquad -U/p_0 = \frac{\partial F_2}{\partial t}$$

$$F_2 = z\delta + F(t,s) = (s - v_0 t)\delta + F(t,s) \qquad F_2 = -\frac{U}{p_0}t + F(\delta, s)$$

$$\frac{U}{p_0} = \frac{U_0}{p_0}\left(1 + \frac{\Delta U}{U}\right) = \frac{c}{\beta_0}(1 + \beta_0^2\delta)$$

$$F_2 = -\frac{c}{\beta_0}(1 + \beta_0^2\delta)t + \delta s + F(s)$$

$$F_2 = \frac{c}{\beta_0}(1 + \beta_0^2\delta)\left(\frac{s}{v_0} - t\right) - \frac{s}{\beta_0}{}^{2} + s$$

$$H = -\frac{qA_s}{p_0} + \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{p}{p_0}\right)^2 - x'^2 - y'^2} + \frac{\partial F_2(t,\delta;s)}{\partial s}$$

Standard Canonical Coordinates

$$F_2 = \frac{c}{\beta_0}(1+\beta_0^2\delta)\left(\frac{s}{v_0} - t\right) - \frac{s}{\beta_0}^{\,2} + s$$

$$H = -\frac{qA_s}{p_0} + \left(1+\frac{x}{\rho}\right)\sqrt{\left(\frac{p}{p_0}\right)^2 - x'^2 - y'^2} + \frac{\partial F_2(t,\delta;s)}{\partial s}$$

$$= -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\sqrt{1+2\delta+\delta^2 - x'^2 - y'^2} + 1 + \delta$$

- Keeping lowest-order terms since we are now (in principle) perturbative in all coordinates

$$\boxed{H = -\frac{qA_s}{p_0} + \frac{1}{2}(x'^2 + y'^2) - \frac{x}{\rho} - \frac{x\delta}{\rho} + \ldots}$$

where **all** magnetic field contributions are from the longitudinal vector field component A_s

Coordinate Systems Aplenty

- Many different codes and approaches use many different 6D coordinate systems
  - E.g. Section 1.7 of the madx documentation:

$$(x, x' \equiv p_x/p_0, y, y' \equiv p_y/p_0, T \equiv -ct, PT \equiv \Delta E/p_s c)$$

  - We follow C-M, which uses the same transverse coordinates, but generally uses

$$\delta \equiv \frac{p - p_0}{p_0}$$

as a longitudinal momentum coordinate, and *z* as the corresponding spatial coordinate (e.g. homework 3-6).

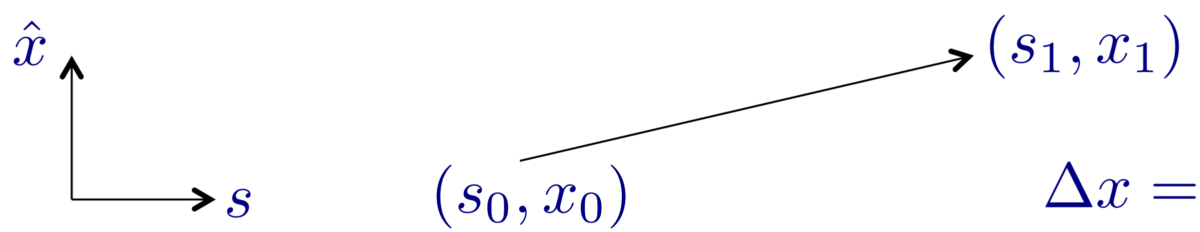
Often 2D and 4D subspaces are used when appropriate.

Drift Transport Matrix

- So now we have nicely intuitive coordinates that are independent in free space:

$$(x, x') \quad (y, y') \quad (z, \delta)$$

- We can write down *M* by inspection for a field-free region for each of the transverse coordinates:



$$\Delta x = x_1 - x_0$$
$$\Delta x' = 0$$

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & s_1 - s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \Rightarrow \boxed{M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}}$$

Transverse transport matrix of drift of length *L*
(is it symplectic? What about z, δ?)

3.7: Symplectic Generators

- As we add transverse magnetic fields, we need a general way to *construct* transport matrices
- Consider magnets with fields that are independent of *s* for now. Recall we only need longitudinal vector field components to describe transverse fields, so

$$A_s = A_s(x, y)$$

  - We can integrate a matrix generator *G* for infinitesimal steps *ds:* $M(ds) = I + G\,ds$
  - For s-independent fields, *G* is constant
  - Integrating gives us a Lie-algebra like result

$$M(s) = \lim_{n\to\infty}\left(I + G\frac{s}{n}\right)^n = \mathrm{e}^{Gs}$$

where $\mathrm{e}^{Gs} = I + (Gs) + \frac{1}{2!}(Gs)^2 + \frac{1}{3!}(Gs)^3 + \ldots$

Symplectic Generator: Dipole

- Consider a pure "simple" dipole field: $\vec{B} = B_0 \hat{y}$
- Our vector potential simplification was

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{1 + x/\rho} \left(\frac{\partial A_s}{\partial y} \right) \hat{x} + \frac{1}{1 + x/\rho} \left(-\frac{\partial A_s}{\partial x} \right) \hat{y}$$

$$B_y = B_0 = -\frac{1}{1 + x/\rho} \left(\frac{\partial A_s}{\partial x} \right)$$

$$\Rightarrow \quad A_s = -B_0 \left(x + \frac{x^2}{2\rho} \right)$$

$$H = -\frac{q A_s}{p_0} + \frac{1}{2}(x'^2 + y'^2) - \frac{x}{\rho} - \frac{x\delta}{\rho} + \ldots$$ From boxed eqn on slide 38

$$H = \frac{q B_0}{p_0} \left(x + \frac{x^2}{2\rho} \right) - \frac{x}{\rho} - \frac{x\delta}{\rho} + \frac{1}{2}(x'^2 + y'^2)$$

Symplectic Generator: Dipole

$$H = \frac{qB_0}{p_0}\left(x + \frac{x^2}{2\rho}\right) - \frac{x}{\rho} - \frac{x\delta}{\rho} + \frac{1}{2}(x'^2 + y'^2)$$

- Remember the curvature, momentum, and field are related! $p_0/q = B_0\rho \quad \Rightarrow \quad qB_0/p_0 = 1/\rho$

$$H = \frac{x^2}{2\rho^2} - \frac{x\delta}{\rho} + \frac{1}{2}(x'^2 + y'^2)$$

$$\frac{dx}{ds} = \frac{\partial H}{\partial x'} = x' \qquad \frac{dx'}{ds} = -\frac{\partial H}{\partial x} = -\frac{x}{\rho^2} + \frac{\delta}{\rho}$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial y'} = y' \qquad \frac{dy'}{ds} = -\frac{\partial H}{\partial y} = 0$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{x}{\rho} \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z} = 0$$

Symplectic Generator: Dipole

$$\frac{dx}{ds} = \frac{\partial H}{\partial x'} = x' \qquad \frac{dx'}{ds} = -\frac{\partial H}{\partial x} = -\frac{x}{\rho^2} + \frac{\delta}{\rho}$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial y'} = y' \qquad \frac{dy'}{ds} = -\frac{\partial H}{\partial y} = 0$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{x}{\rho} \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z} = 0$$

$$x'_1 = x'_0 - \left(\frac{ds}{\rho^2}\right) x + \left(\frac{ds}{\rho}\right) \delta$$

$$\begin{pmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \\ z_1 \\ \delta_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & ds & 0 & 0 & 0 & 0 \\ -ds/\rho^2 & 1 & 0 & 0 & 0 & ds/\rho \\ 0 & 0 & 1 & ds & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -ds/\rho & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{(I + G\, ds)} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ z_0 \\ \delta_0 \end{pmatrix}$$

Symplectic Generator: Dipole

$$G\,ds = \begin{pmatrix} 0 & \rho & 0 & 0 & 0 & 0 \\ -1/\rho & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{ds}{\rho} = K\frac{ds}{\rho} = K\,d\theta$$

$$M(\theta) = \lim_{n\to\infty}\left(I + K\frac{\theta}{n}\right)^n = e^{K\theta} = I + \theta K + \frac{\theta^2}{2!}K^2 + \frac{\theta^3}{3!}K^3 + \dots$$

$$K^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \rho \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = \begin{pmatrix} 0 & -\rho & 0 & 0 & 0 & 0 \\ 1/\rho & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\rho \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} K^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\rho \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -K^2$$

$$K^5 = -K^3 \dots$$

Symplectic Generator: Dipole

$$M(\theta) = \lim_{n\to\infty}\left(I + K\frac{\theta}{n}\right)^n = e^{K\theta} = I + \theta K + \frac{\theta^2}{2!}K^2 + \frac{\theta^3}{3!}K^3 + \ldots$$

$$K = \begin{pmatrix} 0 & \rho & 0 & 0 & 0 & 0 \\ -1/\rho & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \rho \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^3 = \begin{pmatrix} 0 & -\rho & 0 & 0 & 0 & 0 \\ 1/\rho & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\rho \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\rho \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -K^2$$

$$K^5 = -K^3 \ldots$$

$$M(\theta) = I + \theta K + K^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \ldots\right) + K^3\left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \ldots\right)$$

Symplectic Generator: Dipole

$$M(\theta) = I + \theta K + K^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \ldots \right) + K^3 \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \ldots \right)$$

$$= I + \theta K + K^2 (1 - \cos\theta) + K^3 (\theta - \sin\theta)$$

Substituting in powers of K in matrix form:

$$M(\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 1 & -\rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Dipole transport matrix for coordinate system $(x, x', y, y', z, \delta)$

You will do a similar, but more straightforward, exercise for the solenoid in homework.