USPAS Accelerator Physics 2017
University of California, Davis

Observables and the Sigma Matrix, and
Chapter 10: Resonances and Nonlinear Dynamics
(mostly from separate handout)

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Happy birthday to Ilya Prigogene, Samuel Cohen, Virginia Woolf, and Somerset Maugham!
Happy National Irish Coffee Day and National Opposite Day!

A Brief Discussion of Observables

- We have discussed a lot about the (linearized) motion of individual particles through our accelerator lattice. In 1D:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

- But we usually observe RMS properties of the beam, such as by fitting the observed beam size to a Gaussian

$$\sigma_x^2 = \langle \langle x^2 \rangle - \langle x \rangle^2 \rangle$$

- How do we connect the single-particle linear dynamics matrix *M* to the observables like σ_x^2 ?

Beam Size Evolution

- Ignore beam centroid offsets to make the notation easier
- Then $\sigma_x^2 = \langle x^2 \rangle$
- We can come up with a clever way to evaluate this:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad\qquad (x, x')_2 = (x, x')_1 M^T$$

- Multiply and average over all particles

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 (x, x')_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1 (x, x')_1 M^T$$

$$\begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}_2 = M \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}_1 M^T$$

Sigma Matrix

- This new matrix is called the **sigma matrix**. It is a property of the beam since it includes distribution information:

$$\Sigma(s) \equiv \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \epsilon_x \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix}$$

$$\det \Sigma(s) = \epsilon_x$$

$$\Sigma(s_2) = M\, \Sigma(s_1)\, M^T$$

- This describes how observable RMS beam sizes evolve as they move through our lattice
- This equation can also be used to derive the 3x3 matrix equation that shows how $(\beta_x, \alpha_x, \gamma_x)$ evolve through a set of magnets that have total transport matrix *M*
- Symplecticity, unimodularity of *M* constrains this “motion”

Steering Error in Synchrotron Ring

- Short steering error $\Delta x'$ in a ring with periodic matrix M
  - Solve for new periodic solution or design orbit (x_0,x_0')

$$M \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

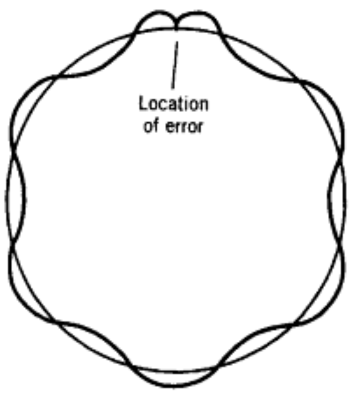
  - Note that (x_0=0,x_0'=0) is not the periodic solution any more!

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \Delta x_0' \end{pmatrix}$$

$$(I-M)^{-1} = (I - e^{(2\pi Q)J})^{-1} = \left(\left[e^{\pi Q J}\left(e^{-\pi Q J} - e^{\pi Q J}\right)\right]\right)^{-1}$$

$$= -\left(2J\sin(\pi Q)\right)^{-1}\left(e^{\pi Q J}\right)^{-1}$$

$$= \frac{1}{2\sin(\pi Q)}\left(J\cos(\pi Q) + I\sin(\pi Q)\right)$$



New closed orbit

$$x_0 = \frac{\beta_0\,\Delta x_0'}{2}\tan(\pi Q) \qquad \Rightarrow \infty \text{ if } Q = n/2$$

$$x_0' = \frac{\Delta x_0'}{2}\left[1 - \alpha_0 \cot(\pi Q)\right]$$

resonances

Steering Error in Synchrotron Ring

- We can use the general propagation matrix to find the new closed orbit displacement at all locations around the synchrotron

$$x(s) = \frac{\Delta x_0' \sqrt{\beta_0 \beta(s)}}{2\sin(\pi Q)} \cos[\Delta\Psi - \pi Q]$$

  - This displacement of the closed orbit changes its path length
  - If the revolution (RF) frequency is constant, then the beam energy changes, and there is an extra (small) term in the closed orbit displacement

$$x(s) = \frac{\Delta x_0' \sqrt{\beta_0 \beta(s)}}{2\sin(\pi Q)} \cos[\Delta\Psi - \pi Q] + \Delta x_0' \frac{\eta_0 \eta(s)}{\alpha_p C}$$

where $\alpha_p \equiv \left(\frac{dL}{L} / \frac{dp}{p}\right)$ is the momentum compaction and C is the accelerator circumference.

Closed orbit and orbit correction

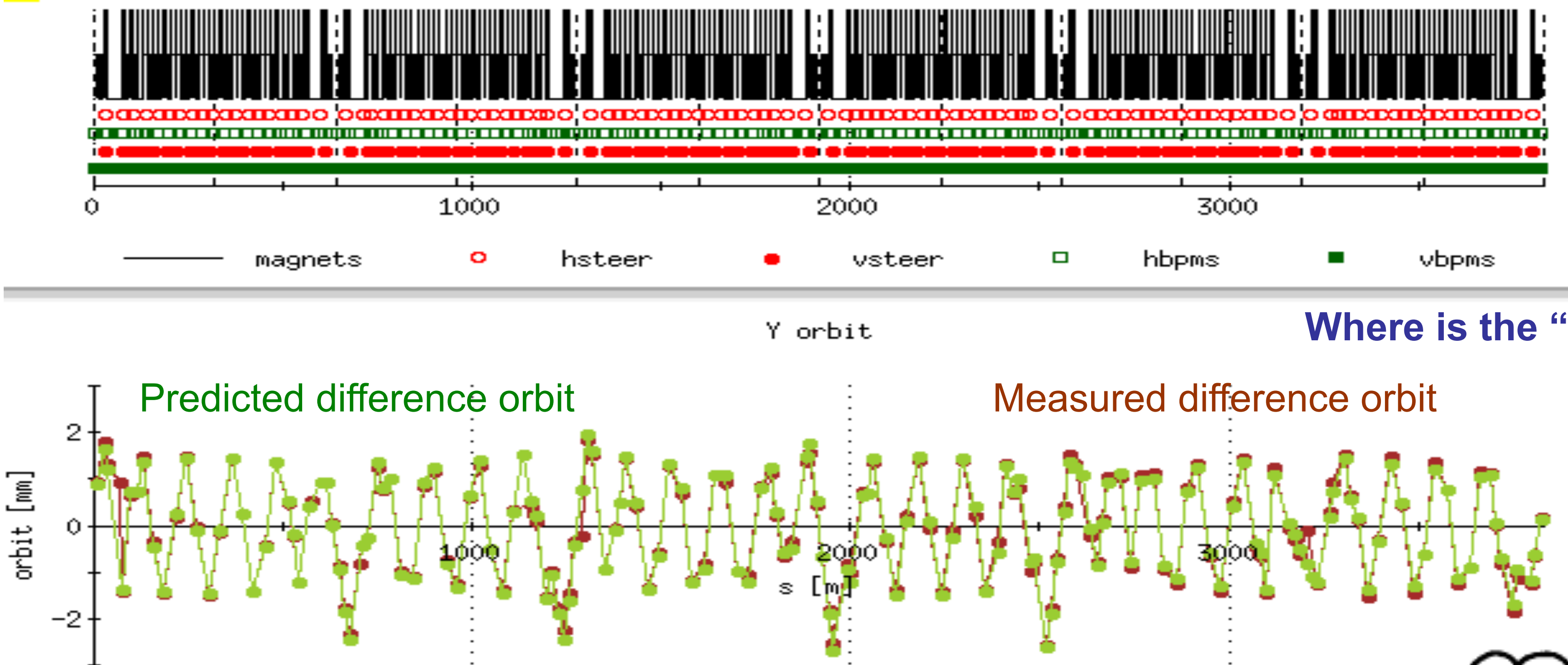
- Generally we can integrate all sources of steering error in the accelerator to find the general new "closed orbit"

$$x(s) = \frac{1}{2\sin(\pi Q)} \int \Delta x'(s_1) \sqrt{\beta(s)\beta(s_1)} \cos[|\Psi(s_0) - \Psi(s_1)| - \pi Q]$$

  - This is now effectively the perturbed design orbit
  - It is a fixed point of the one-turn map so betatron oscillations now occur around this orbit, not (x,x')=(0,0)

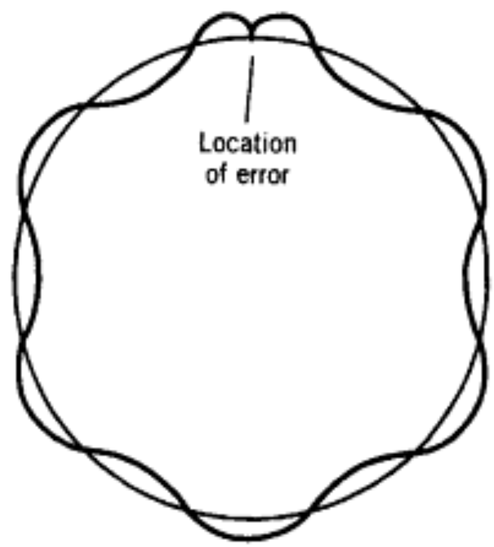
- Orbit correction is the process of adding in extra small dipole fields to move this closed orbit back to zero

RHIC Vertical Difference Orbit

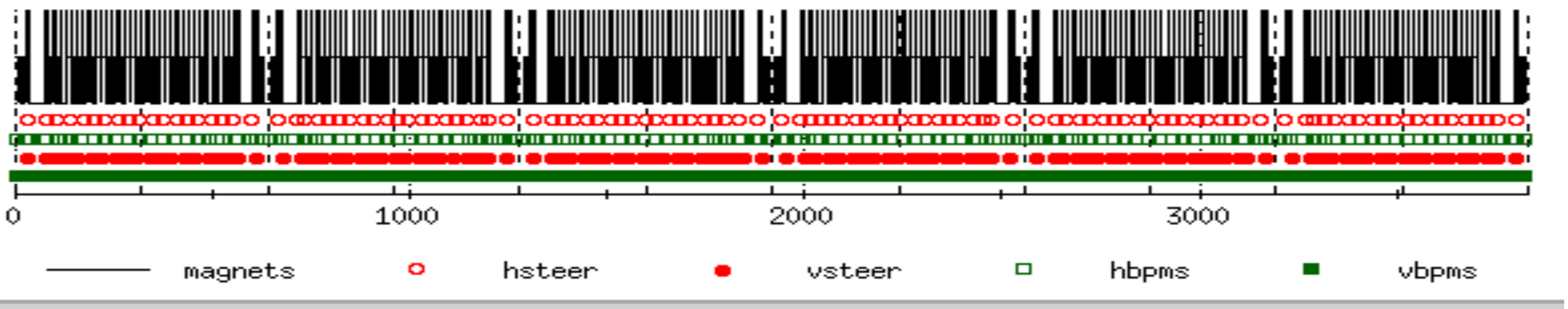


Where is the "cusp"?

- Ring difference orbits also measure optics
- Compare against modeled/design orbit
  - Integer part of tune readily visible: Q~30
  - Can easily find reversed, broken BPMs
  - Can also find (or eliminate) focusing errors

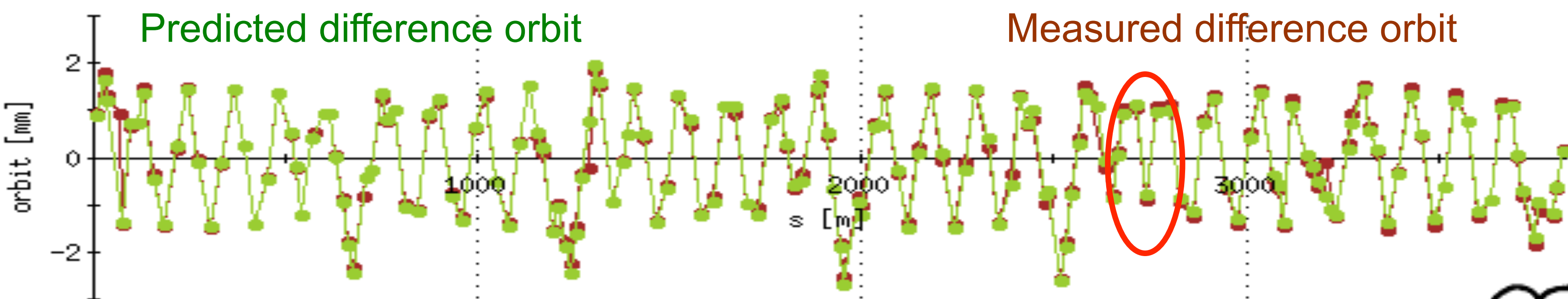


RHIC Vertical Difference Orbit



Y orbit

Where is the “cusp”?



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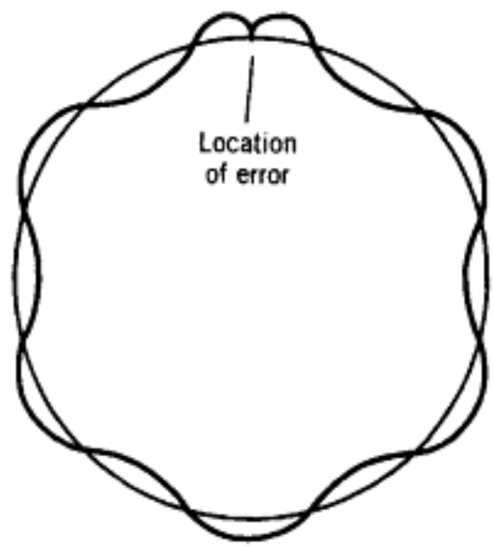


Focusing Error in Synchrotron Ring

- Short focusing error in a ring with periodic matrix M
  - Now solve for Tr M to find effects on tune Q

$$M_{\text{new}} = M \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

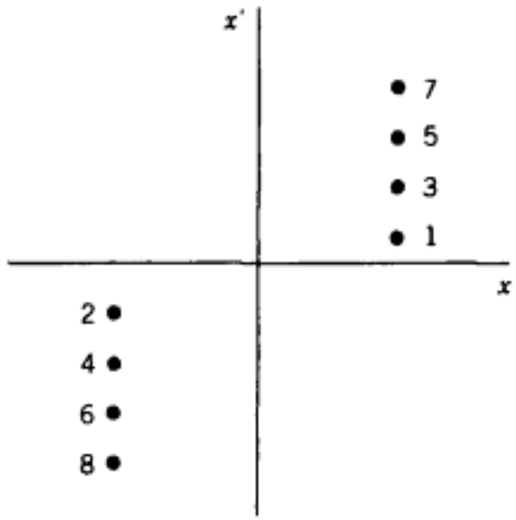
$$\frac{1}{2}\text{Tr } M = \cos(2\pi Q_{\text{new}}) = \cos(2\pi Q_0) - \frac{1}{2}\frac{\beta_0}{f}\sin(2\pi Q_0)$$

  - For small errors $Q_{\text{new}} = Q_0 + \Delta Q$ we can expand to find

$$\Delta Q \approx \frac{1}{4\pi}\frac{\beta_0}{f}$$

  - Can be used for a simple measurement of β_0 at the quadrupole

  - Quadrupole errors also cause resonances when $Q = k/2$: **half-integer resonances**



10.2: Linear coupling resonances

- Let's start following the book and think more in terms of differential equations
  - Easier to add nonlinear potentials or "driving terms"
  - We are assuming these extra potentials are perturbative
  - Move to a time basis where the "free" motion is simple harmonic oscillators with frequencies (Q_x, Q_y)
    - Like Floquet transformation in homework from last week
    - τ is an azimuthal coordinate around the ring going from 0 to 2π

Unperturbed "free" motion

$$\frac{d^2x}{d\theta^2} + Q_{\mathrm{H}}^2\, x = 0$$
$$\frac{d^2y}{d\theta^2} + Q_{\mathrm{V}}^2\, y = 0$$

Perturbed linearly coupled motion

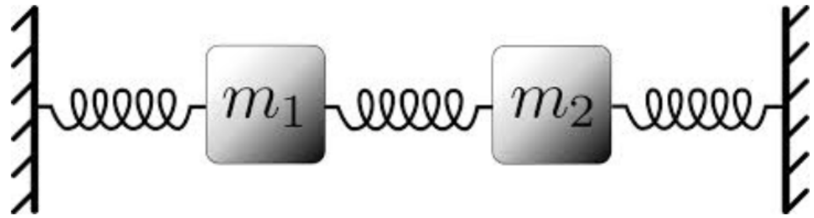
$$\frac{d^2x}{d\theta^2} + Q_{\mathrm{H}}^2\, x = \epsilon\, \cos(m\theta)\, y$$
$$\frac{d^2y}{d\theta^2} + Q_{\mathrm{V}}^2\, y = \epsilon\, \cos(m\theta)\, x$$

10.2: Linear coupling resonances

Perturbed linearly coupled motion

$$\frac{d^2x}{d\theta^2} + Q_{\rm H}^2\, x = \epsilon\, \cos(m\theta)\, y$$

$$\frac{d^2y}{d\theta^2} + Q_{\rm V}^2\, y = \epsilon\, \cos(m\theta)\, x$$



- Fourier components of linear coupled harmonic oscillators
  - Errors that create coupling must be periodic in θ
  - So must be expandable in a Fourier series in θ
  - Classical mechanics: sigma and pi eigenmodes with two separate frequencies
  - Gives rise to coupled resonance conditions:

$$Q_H + Q_V = m$$ Sum resonance (unstable)

$$|Q_H - Q_V| = m$$ Difference resonance (stable)