USPAS Accelerator Physics 2017
University of California, Davis

Chapter 5: Strong Focusing

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Happy Birthday to Yoichiro Nambu, Paul Ehrenfest, AA Milne, and Cary Grant!
Happy National Peking Duck Day, Thesaurus Day, and Winnie the Pooh Day!

Overview

- Doublets and simple focusing
- 5.1: Transfer matrix approach and linear stability
- 5.2: Analytical approach
- 5.3: Emittances (revisit Friday PM)
- 5.4: Adiabatic Invariants
- 5.5: Dispersion
- 5.6: Momentum compaction
- Solutions to some problems

Strong Focusing

- String magnets together and look at optical systems
- Real transport systems are generally made of long repeated periods of magnet groups
  - Keep beam together while getting from here to there
- Fundamental problems
  - Quadrupoles (linear focusing) defocus in opposite plane
  - Focusing in both planes with weak focusing is too weak
- How do we build a focusing system out of magnets and drifts that we know about?
  - Easiest analysis: string together transport matrices

Magnet Transport Matrix Review

Drift
$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Thick quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} \cos\sqrt{k}l & \frac{1}{\sqrt{k}}\sin\sqrt{k}l \\ -\sqrt{k}\sin\sqrt{k}l & \cos\sqrt{k}l \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad \begin{pmatrix} y \\ y' \end{pmatrix}_2 = \begin{pmatrix} \cosh\sqrt{k}l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}l \\ \sqrt{k}\sinh\sqrt{k}l & \cosh\sqrt{k}l \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_1$$

Thin quadrupole
$f=(kl)^{-1}$>>length l

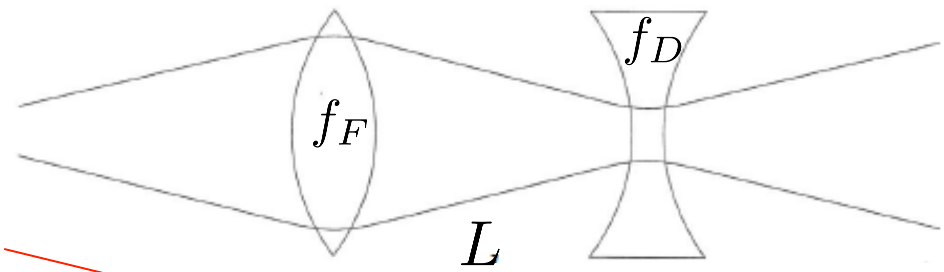
$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} 1 & 0 \\ \mp kl & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Sector dipole of bend angle θ and radius ρ

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix}_2 = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 1 & -\rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix}_1$$

Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift L



Thin quadrupole matrices

$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$f_F, f_D > 0$

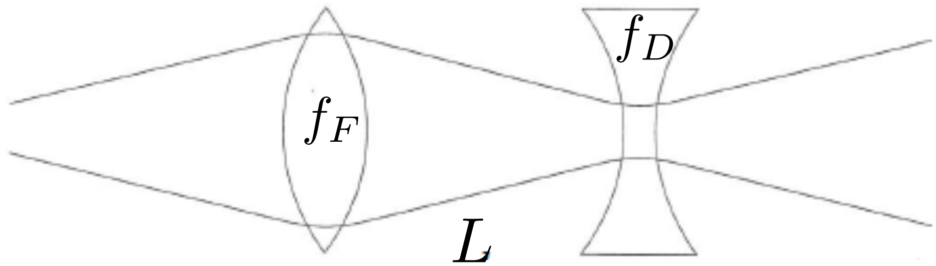
$$\frac{1}{f_{\text{doublet}}} = \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \qquad \text{(C\&M 5.1 with } f_F = -f_D)$$

$$f_D = f_F = f \quad \Rightarrow \quad \frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2}$$

There is **net focusing** given by this **alternating gradient** system

A fundamental point of optics, and of accelerator **strong focusing**

Strong Focusing: Another View



$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$$\text{incoming paraxial ray} \qquad \begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \end{pmatrix} x_0$$

For this to be focusing, x' must have opposite sign of x where these are coordinates of transformation of incoming paraxial ray

$$\boxed{f_F = f_D \quad x' < 0 \quad \mathbf{BUT} \quad x > 0 \text{ iff } f_F > L}$$

Equal strength doublet is net focusing under condition that each lens' focal length is greater than distance between them

Transport Matrix Stability Criteria

- For long systems (rings) we want $M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ stable as $n \to \infty$
  - If 2x2 M has eigenvectors (V_1, V_2) and eigenvalues (λ_1, λ_2):

$$M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = A\lambda_1^n V_1 + B\lambda_2^n V_2$$

  - M is also unimodular (det M=1) so $\lambda_{1,2} = e^{\pm i\mu}$ with complex μ
  - For $\lambda_{1,2}^n$ to remain bounded, μ must be real
  - We can always transform M into diagonal form with the eigenvalues on the diagonal (since det M=1); this does not change the trace of the matrix

$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = \mathrm{Tr}\, M$$

  - The **stability requirement** for these types of matrices is then

$$\mu \text{ real} \qquad \Rightarrow \qquad \boxed{-1 \leq \frac{1}{2}\,\mathrm{Tr}\, M \leq 1}$$

More Math: Hill's Equation

- Let's go back to our quadrupole equations of motion for $R \to \infty$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)$$

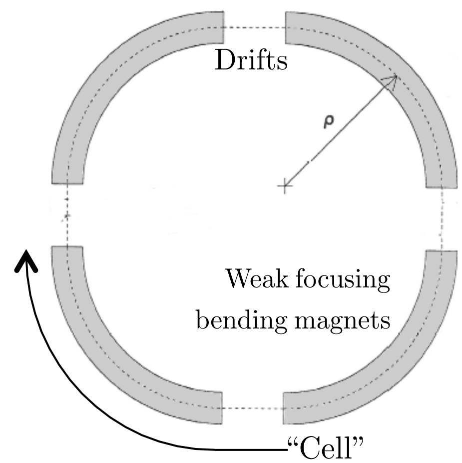
What happens when we let the focusing K vary with s?

Also assume K is **periodic** in s with some periodicity C

$$x'' + K(s)x = 0 \qquad K(s) \equiv \frac{1}{(B\rho)} \left(\frac{\partial B_y}{\partial x} \right)(s) \qquad K(s+C) = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated "cell" of the layout

The simple harmonic oscillator equation with a **periodically** varying spring constant K(s) is known as **Hill's Equation**



Hill's Equation Solution Ansatz

$$x'' + K(s)x = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)(s)$$

- Solution is a quasi-periodic harmonic oscillator

$$x(s) = A\,w(s)\,\cos[\Psi(s) + \Psi_0]$$

where w(s) is periodic in C but the phase Ψ(s) is not!!

Substitute this educated guess ("ansatz") to find

$$x' = Aw'\cos[\Psi + \Psi_0] - Aw\Psi'\sin[\Psi + \Psi_0]$$

$$x'' = A(w'' - w\Psi'^2)\cos[\Psi + \Psi_0] - A(2w'\Psi' + w\Psi'')\sin[\Psi + \Psi_0]$$

$$x'' + K(s)x = -A(2w'\Psi' + w\Psi'')\sin(\Psi + \Psi_0) + A(w'' - w\Psi'^2 + Kw)\cos(\Psi + \Psi_0) = 0$$

For $w(s)$ and $\Psi(s)$ to be independent of Ψ_0, coefficients of the sin and cos terms must vanish identically

Courant-Snyder Parameters

$$2ww'\Psi' + w^2\Psi'' = (w^2\Psi')' = 0 \quad \Rightarrow \quad \Psi' = \frac{k}{w(s)^2}$$

$$w'' - (k^2/w^3) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2$$

- Notice that in both equations $w^2 \propto k$ so we can scale this out and define a new set of functions, **Courant-Snyder Parameters** or **Twiss Parameters**

$$\boxed{\begin{aligned} \beta(s) &\equiv \frac{w^2(s)}{k} \\ \alpha(s) &\equiv -\frac{1}{2}\beta'(s) \\ \gamma(s) &\equiv \frac{1+\alpha(s)^2}{\beta(s)} \end{aligned}} \quad \Rightarrow$$

$$\Psi' = \frac{1}{\beta(s)} \qquad \boxed{\Psi(s) = \int \frac{ds}{\beta(s)}}$$

$$\boxed{K\beta = \gamma + \alpha'}$$

$\beta(s), \alpha(s), \gamma(s)$ are all periodic in C

$\Psi(s)$ is **not** periodic in C

Towards The Matrix Solution

- What is the matrix for this Hill's Equation solution?

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

Take a derivative with respect to s to get $x' \equiv \frac{dx}{ds}$

$$\Psi' = \frac{1}{\beta(s)} \qquad x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

Now we can solve for A and B in terms of initial conditions $(x(0), x'(0))$

$$x_0 \equiv x(0) = A\sqrt{\beta(0)} \qquad x'_0 \equiv x'(0) = \frac{1}{\sqrt{\beta(0)}}[B - \alpha(0)A]$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x'_0 + \alpha(0)x_0]$$

And take advantage of the periodicity of β, α to find $x(C), x'(C)$

Hill's Equation Matrix Solution

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

$$x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x'_0 + \alpha(0)x_0]$$

$$x(C) = [\cos\Psi(C) + \alpha(0)\sin\Psi(C)]x_0 + \beta(0)\sin\Psi(C)x'_0$$

$$x'(C) = -\gamma(0)\sin\Psi(C)x_0 + [\cos\Psi(C) - \alpha(0)\sin\Psi(C)]x'_0$$

We can write this down in a matrix form where $\mu = \Psi(C) - \Psi(0)$ is the betatron phase advance through one period C

$$\boxed{\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}}$$

$$\boxed{\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}}$$ phase advance per cell

Interesting Observations

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

phase advance per cell

- μ is independent of s: this is the **betatron phase advance** of this periodic system
- Determinant of matrix M is still 1!
- Looks like a rotation and some scaling
- M can be written down in a **beautiful** and **deep** way

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

remember $x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$

Convenient Calculations

- If we know the transport matrix M, we can find the lattice parameters (periodic in C)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance per cell} \quad \cos\mu = \frac{1}{2}\,\mathrm{Tr}\,M$$

$$\beta(0) = \beta(C) = \frac{m_{12}}{\sin\mu}$$

$$\alpha(0) = \alpha(C) = \frac{m_{11} - \cos\mu}{\sin\mu}$$

$$\gamma(0) \equiv \frac{1+\alpha^2(0)}{\beta(0)}$$

General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using lattice parameters and $\Delta\Psi = \Psi(s_2) - \Psi(s_1)$

$$M_{s_1\to s_2} = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\Psi + \alpha(s_1)\sin\Delta\Psi] & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\Psi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\Psi+[1+\alpha(s_1)\alpha(s_2)]\sin\Delta\Psi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\Psi - \alpha(s_2)\sin\Delta\Psi] \end{pmatrix}$$

(C&M Eqn 5.52)

- This does not have a pretty form like the periodic matrix

However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

A common use of this matrix is the m_{12} term:

$$\boxed{\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\Psi)\, x'(s_1)}$$

Effect of angle kick on downstream position

(Deriving the Non-Periodic Transport Matrix)

$$x(s) = Aw(s)\cos\Psi(s) + Bw(s)\sin\Psi(s)$$

$$x'(s) = A\left(w'(s)\cos\Psi(s) - \frac{\sin\Psi(s)}{w(s)}\right) + B\left(w'(s)\sin\Psi(s) + \frac{\cos\Psi(s)}{w(s)}\right)$$

Calculate A, B in terms of initial conditions (x_0, x_0') and (w_0, Ψ_0)

$$A = \left(w_0'\sin\Psi_0 + \frac{\cos\Psi_0}{w_0}\right)x_0 - (w_0\sin\Psi_0)x_0'$$

$$B = -\left(w_0'\cos\Psi_0 - \frac{\sin\Psi_0}{w_0}\right)x_0 + (w_0\cos\Psi_0)x_0'$$

Substitute (A,B) and put into matrix form: $\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

$$m_{11}(s) = \frac{w(s)}{w_0}\cos\Delta\Psi - w(s)w_0'\sin\Delta\Psi$$

$$\Delta\Psi \equiv \Psi(s) - \Psi_0$$

$$w(s) = \sqrt{\beta(s)}$$

$$m_{12}(s) = w(s)w_0\sin\Delta\Psi$$

$$m_{21}(s) = -\frac{1 + w(s)w_0w'(s)w_0'}{w(s)w_0}\sin\Delta\Psi - \left[\frac{w_0'}{w(s)} - \frac{w'(s)}{w_0}\right]\cos\Delta\Psi$$

$$m_{22}(s) = \frac{w_0}{w(s)}\cos\Delta\Psi + w_0w'\sin\Delta\Psi$$

Non-Periodic Transport Matrix

- C-M 5.52, a very general form from all that math:

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}[\cos\mu(s) + \alpha_0 \sin\mu(s)] & \sqrt{\beta_0\beta(s)}\sin\mu(s) \\ -\frac{[\alpha(s)-\alpha_0]\cos\mu(s)+[1+\alpha_0\alpha(s)]\sin\mu(s)}{\sqrt{\beta_0\beta(s)}} & \sqrt{\frac{\beta_0}{\beta(s)}}[\cos\mu(s) - \alpha(s)\sin\mu(s)] \end{pmatrix}$$

  - This is how one can parameterize uncoupled x or y transport from one location s_0 to another location s where the Twiss parameters of those locations and their relative phase advance is known
    - Such as from a table from a modeling code like madx
  - Most modeling codes can also output matrix elements between specific locations at starts or ends of magnets
    - This still follows the (C S ; C' S') matrix form!

Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi}-\text{periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\Psi(s)+\Psi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \Psi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

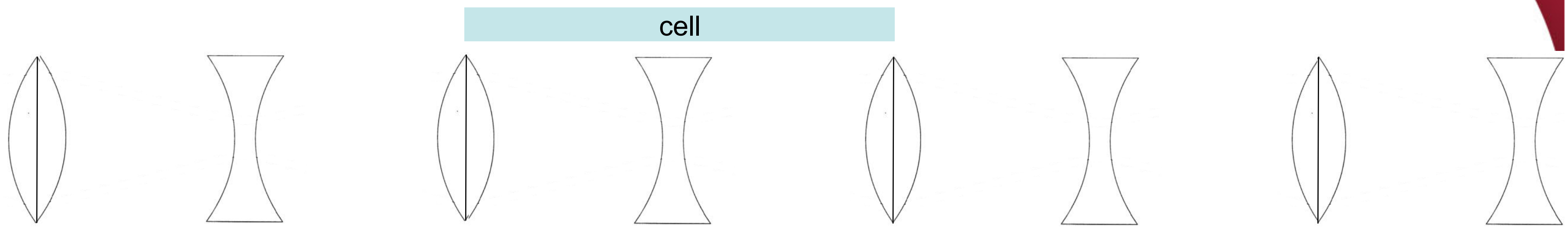
betatron phase advance $\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$ $\quad \mathrm{Tr}\, M = 2\cos\mu$

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

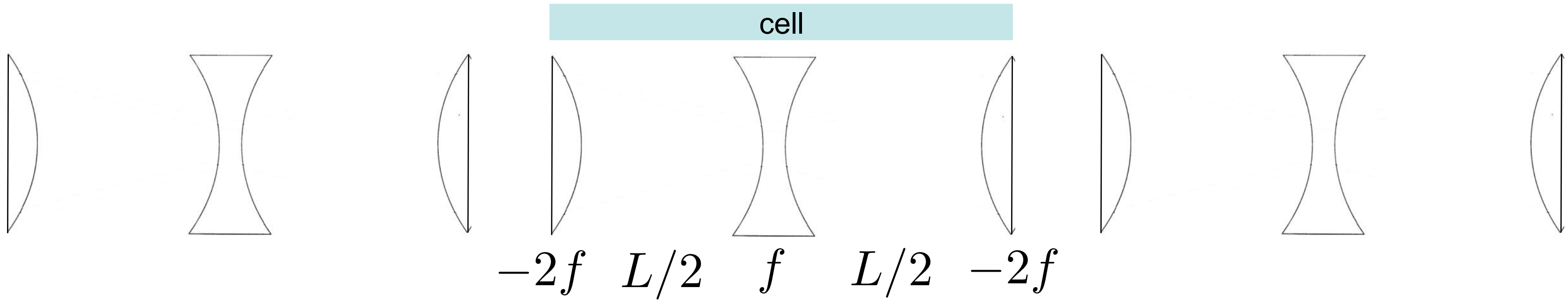
(Lunch break)

Lattice Optics: FODO Lives!



- Most accelerator lattices are designed in modular ways
  - Design and operational clarity, separation of functions
- One of the most common modules is a FODO module
  - Alternating focusing and defocusing “strong” quadrupoles
  - Spaces between are combinations of drifts and dipoles
  - Strong quadrupoles dominate the focusing
  - Periodicity is one FODO “cell” so we’ll investigate that motion
  - Horizontal beam size largest at centers of focusing quads
  - Vertical beam size largest at centers of defocusing quads

Periodic Example: FODO Cell Phase Advance



- Select periodicity between centers of **focusing** quads
  - A natural periodicity if we want to calculate maximum β(s)

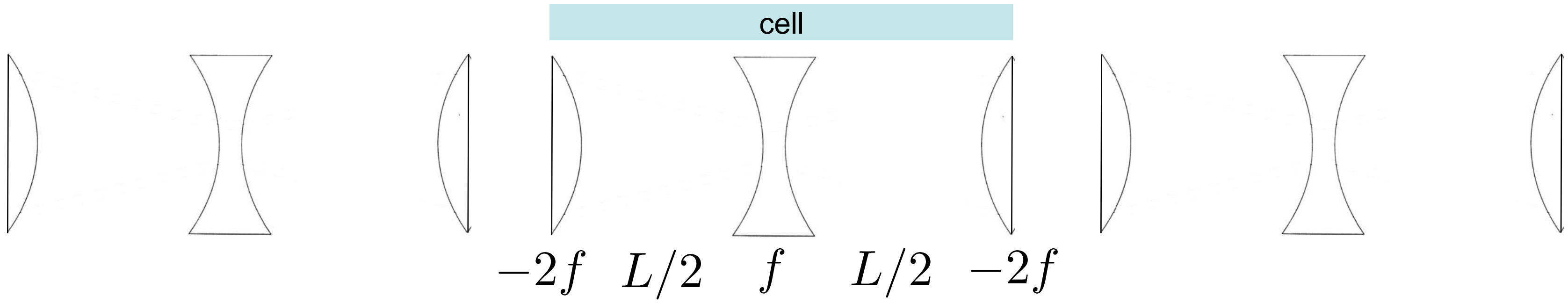
$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\mu = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\mu = 1 - 2\sin^2\frac{\mu}{2} \quad \Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f}$$

  - μ only has real solutions (stability) if $\frac{L}{4} < f$

Periodic Example: FODO Cell Beta Max/Min



- What is the maximum beta function, $\hat{\beta}$?
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad m_{12} = \beta \sin \mu$$

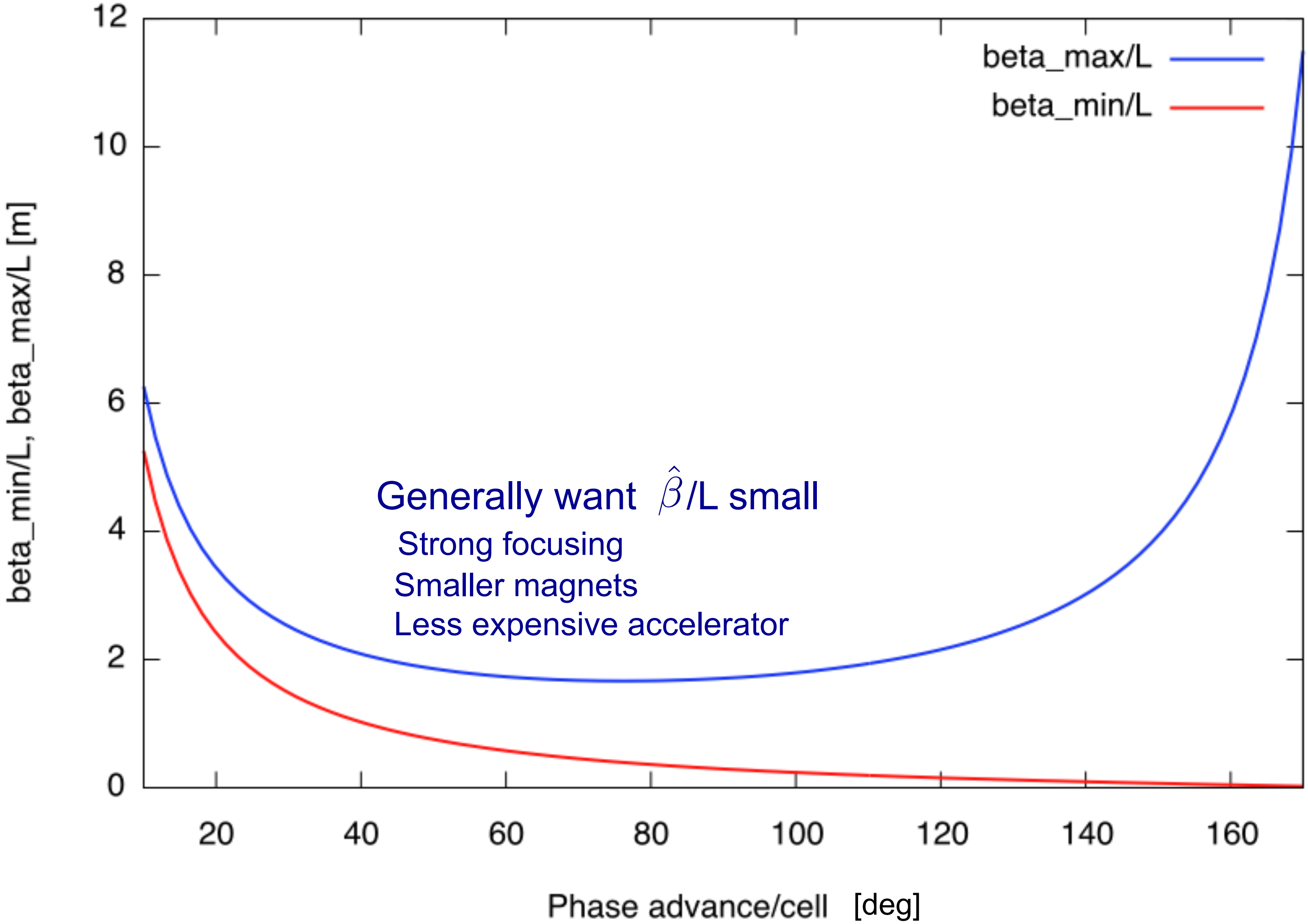
$$\hat{\beta} \sin \mu = \frac{L^2}{4f} + L = L \left(1 + \sin \frac{\mu}{2}\right)$$

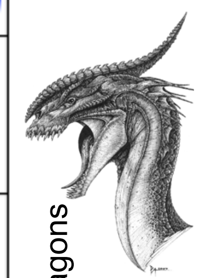
$$\boxed{\hat{\beta} = \frac{L}{\sin \mu} \left(1 + \sin \frac{\mu}{2}\right)}$$

  - Follow a similar strategy reversing F/D quadrupoles to find the minimum β(s) within a FODO cell (center of D quad)

$$\boxed{\check{\beta} = \frac{L}{\sin \mu} \left(1 - \sin \frac{\mu}{2}\right)}$$

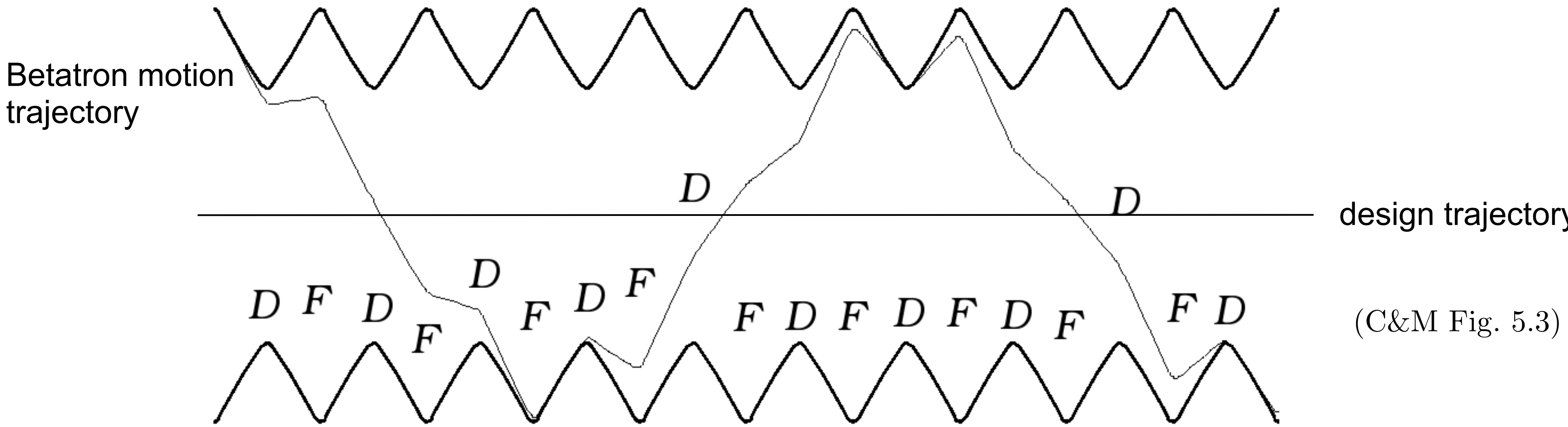
FODO Betatron Functions vs Phase Advance





Here there be dragons

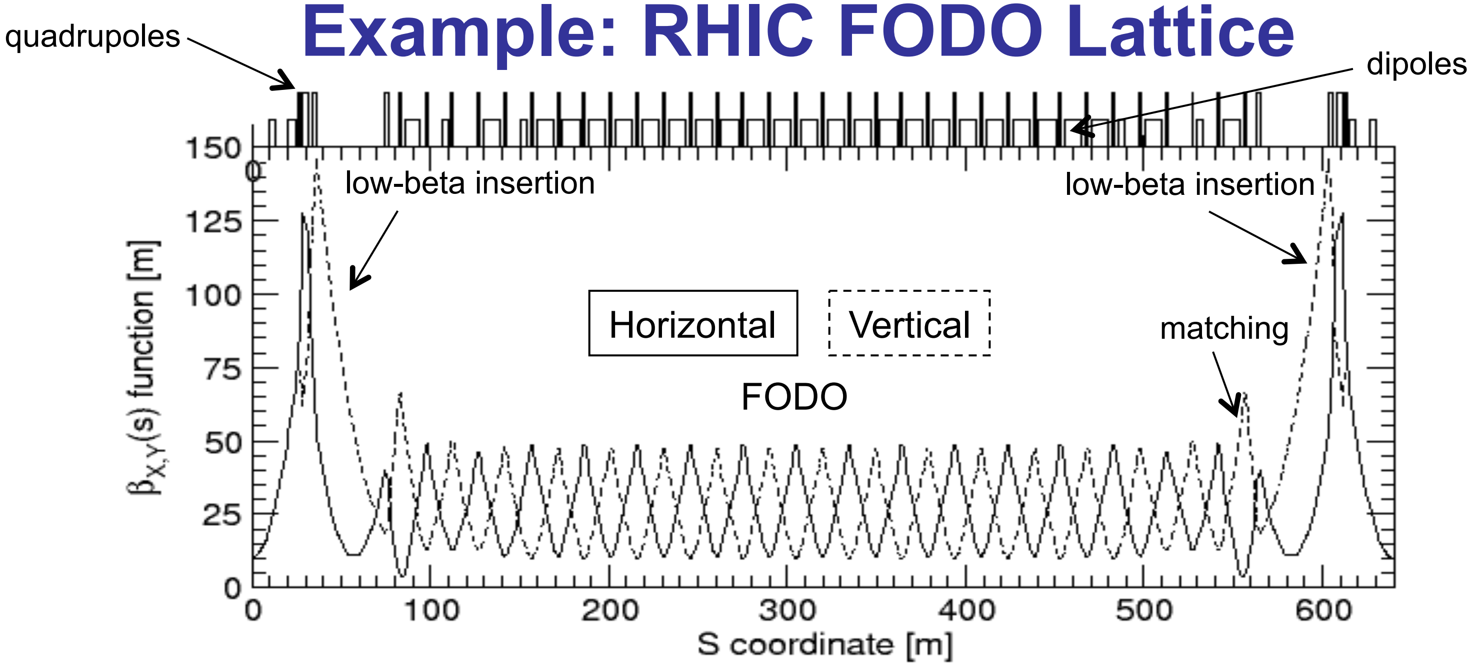
FODO Beta Function, Betatron Motion



- This is a picture of a FODO lattice, showing contours of $\pm\sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$
  - This also shows a particle oscillating through the lattice
  - Note that $\sqrt{\beta(s)}$ provides an "envelope" for particle oscillations
    - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
  - Min beta is at defocusing quads, max beta is at focusing quads
  - 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \quad \mu = 360^\circ/6.5 \approx 55^\circ$$

Example: RHIC FODO Lattice



- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about L=30 m
  - Phase advance per FODO cell is about $\mu = 77^\circ = 1.344\ \mathrm{rad}$

$$\hat{\beta} = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right) \approx 53\ \mathrm{m}$$

☺

$$\check{\beta} = \frac{L}{\sin\mu}\left(1 - \sin\frac{\mu}{2}\right) \approx 8.7\ \mathrm{m}$$

Propagating Lattice Parameters

- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos\mu + J \sin\mu = \begin{pmatrix} \cos\mu + \alpha(s_1)\sin\mu & \beta(s_1)\sin\mu \\ -\gamma(s_1)\sin\mu & \cos\mu - \alpha(s_1)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Homework ☺

$$\boxed{\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}}$$

Propagating Lattice Parameters

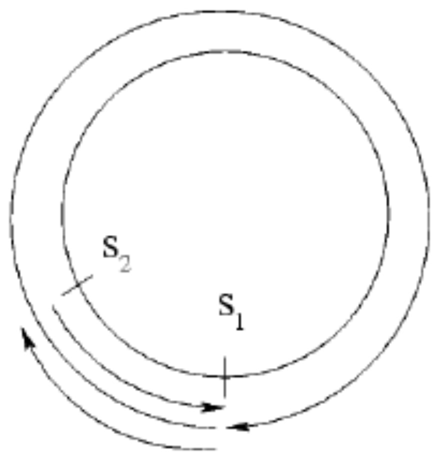
- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos\mu + J \sin\mu = \begin{pmatrix} \cos\mu + \alpha(s_1)\sin\mu & \beta(s_1)\sin\mu \\ -\gamma(s_1)\sin\mu & \cos\mu - \alpha(s_1)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The J(s) matrices at s_1, s_2 are related by

$$J(s_2) = M(s_1, s_2) J(s_1) M^{-1}(s_1, s_2)$$

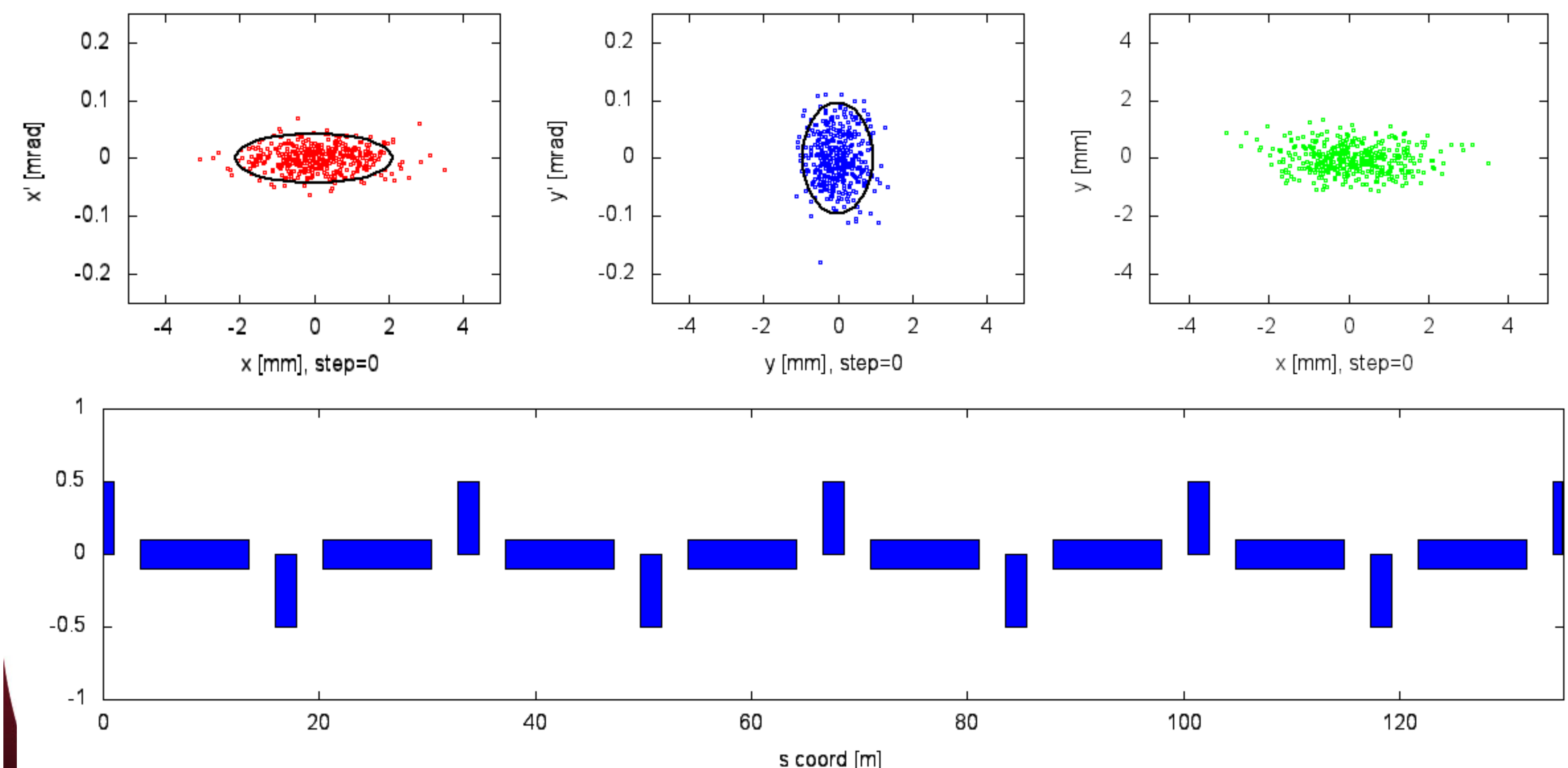
Then expand, using det M=1



Your homework is to then show…

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$

=========== What's the Ellipse? ===========



- Area of an ellipse that envelops a given percentage of the beam particles in phase space is related to the **emittance**

We can express this in terms of our lattice functions!

Invariants and Ellipses

$$x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

- We assumed A was constant, an **invariant of the motion**

  A can be expressed in terms of initial coordinates to find

  $$\boxed{\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2}$$

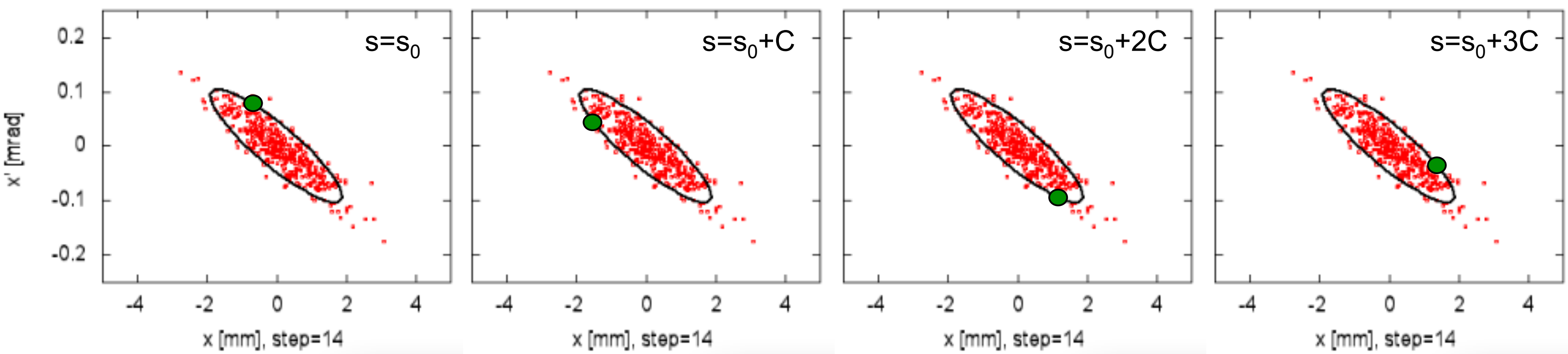
  This is known as the **Courant-Snyder invariant**: for all s,

  $$\mathcal{W} = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

  Similar to total energy of a simple harmonic oscillator

  $\mathcal{W}$ looks like an elliptical area in (x, x') phase space

  Our matrices look like scaled rotations (ellipses) in phase space



Emittance

- The area of the ellipse inscribed by any given particle in phase space as it travels through our accelerator is called the **emittance** ϵ: it is "constant" and given by

$$\boxed{\epsilon = \pi \mathcal{W} = \pi[\gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2]}$$

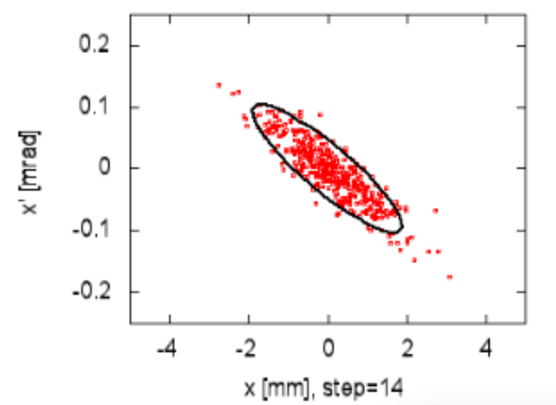
Emittance is often quoted as the area of the ellipse that would contain a certain fraction of all (Gaussian) beam particles

e.g. RMS emittance contains 39% of 2D beam particles

Related to RMS beam size σ_{RMS}

$$\sigma_{\mathrm{RMS}} = \sqrt{\epsilon\beta(s)}$$

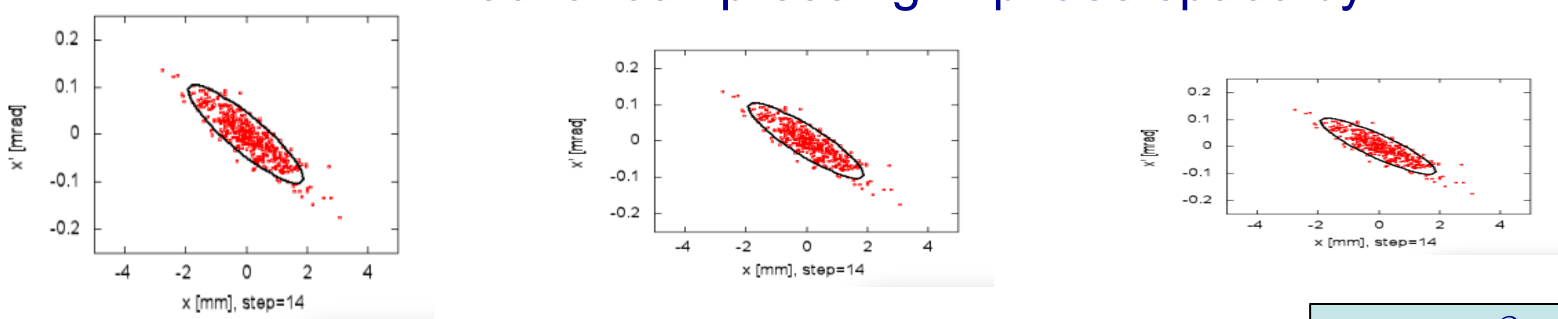
RMS beam size depends on s!



RMS emittance convention is fairly standard for electron rings, with units of mm-mrad

Adiabatic Damping and Normalized Emittance

- But we introduce electric fields when we accelerate
  - When we accelerate, invariant emittance is not invariant!
  - We are defining areas in (x, x') phase space
  - The definition of x doesn't change as we accelerate
  - But $x' \equiv dx/ds = p_x/p_0$ **does** since p_0 changes!
  - p_0 scales with relativistic beta, gamma: $p_0 \propto \beta\gamma$
  - This has the effect of compressing x' phase space by $\beta\gamma$

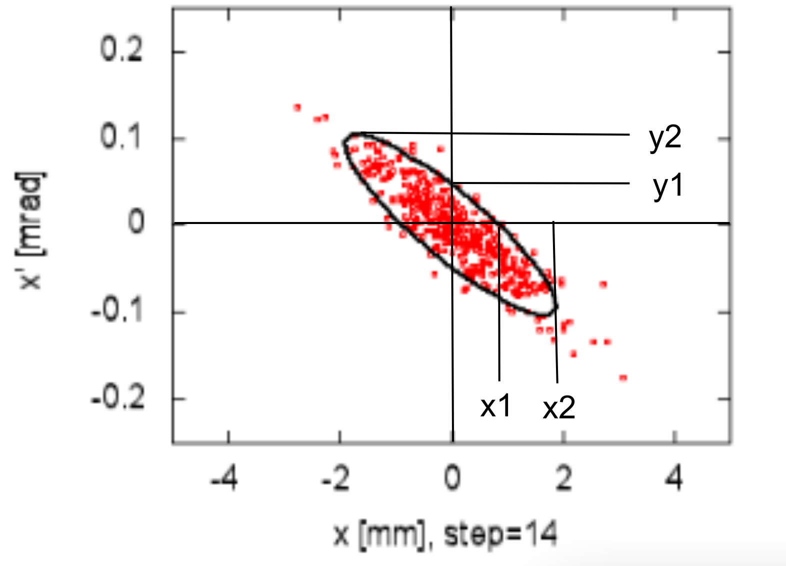


  - **Normalized emittance** is the invariant in this case $\boxed{\epsilon_{\mathrm{N}} \equiv \beta\gamma\epsilon}$

    unnormalized emittance goes down as we accelerate

    This is called **adiabatic damping**, important in, e.g., linacs

Phase Space Ellipse Geography



CM Fig 5.2

- Now we can figure out some things from a phase space ellipse at a given s coordinate:

$$x_1 = \sqrt{\mathcal{W}/\gamma(s)} \qquad x_2 = \sqrt{\mathcal{W}\beta(s)}$$

$$y_1 = \sqrt{\mathcal{W}/\beta(s)} \qquad y_2 = \sqrt{\mathcal{W}\gamma(s)}$$

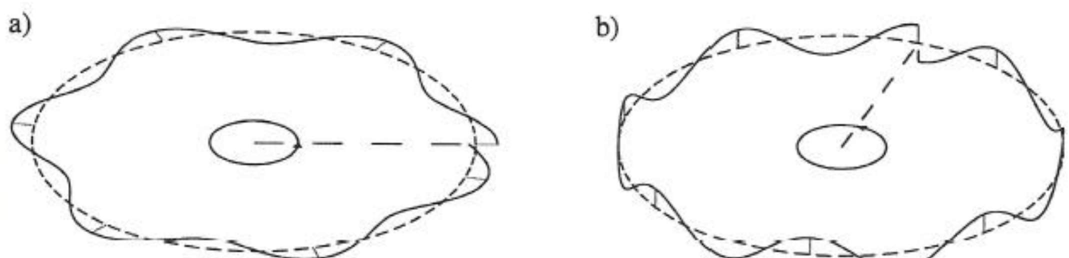
Rings and Tunes

- A synchrotron is by definition a periodic focusing system
  - It is very likely made up of many smaller periodic regions too
  - We can write down a periodic **one-turn matrix** as before

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

  - We define **tune** as the total betatron phase advance in one revolution around a ring divided by the total angle 2π

$$Q_{x,y} = \frac{\Delta\mu_{x,y}}{\Delta\theta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$



Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

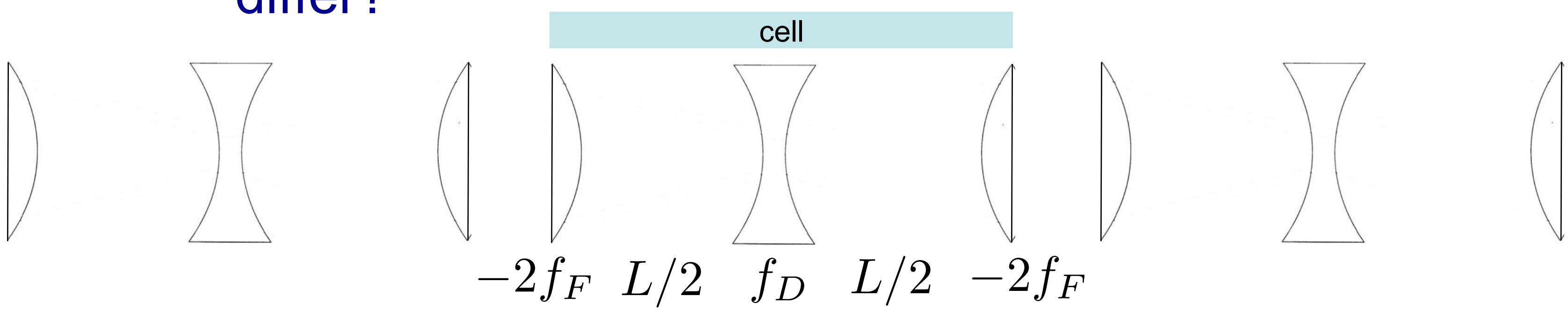
Tunes

- There are horizontal and vertical tunes
  - turn by turn oscillation frequency
- Tunes are a direct indication of the amount of focusing in an accelerator
  - Higher tune implies tighter focusing, lower $\langle \beta_{x,y}(s) \rangle$
- Tunes are a critical parameter for accelerator performance
  - Linear stability depends greatly on phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted by changing groups of quadrupoles

$$\boxed{M_{\text{one turn}} = I \cos(2\pi Q) + J \sin(2\pi Q)}$$

6.2: Stability Diagrams

- Designers often want or need to change the focusing of the two transverse planes in a FODO structure
  - What happens if the focusing/defocusing strengths differ?



  - Recalculate the M matrix and use dimensionless quantities

$$F \equiv \frac{L}{2f_F} \qquad D \equiv \frac{L}{2f_D}$$

then take the trace for stability conditions to find

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

Stability Diagrams II

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

- For stability, we must have $-1 < \cos\mu < 1$
- Using $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$, stability limits are where

$$\sin^2\frac{\mu}{2} = 0 \qquad \sin^2\frac{\mu}{2} = 1$$

- These translate to a FODO “necktie” stability diagram

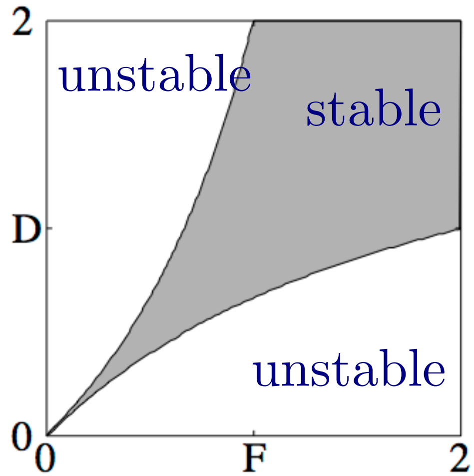


Figure. 6.1 Stability or “necktie” diagram for an alternate focusing lattice. The shaded area is the region of stability.

Onwards to the computer lab…

Introduction to MAD-X

G. Sterbini, CERN
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http://indico.cern.ch/event/286275/contributions/651708/attachments/531417/732835/juas.pdf

http://toddsatogata.net/2017-USPAS/lab/lab1.pdf