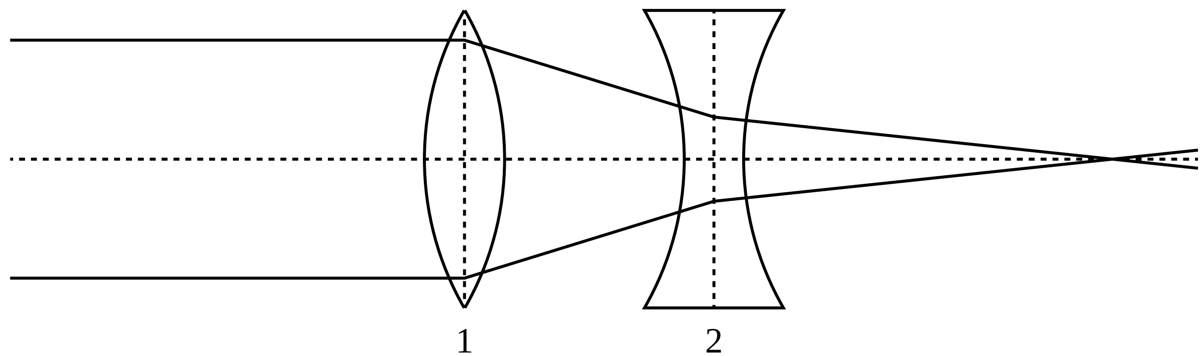
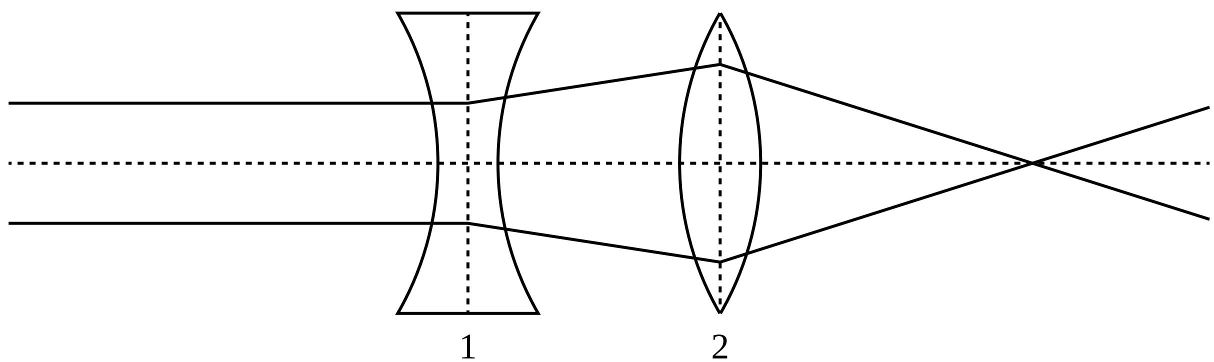
Strong focusing

Weak focusing rings tend to have beams with large transverse dimensions.

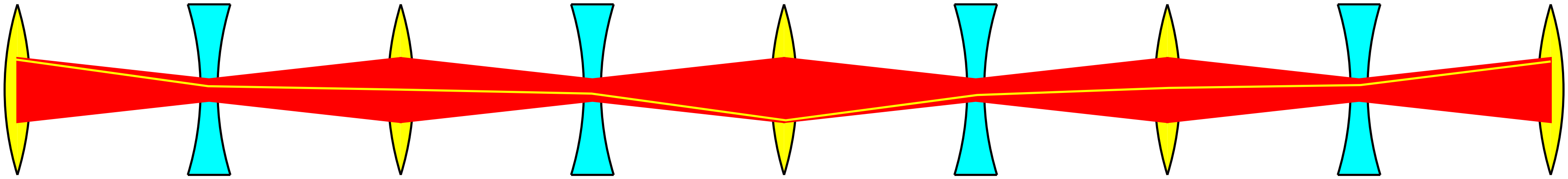
By using alternating focusing and defocusing lenses (or gradient dipoles) we can achieve stronger focusing with higher betatron tunes.



Converging double lens: parallel beam from left. $f_1 = 130, \quad f_2 = -70.$



Converging double lens with quad strengths inverted. $f_1 = -130, \quad f_2 = 70.$



FODO lattice with periodic cells. Cell: $\mathbf{Q}_\mathrm{F}$ **drift** $\mathbf{Q}_\mathrm{D}$ **drift.**

- Consider a general 2×2 transport matrix for periodic cell

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

  - Transport matrix for 5-cells: $\mathbf{M}^5$.

- For a ring, we might just as well take our unit cell to be the 1-turn map.
- Eigenvalues can tell us about stability.

Some general properties of square matrices

Characteristic polynomial of an $m \times m$ matrix $\mathbf{M}$:

$$P_{\mathbf{M}}(\lambda) = |\mathbf{M} - \lambda\mathbf{I}| = \sum_{j=0}^{m} A_j \lambda^j = \prod_{j=1}^{m} (\lambda - \lambda_j).$$

- Characteristic equation: $P_{\mathbf{M}}(\lambda) = 0.$
- Cayley-Hamilton theorem: $P_{\mathbf{M}}(\mathbf{M}) = 0.$
- $A_m = 1.$
- $A_0 = P_{\mathbf{M}}(0) = |\mathbf{M}|.$
- $A_1 = -\mathrm{tr}(\mathbf{M}) = \sum_{j=1}^{m} M_{jj}.$
- Since $P_{\mathbf{WMW}^{-1}}(\lambda) = P_{\mathbf{M}}(\lambda)$, all A_j are invariant under similarity transformations of the form: $\mathbf{M} \rightarrow \mathbf{WMW}^{-1}.$

Eigenvalues of a 2x2 matrix

- The eigenvalues of the 1-turn (or periodic cell) matrix can tell us about the stability.
  - Characteristic equation: $\lambda^2 - (a+b)\lambda + 1 = 0$,
  - Since $\lambda_1\lambda_2 = |\mathbf{M}| = 1$, we must have $\lambda_1 = \frac{1}{\lambda_2}$.
  - Matrix is real, characteristic equation has only real coefficients. So either
    - both roots are real,
    - or $\lambda_1 = \lambda_2^*$, (when $\lambda_1, \lambda_2 \notin \mathbb{R}$.)

$$\lambda_1 = \frac{1}{2}\left[\mathrm{tr}(\mathbf{M}) + \sqrt{(\mathrm{tr}(\mathbf{M}))^2 - 4}\right], \qquad \lambda_2 = \frac{1}{2}\left[\mathrm{tr}(\mathbf{M}) - \sqrt{(\mathrm{tr}(\mathbf{M}))^2 - 4}\right].$$

- Note: Some symplectic matrices are *defective*, e. g.: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ has only one normalized eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Liapunov stability

Definition of Liapunov stability:

A point $\vec{x}_0$ is called a fixed point of the map $\mathbf{M}$, if $\mathbf{M}\vec{x}_0 = \vec{x}_0$. A system is said to be *Liapunov stable* about a fixed point, $\vec{x}_0$, if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\vec{x} - \vec{x}_0\| < \delta$, then $\|\mathbf{M}^n\vec{x} - \mathbf{M}^n\vec{x}_0\| < \epsilon$ for all $0 < n < \infty$.

- That is, there is some neighborhood about the fixed point which remains bounded for all time.

- Some obvious symplectic unstable examples:

$$\text{thin quad}: \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix},$$

$$\text{drift}: \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix},$$

$$\text{shear}: \begin{pmatrix} 1.1 & 0 \\ 0 & \frac{1}{1.1} \end{pmatrix}.$$

Example of a general 2×2 symplectic matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with fixed point} \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Identity matrix is trivially stable.
- Assuming eigenvalues are not degenerate, and that $b \neq 0$ we can find two independent eigenvectors:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b \\ x \end{pmatrix} = \begin{pmatrix} ab + bx \\ cb + dx \end{pmatrix} = \begin{pmatrix} \lambda b \\ \lambda x \end{pmatrix}$$

$$x = \lambda - a \qquad \text{from top row,}$$

$$x = \frac{bc}{\lambda - d} \qquad \text{from bottom row.}$$

So we must have $\quad bc = (\lambda - a)(\lambda - d) = \lambda^2 - (a+d)\lambda + ad,$

which is identical to the characteristic function, since $ad - bc = 1$.

Hence we may write our independent eigenvectors as

$$\vec{v}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$

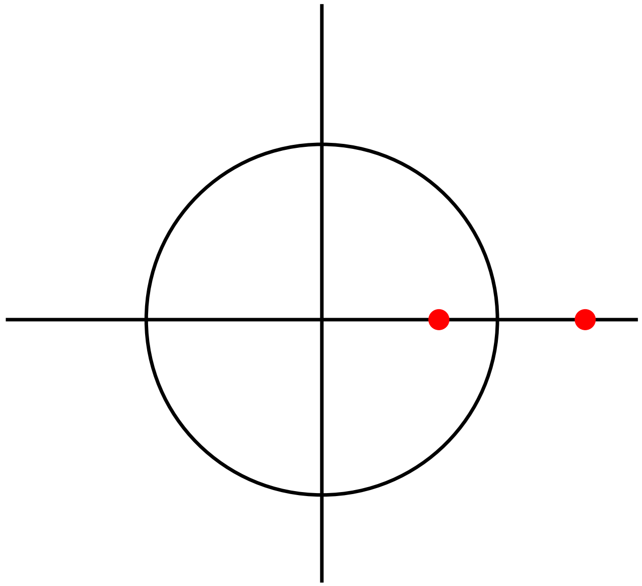
Expand $\vec{x} \neq \vec{x}_0$ in terms of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{x} = A\vec{v}_1 + B\vec{v}_2.$$

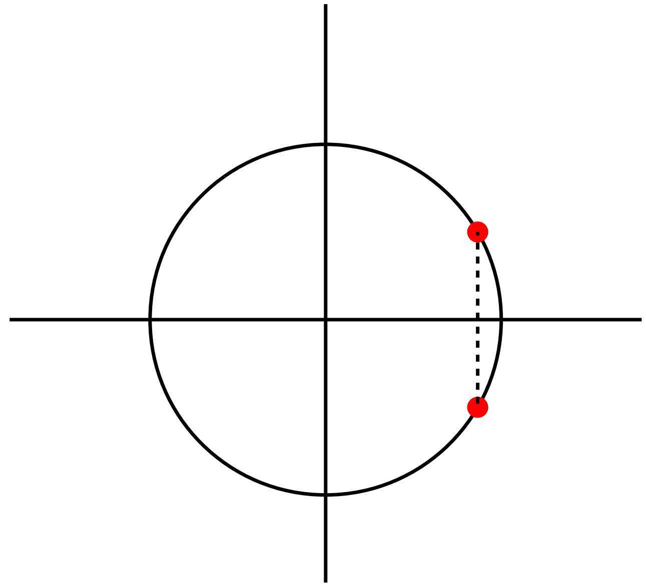
After n turns we have a deviation

$$D_n = ||\mathbf{M}^n\vec{x} - \mathbf{M}^n\vec{x}_0|| = ||A\lambda_1^n\vec{v}_1 + B\lambda_2^n\vec{v}_2|| < |\lambda_1|^n\,|A|\,||\vec{v}_1|| + |\lambda_2|^n\,|B|\,||\vec{v}_2||$$

Since $\lambda_2 = \frac{1}{\lambda_1}$, the motion can be unstable if either $|\lambda_j| \neq 1$.



Real and unstable.



Complex and stable.

- Real eigenvalues: $\lambda_1 = \dfrac{1}{\lambda_2}$, when $\mathrm{tr}(\mathbf{M}) \geq 2$.
- Non-real eigenvalues: $\lambda_1 = \dfrac{1}{\lambda_2}$, and $\lambda_1 = \lambda_2^*$ $\Rightarrow$ on unit circle, i. e. complex and not real when $|\mathrm{tr}(\mathbf{M})| < 2$.

$$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{tr}(\mathbf{M}) \pm \sqrt{(\mathrm{tr}(\mathbf{M}))^2 - 4}\right].$$

For $|\text{tr}(\mathbf{M})| < 2$, write: $\cos\mu = \frac{1}{2}\text{tr}(\mathbf{M})$, for some real angle μ.

Then we have

$$\lambda_\pm = \cos\mu \pm \sqrt{\cos^2\mu - 1} = \cos\mu \pm i\sin\mu = e^{\pm i\mu}.$$

For $|\text{tr}(\mathbf{M})| > 2$, we can write with $\mu > 0$:

$$\lambda_\pm = \cosh\mu \pm \sqrt{\cosh^2\mu - 1} = \cosh\mu \pm \sinh\mu = e^{\pm\mu}.$$

For $|\text{tr}(\mathbf{M})| = 2$, only $\mathbf{M} = \pm\mathbf{I}$ are stable.

But this is right on the borderline of stability. (Not good.)

Consider the stable motion case with $-2 < \mathrm{tr}(\mathbf{M}) = 2\cos\mu < 2$ for

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define $k = (a-d)/2$. Then the diagonal elements can be written as

$$a = \cos\mu + k, \quad \text{and} \quad d = \cos\mu - k.$$

$$ad = \cos^2\mu - k^2.$$

$$1 = ad - bc.$$

$$bc = ad - 1 = \cos^2\mu - k^2 - 1. = -k^2 - \sin^2\mu.$$

$$\sin\mu \neq 0, \quad \text{since} \quad |\cos\mu| < 1.$$

Replace k by $\alpha\sin\mu$ for some real parameter α, so

$$bc = -(1+\alpha^2)\sin^2\mu, \quad \text{and partition } bc \text{ by a parameter } \beta \text{ so that}$$

$$b = \beta\sin\mu,$$

$$c = -\frac{1+\alpha^2}{\beta}\sin\mu.$$

So for the stable region with real parameters α, β, γ, and angle μ:

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix},$$

where $\gamma = \dfrac{1+\alpha^2}{\beta}$.

Since the sign of μ is ambiguous, we may take both β and γ to be positive.

Define $\quad \mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad$ then

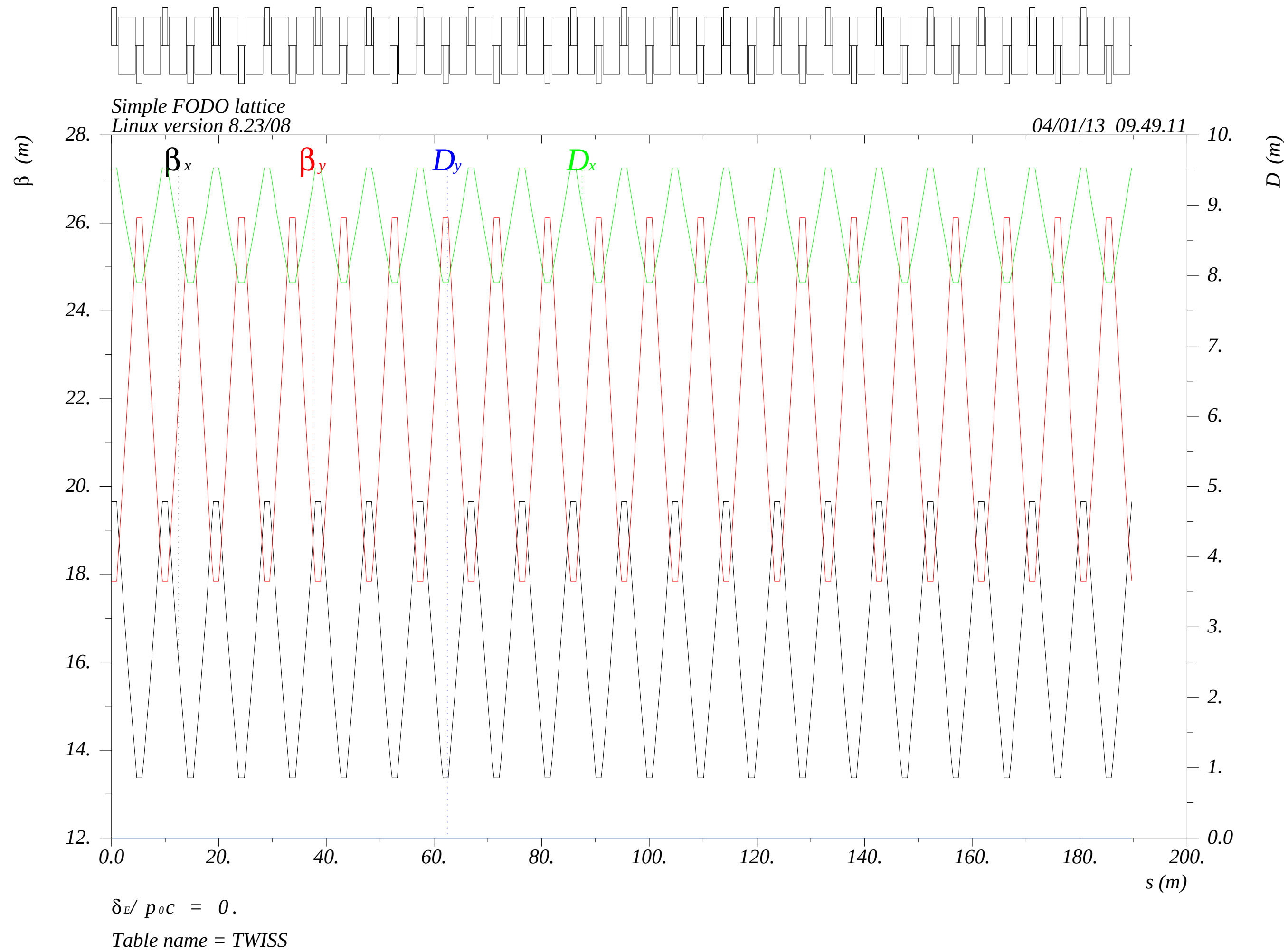
$$|\mathbf{J}| = \gamma\beta - \alpha^2 = 1.$$

$$\mathbf{J}^2 = \begin{pmatrix} \alpha^2 - \gamma\beta & 0 \\ 0 & \alpha^2 - \gamma\beta \end{pmatrix} = -\mathbf{I}.$$

$$\mathbf{M} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu = e^{\mathbf{J}\mu}.$$

α, β, and γ are frequently called *Twiss parameters*, though why is unknown.

FODO cell example



$$L = 189.664 \text{ m},$$
$$Q_{\rm H} = 1.8765,$$
$$Q_{\rm V} = 1.4106$$
$$\beta_{\rm H0} = 19.655 \text{ m}$$
$$\alpha_{\rm H0} = -0.966$$
$$\beta_{\rm V0} = 17.840 \text{ m}$$
$$\alpha_{\rm V0} = 0.913$$

Calculated and plotted with MAD8.

$$\mathbf{M}_{\mathrm{turn}} = \begin{pmatrix} 1.390329 & -13.764241 & 0.000000 & 0.000000 \\ 0.068876 & 0.037379 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & -0.360158 & 9.499991 \\ 0.000000 & 0.000000 & -0.054739 & -1.332692 \end{pmatrix}.$$

$$\begin{aligned}
\cos\mu_{\mathrm{H}} &= \frac{1.390329 + 0.037379}{2} = 0.77581, \\
\mathrm{sign}(\sin\mu_{\mathrm{H}}) &= -1, \qquad (\text{since } \beta_{\mathrm{H}} > 0), \\
\mu_{\mathrm{H}} &= 2\pi - \cos^{-1}(0.77581) \qquad + 2\pi n \qquad = 5.5074 \qquad + 2\pi n, \\
\sin\mu_{\mathrm{H}} &= -0.70029, \\
Q_{\mathrm{H}} &= \frac{5.5074}{2\pi} \qquad + n = 0.87653 + n, \\
\beta_{\mathrm{H}} &= \frac{-13.764241}{\sin\mu_{\mathrm{H}}} = 19.655 \text{ m}, \\
\alpha_{\mathrm{H}} &= \frac{1.390329 - 0.037379}{2\sin\mu_{\mathrm{H}}} = -0.966, \\
\gamma_{\mathrm{H}} &= -\frac{0.068876}{2\sin\mu_{\mathrm{H}}} = -0.098353.
\end{aligned}$$

Analytic approach

Recall our equations of motion from Chapter 3:

$$x'' + k_x(s)x = \frac{\delta}{\rho(s)}, \quad \text{and}$$

$$y'' + k_x(s)y = 0,$$

where

$$k_x(s) = \frac{1}{\rho^2} + \frac{q}{p_0}\frac{\partial B_y}{\partial x}, \quad \text{and}$$

$$k_y(s) = -\frac{q}{p_0}\frac{\partial B_y}{\partial x}.$$

Hill's equation

For now with $\delta = 0$, let's use for either plane:

$$\frac{d^2 z}{ds^2} + k(s)z = 0.$$

For a ring, we have a periodic guide field

$$k(s + L) = k(s).$$

where L is the length of the period, such as the circumference of the ring.

- Quasiperiodic solutions. From Floquet's theorem, solutions have form:

$$z(s) \sim A\,w(s)\,e^{i\Psi(s)}, \quad \text{where}$$

  - $w(s)$ has the same periodicity as $k(s)$, and
  - $\Psi(s)$ is a nonperiodic phase of the oscillations, aka *betatron phase.*

$$
\begin{aligned}
z(s) &= w(s)\, e^{i\Psi(s)}. \\
z'(s) &= (w' + iw\Psi')e^{i\Psi(s)}. \\
z''(s) &= (w'' + iw'\Psi' + iw\Psi'')e^{i\Psi} + (w' + iw\Psi')i\Psi' e^{i\Psi} \\
&= (w'' - w\Psi'^2)e^{i\Psi} + i(2w'\Psi' + w\Psi'')e^{i\Psi}. \\
0 = \frac{z'' + kz}{e^{i\Psi}} &= (w'' - w\Psi'^2 + kw) + i(2w'\Psi' + w\Psi'')
\end{aligned}
$$

1. $w'' + kw - w\Psi'^2 = 0.$ (Called the *eikonal equation.*)

2. $\dfrac{2w'}{w} + \dfrac{\Psi''}{\Psi'} = 0.$ integrate: $2\log w + \log \Psi' = \text{arbitrary const} \to 0.$

3. $\Psi' = \dfrac{1}{w^2}.$

4. 1 and 3 combine to give: $w'' + kw - \dfrac{1}{w^3} = 0.$

General solution in noncomplex form:

$$z = Aw\,\cos\Psi + Bw\sin\Psi.$$

$$z' = A(w'\cos\Psi - w\Psi'\sin\Psi) + B(w'\sin\Psi + w\Psi'\cos\Psi).$$

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \mathbf{F}\begin{pmatrix} A \\ B \end{pmatrix}, \quad \text{with}$$

$$\mathbf{F} = \begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - w\Psi'\sin\Psi & w'\sin\Psi + w\Psi'\cos\Psi \end{pmatrix}.$$

Initial conditions:

$$\begin{pmatrix} z_0 \\ z_0' \end{pmatrix} = \mathbf{F}_0\begin{pmatrix} A \\ B \end{pmatrix}, \qquad \Rightarrow \qquad \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{F}_0^{-1}\begin{pmatrix} z_0 \\ z_0' \end{pmatrix}.$$

Propagate to s:

$$\begin{pmatrix} z(s) \\ z'(s) \end{pmatrix} = \mathbf{F}(s)\mathbf{F}_0^{-1}\begin{pmatrix} z_0 \\ z_0' \end{pmatrix},$$

so we have

$$\mathbf{M}(s) = \mathbf{F}(s)\mathbf{F}_0^{-1}.$$

$$\mathbf{F} = \begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - w\Psi'\sin\Psi & w'\sin\Psi + w\Psi'\cos\Psi \end{pmatrix}.$$

$$\begin{aligned} |\mathbf{F}| &= w\cos\Psi(w'\sin\Psi + w\Psi'\cos\Psi) - w\sin\Psi(w'\cos\Psi - w\Psi'\sin\Psi) \\ &= w^2\Psi'(\cos^2\Psi + \sin^2\Psi) \\ &= 1. \end{aligned}$$

$$\mathbf{F}^{-1} = \begin{pmatrix} w'\sin\Psi + w\Psi'\cos\Psi & -w\sin\Psi \\ -w'\cos\Psi + w\Psi'\sin\Psi & w\cos\Psi \end{pmatrix}.$$

$$\begin{aligned} \mathbf{M} = &\begin{pmatrix} w\cos\Psi & w\sin\Psi \\ w'\cos\Psi - w\Psi'\sin\Psi & w'\sin\Psi + w\Psi'\cos\Psi \end{pmatrix} \\ &\times \begin{pmatrix} w_0'\sin\Psi_0 + w_0\Psi_0'\cos\Psi_0 & -w_0\sin\Psi_0 \\ -w_0'\cos\Psi_0 + w_0\Psi_0'\sin\Psi_0 & w_0\cos\Psi_0 \end{pmatrix} \end{aligned}$$

After grundgy algebra:

$$a(s) = \frac{w(s)}{w_0}\cos\left[\Psi(s)-\Psi_0\right] - w(s)w_0'\sin\left[\Psi(s)-\Psi_0\right],$$

$$b(s) = w(s)w_0\sin\left[\Psi(s)-\Psi_0\right],$$

$$c(s) = -\frac{1+w(s)w_0w'(s)w_0'}{w(s)w_0}\sin\left[\Psi(s)-\Psi_0\right] - \left[\frac{w_0'}{w(s)} - \frac{w'(s)}{w_o}\right]\cos\left[\Psi(s)-\Psi_0\right],$$

and

$$d(s) = \frac{w_0}{w(s)}\cos\left[\Psi(s)-\Psi_0\right] + w_0w'(s)\sin\left[\Psi(s)-\Psi_0\right].$$

Look at full turn matrix starting from $s = 0$ to $s = L$ with $\Psi_0 = 0$, $\Psi(L) = \mu$, and $w(L) = w(0) = w_0$.

For simplicity, we can drop the arguments:

$$a = \cos\mu - ww' \sin\mu,$$

$$b = w^2 \sin\Psi,$$

$$c = -\frac{1 + w^2 w'^2}{w^2} \sin\Psi - \left[\frac{w'}{w} - \frac{w'}{w}\right] \cos\Psi(L),$$

$$d = \cos\Psi + ww' \sin\Psi.$$

Compare with

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}.$$

$$\beta = w^2,$$

$$\alpha = -ww' = -\frac{\beta'}{2},$$

$$\gamma = \frac{1 + w^2 w'^2}{w^2} = \frac{1 + \alpha^2}{\beta}.$$

Since the matrix $\mathbf{M}(s+L|s)$ from s to $s+L$ must be identical for each turn,

The Twiss parameters must also be periodic functions of the ring:

$$
\begin{aligned}
k(s+L) &= k(s), \\
w(s+L) &= w(s),
\end{aligned}
$$

$$
\begin{aligned}
\beta(s+L) &= \beta(s), \\
\alpha(s+L) &= \alpha(s), \\
\gamma(s+L) &= \gamma(s),
\end{aligned}
$$

- The $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ functions are also called *Courant-Snyder functions.*
- $\sqrt{\beta(s)}$ is also called the *envelope function.*

$$
z(s) = A\sqrt{\beta(s)}\cos\left(\Psi(s)\right).
$$

More generally, for a nonperiodic portion of the ring or a nonperiodic beam line:

$$\begin{aligned}\mathbf{M}(s) =& \mathbf{F}(s)\mathbf{F}_0^{-1} \\ =& \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\,[\cos\mu(s) + \alpha_0\sin\mu(s)] & \sqrt{\beta_0\beta(s)}\sin\mu(s) \\ -\frac{[\alpha(s)-\alpha_0]\cos\mu(s) + [1+\alpha_0\alpha(s)]\sin\mu(s)}{\sqrt{\beta_0\beta(s)}} & \sqrt{\frac{\beta_0}{\beta(s)}}\,[\cos\mu(s) - \alpha(s)\sin\mu(s)] \end{pmatrix} \\ =& \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix},\end{aligned}$$

with

$$C(0) = S'(0) = 1, \qquad \text{and} \qquad S(0) = C'(0) = 0.$$

Betatron tunes

For the full turn we have

$$\mu = \int_s^{s+L} \Psi' \, ds = \int_s^{s+L} \frac{1}{w^2} \, ds = \int_s^{s+L} \frac{1}{\beta(s)} \, ds = 2\pi Q.$$

For x and y, respectively: $Q_{\mathrm{H}} = \frac{1}{2\pi} \oint \frac{ds}{\beta_{\mathrm{H}}(s)}$, and $Q_{\mathrm{V}} = \frac{1}{2\pi} \oint \frac{ds}{\beta_{\mathrm{V}}(s)}$.

- Note that I can start the full turn anywhere along the circumference. The integral will always be the same.

If the ring consists of N cells, each of period L:

$$Q = \frac{N}{2\pi} \int_s^{s+L} \frac{ds}{\beta(s)}.$$

Fractional tune

$$\cos\mu = \frac{a+b}{2} = \frac{\mathrm{tr}(\mathbf{M})}{2}$$

- Fractional tune: $q = \frac{1}{2\pi}\cos^{-1}\left(\frac{\mathrm{tr}(\mathbf{M})}{2}\right).$
- The total tune $Q = n \pm q$ for some integer n.
- 4×4 matrix with uncoupled motion:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathrm{V}} \end{pmatrix}. \qquad (2\times2 \text{ blocks.})$$

$$q_{\mathrm{H}} = \frac{1}{2\pi}\cos^{-1}\left(\frac{\mathrm{tr}(\mathbf{M}_{\mathrm{H}})}{2}\right).$$

$$q_{\mathrm{V}} = \frac{1}{2\pi}\cos^{-1}\left(\frac{\mathrm{tr}(\mathbf{M}_{\mathrm{V}})}{2}\right).$$

Courant-Snyder invariant

$$z(s) = \sqrt{\frac{\beta}{\beta_0}}[\cos\Psi + \alpha_0 \sin\Psi]\, z_0 + \sqrt{\beta\beta_0}\,\sin\Psi\, z_0'$$

$$= \beta^{\frac{1}{2}}\left[\frac{z_0}{\sqrt{\beta_0}}\cos\Psi + \left(\frac{z_0}{\sqrt{\beta_0}}\alpha_0 + \sqrt{\beta_0}\, z_0'\right)\sin\Psi\right].$$

We can write this in the form

$$z(s) = \sqrt{\mathcal{W}\beta(s)}\,\cos[\Psi(s) + \Psi_0],$$

where

$$\sqrt{\mathcal{W}}\,\cos\Psi_0 = \frac{z_0}{\sqrt{\beta_0}},$$

$$\sqrt{\mathcal{W}}\,\sin\Psi_0 = -\left(\frac{z_0}{\sqrt{\beta_0}}\alpha_0 + \sqrt{\beta_0}\, z_0'\right).$$

Squaring and adding gives the constant:

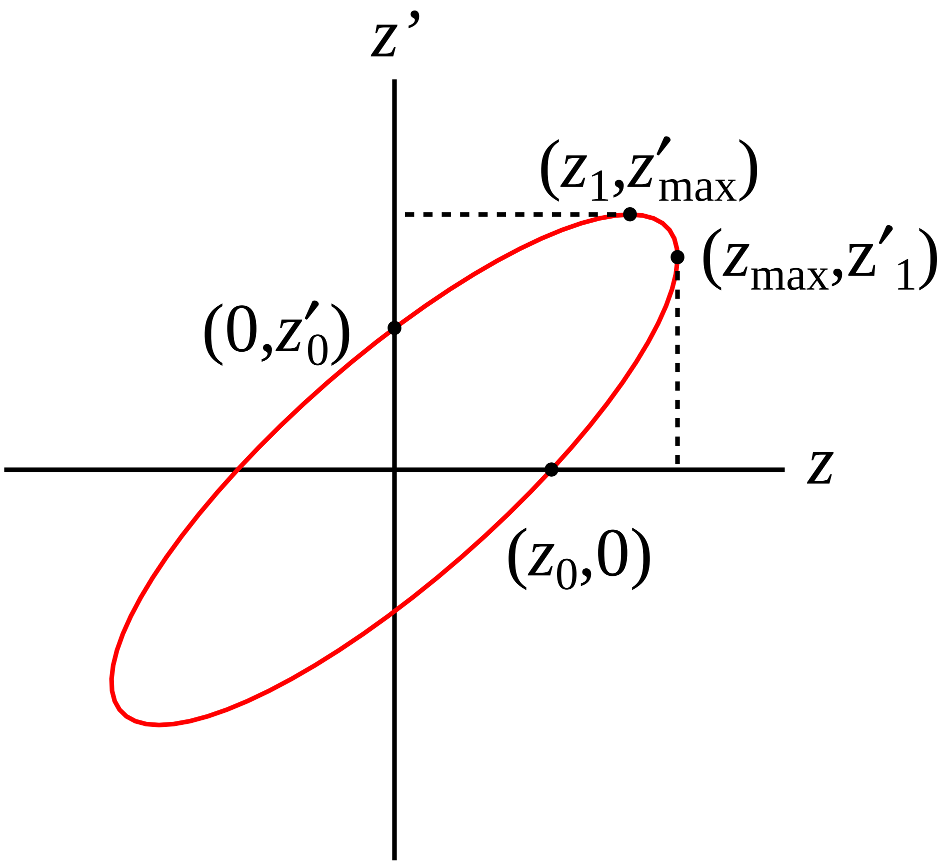
$$\mathcal{W} = \frac{1}{\beta_0}\,\left[(1+\alpha_0^2)z_0^2 + 2\alpha_0\beta_0 z_0 z_0' + \beta_0^2 z_0'^2\right] = \gamma_0\, z_0^2 + 2\alpha_0\, z_0 z_0' + \beta_0\, z_0'^2.$$

For simplicity, write $\Psi(s) + \Psi_0 = \psi$, $\beta(s) = \beta$, and $\alpha(s) = \alpha$.

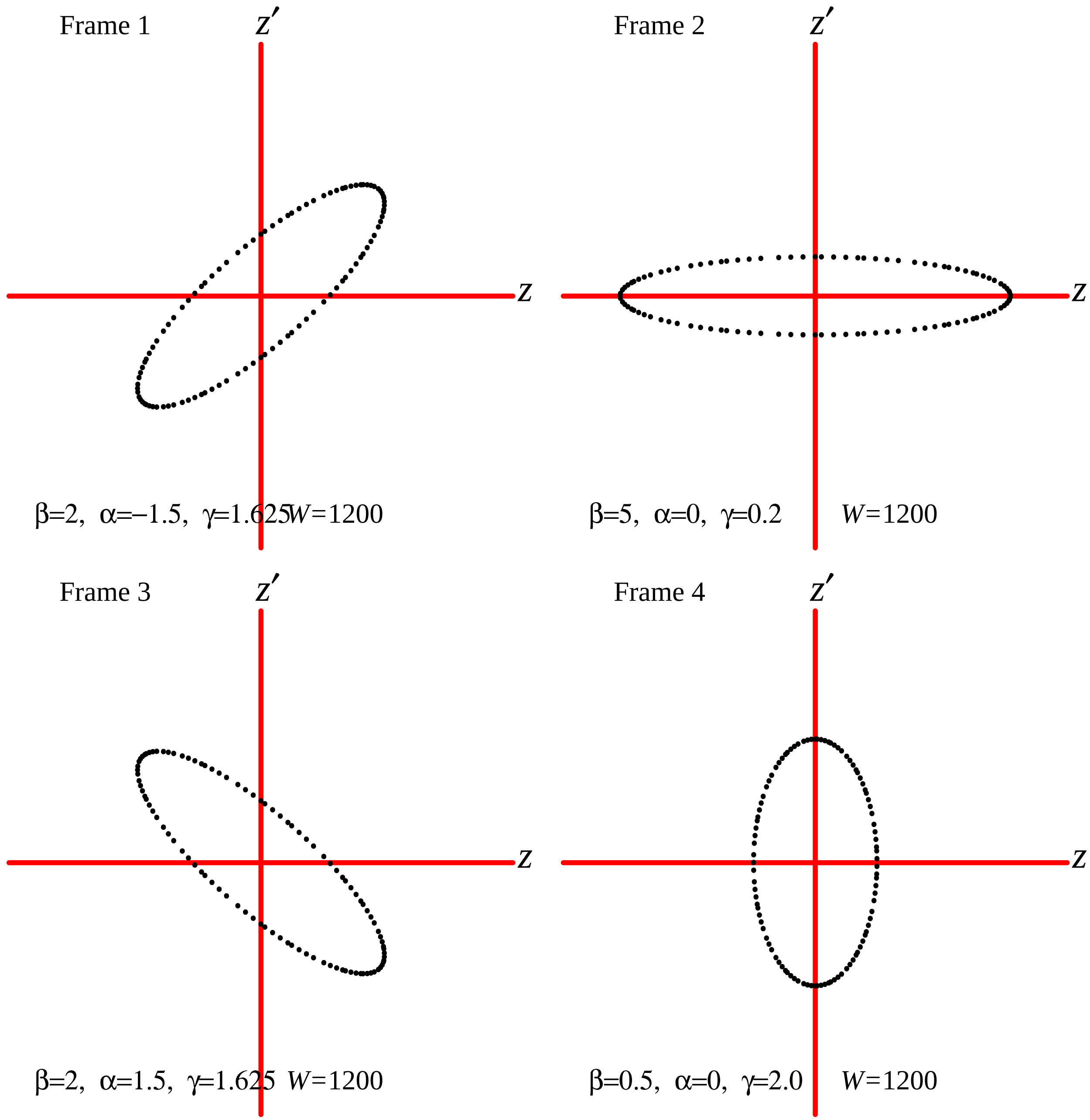
$$\begin{aligned}
z(s) &= \sqrt{\mathcal{W}\beta}\, \cos\psi. \\
\sqrt{\mathcal{W}}\, \cos\psi &= \frac{z}{\sqrt{\beta}}. \\
z'(s) &= \frac{1}{2}\sqrt{\frac{\mathcal{W}}{\beta}}\,\beta' \cos\psi + \sqrt{\mathcal{W}\beta}\,\psi'\, \sin\psi = -\sqrt{\frac{\mathcal{W}}{\beta}}\,\{\alpha\, \cos\psi + \sin\psi\}\,. \\
\sqrt{\mathcal{W}}\, \sin\psi &= -\sqrt{\beta}\, z' - \alpha\sqrt{\mathcal{W}}\, \cos\psi = -\sqrt{\beta}\, z' - \frac{\alpha z}{\sqrt{\beta}} \\
\mathcal{W} &= \frac{z^2}{\beta} + \frac{(\beta z' + \alpha z)^2}{\beta} = \frac{1}{\beta}\left[(1+\alpha^2)z^2 + 2\alpha\beta\, zz' + \beta^2 z'^2\right] \\
&= \gamma\, z^2 + 2\alpha\, zz' + \beta\, z'^2.
\end{aligned}$$

Courant-Snyder invariant

- $\boxed{\gamma(s)\, z(s)^2 + 2\alpha(s)\, z(s) z'(s) + \beta(s)\, z'(s)^2 = \mathcal{W} \text{ is an invariant.}}$



- Area$= \pi\mathcal{W} = \pi z_0 z'_{\text{max}} = \pi z_{\text{max}} z'_0$.
- $z_{\text{max}} = \sqrt{\mathcal{W}\beta}$ at $z'_1 = -\alpha\sqrt{\frac{\mathcal{W}}{\beta}}$.
- $z'_{\text{max}} = \sqrt{\mathcal{W}\gamma}$ at $z_1 = -\alpha\sqrt{\frac{\mathcal{W}}{\gamma}}$.
- $z_0 = \sqrt{\frac{\mathcal{W}}{\gamma}}$.
- $z'_0 = \sqrt{\frac{\mathcal{W}}{\beta}}$.



Linear tracking

- Single particle.
- 4 cases with same $\mathcal{W}$
- 100 turns each.
- Ellipse areas equal.
- Different β and α.
- Frac. tune $q = 0.115$.

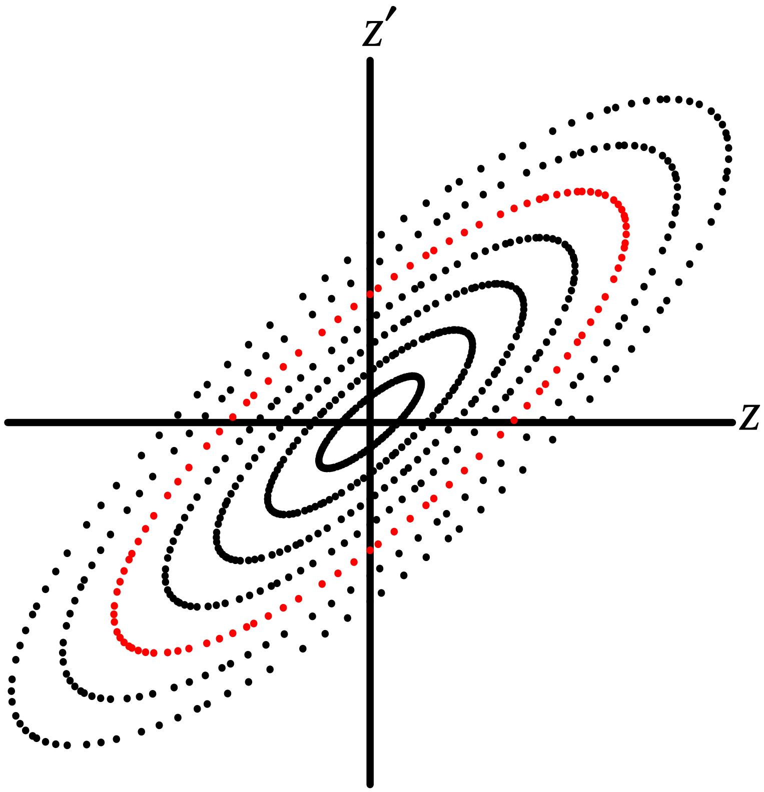
Notes for the curious:

- Units for $\mathcal{W}$ and β: pt (1/72 inch)
- Why?
  - It's PostScript.
  - I'm demented.

Reminder to Waldo: Run demos:

- `http://www.toddsatogata.net/fodo1.gif`
- `fodo1.tcs|gnuplot`
- `split2 -i nodisp.init -f coast.cdr`

Emittance



Tracking of 7 particles for 100 turns each.

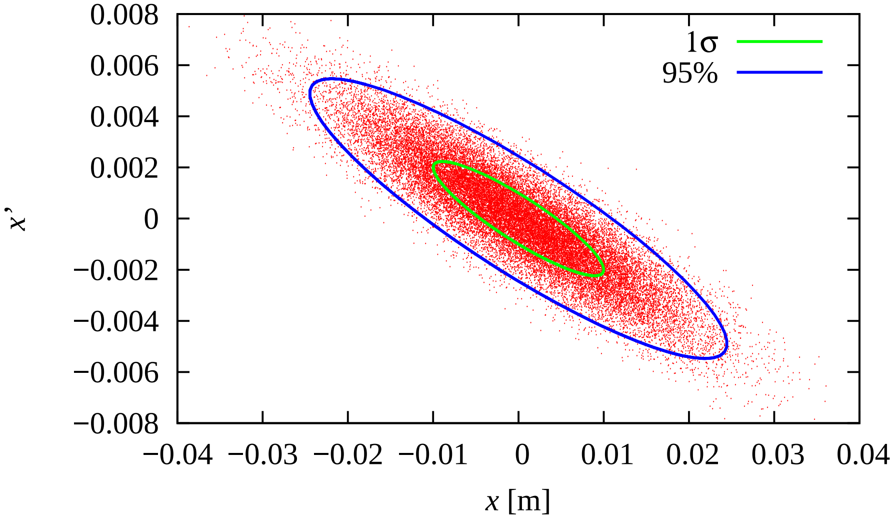
- Note similar ellipses – diff. areas.
- Ellipses don't cross.
- Any particles within red ellipse stay withint the blue ellipse.
- Any particles outside stay outside.

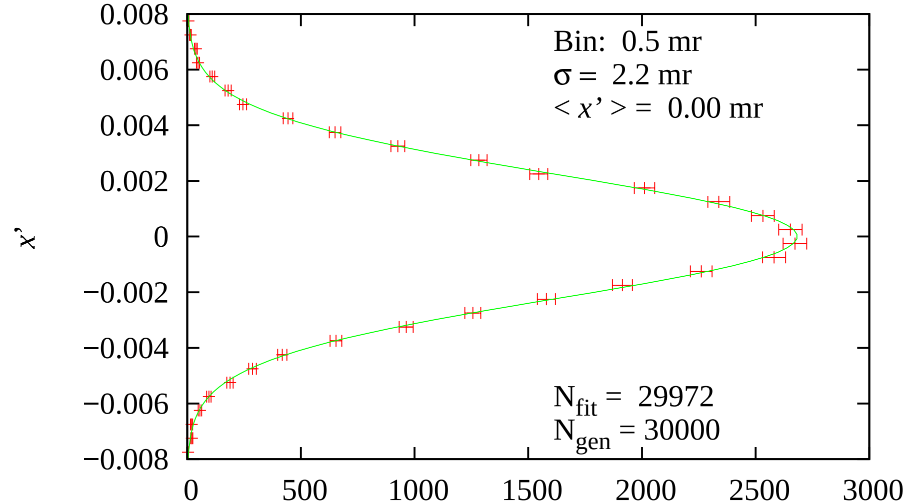
We can pick some fraction f of the beam and define the area of the corresponding ellipse as $\pi\epsilon_f$.

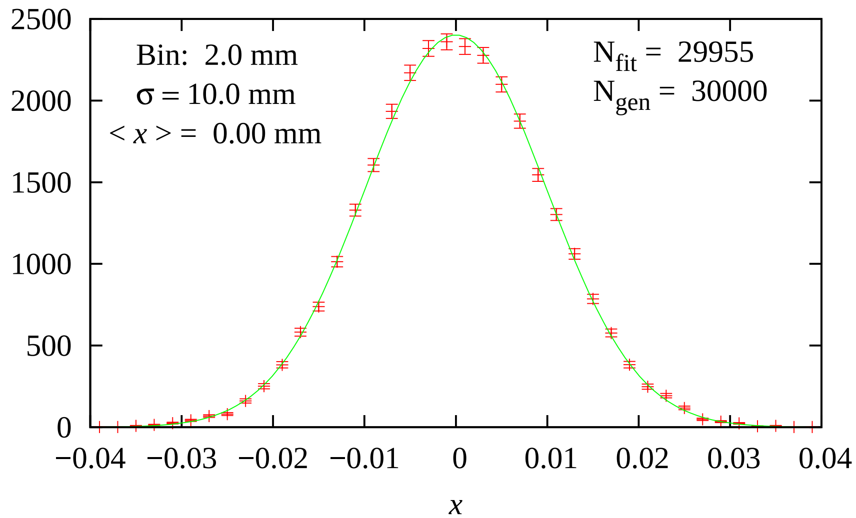
- ϵ_f is called the *emittance*.

- Conventions (#σ for Gaussian approx.): rms (1σ), 90% (4.6σ), 95% (6σ). Convention depends on the which lab, accelerator, or book,
  e. g., electron rings: rms; RHIC: 95%, Conte & MacKay: 90%.

2-d Gaussian distribution







$$N_{\rm gen} = 30000 \quad \text{particles}$$
$$\beta = 10 \text{ m.}$$
$$\alpha = 2$$
$$\gamma = 0.5$$
$$\varepsilon_{\rm rms} = 1\ \mu\text{ m.} \quad \text{(only 39.35\%)}$$

Adiabatic invariants

- As the beam is accelerated, the particles gain energy.
- Since the energy increase is much slower than the betatron oscillations, we may use the *adiabatic approximation*,

  i. e. slow approximately uniform acceleration of particles,
  with no heating of the beam in its rest system.
- Using the Poincaré–Cartan theorem, we have

$$I = \oint p\,dq \approx \text{constant},$$

  where p and q are canonical conjugate variables.
- For horizontal motion, one has $q = x$, and

$$p_x = \gamma m \dot{x} = \gamma m \frac{ds}{dt}\frac{dx}{ds} = (\beta\gamma mc)x' = px'.$$

Note that here β and γ are the relativistic factors and not the envelope functions.

- The horizontal invariant becomes

$$I_{\rm H} = p\oint x'\,dx = p(\text{Area}) = p\pi\epsilon_{\rm H} = \pi mc\beta\gamma\epsilon_{\rm H} = \pi mc\epsilon^*_{\rm H}.$$

- Similarly the vertical invariant is

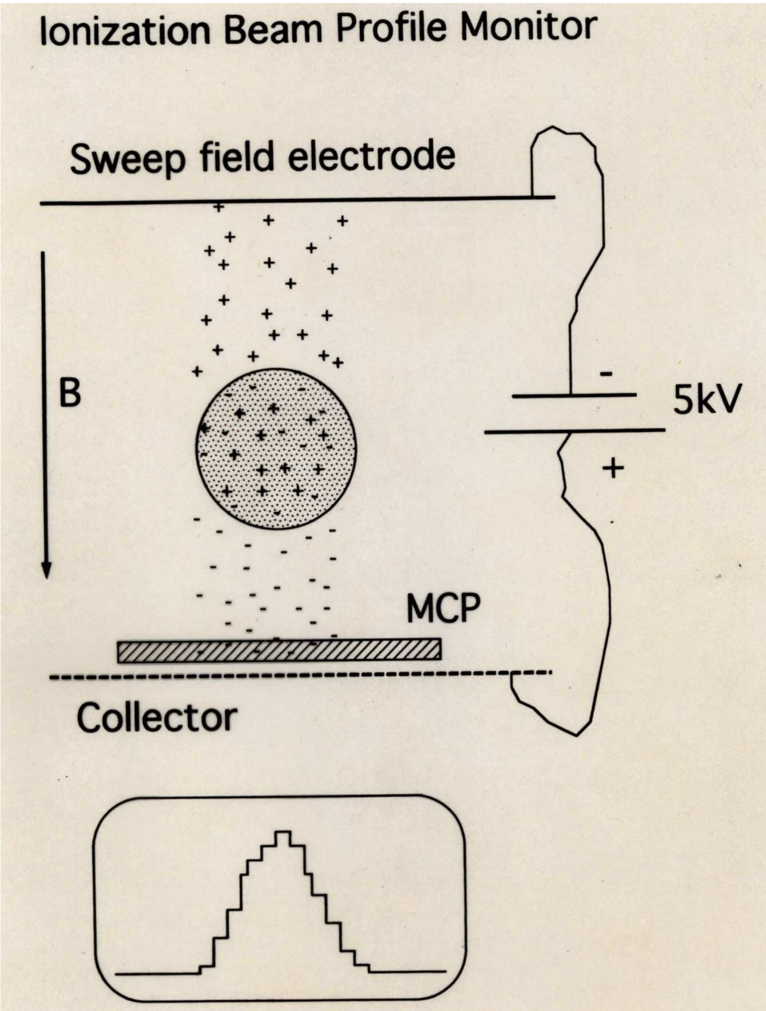
$$I_{\rm V} = \pi mc\beta\gamma\epsilon_{\rm V} = \pi mc\epsilon^*_{\rm V}.$$

- The *normalized emittances* are given by

$$\pi\epsilon^*_{\rm H} = \beta\gamma\pi\epsilon_{\rm H}, \quad \text{and} \quad \pi\epsilon^*_{\rm V} = \beta\gamma\pi\epsilon_{\rm V}.$$

If the energy is adiabatically increased, the area of the ellipse in the xp_x-plane will remain constant, but the area of the ellipse in the xx'-plane will shrink. This effect of shrinking area in the xx'-plane is called *adiabatic damping*.

Ionization profile monitor (IPM)



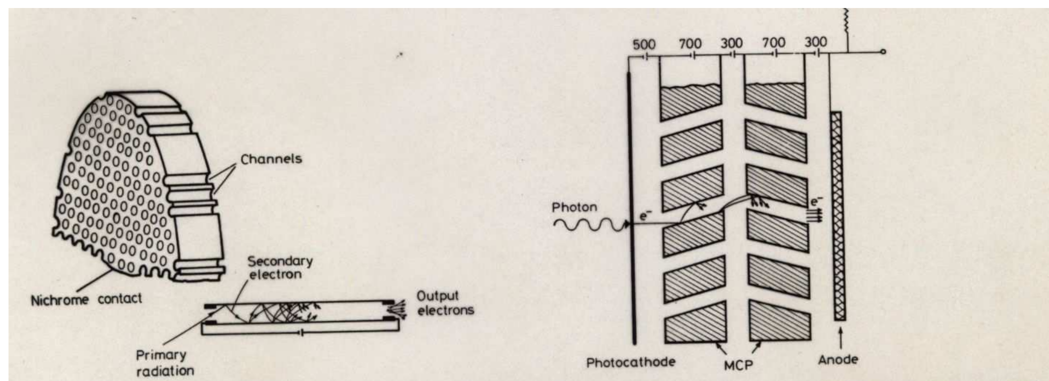
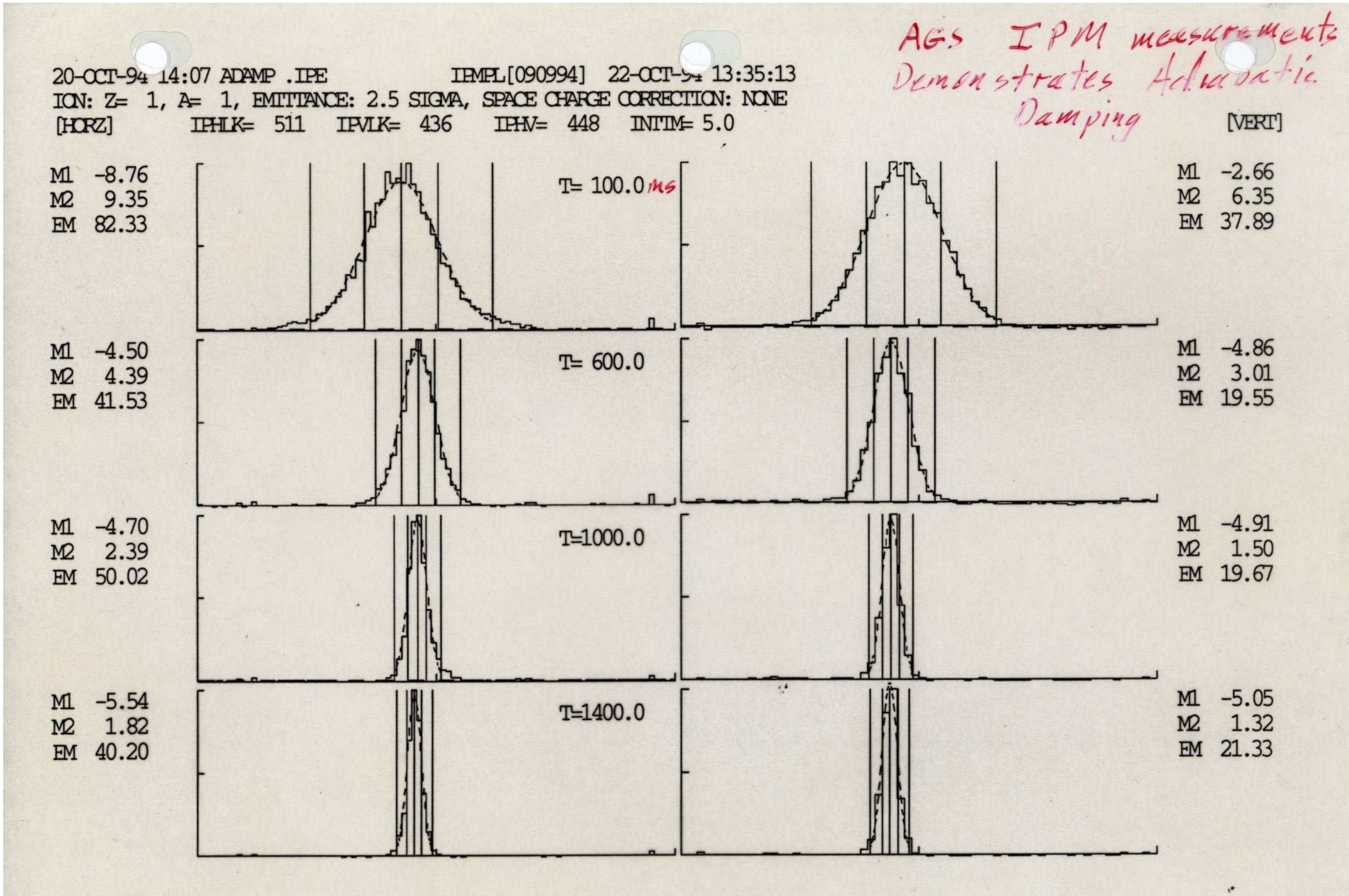


Fig. 8.6. Schematic diagram of a microchannel plate. The many channels act as continuous dynodes (from *Dhawan* [8.4]; picture © 1975 IEEE)

Fig. 8.7. Chevron configuration in a microchannel plate photomultiplier (after *Dhawan* [8.4]). A further increase in gain may be obtained by adding a third plate to form a "Z" configuration (picture © 1975 IEEE)

AGS IPM measurment of Adiabatic Damping



Dispersion

Equation of motion:

$$\frac{d^2x}{ds^2} + k(s)x = \frac{1}{\rho(s)}\delta, \qquad \text{with} \qquad \delta = \frac{\Delta p}{p_0}.$$

Write solution in the form:

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta_0,$$

where

$$\begin{aligned} C(0) &= S'(0) = 1, && \text{(cosine like)} \\ S(0) &= C'(0) = 0, && \text{(sine like)} \\ D(0) &= D'(0) = 0. \end{aligned}$$

The slope of the trajectory is then clearly given by

$$x'(s) = C'(s)\,x_0 + S'(s)\,x_0' + D'(s)\,\delta_0.$$

Matrix equation assuming no electric fields (i. e. constant energy):

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \mathbf{M} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}.$$

Consider $\mathbf{M}$ as the 1-turn matrix. Its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{tr}(\mathbf{M}_{2\times 2}) \pm \sqrt{\left(\mathrm{tr}(\mathbf{M}_{2\times 2}\right)^2 - 4}\right] = e^{\pm i\mu}, \qquad \text{and} \qquad \lambda_3 = +1.$$

Let us write the eigenvector corresponding to λ_3 as

$$\begin{pmatrix} \eta\delta \\ \eta'\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} \delta.$$

The periodic function η is called the *dispersion function.*

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

Solving this for η and η' yields

$$\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1-\cos\mu)},$$

$$\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1-\cos\mu)}.$$

- In general, the matrix's dispersion term $D(s) \neq \eta(s)$, but rather

$$\eta(s) = \frac{dx}{d\delta}, \qquad \text{whereas} \qquad D(s) = \frac{\partial x}{\partial \delta}.$$

(Some authors have muddled this point by using $D(s)$ for both quantities.)

- $x_p(s) = \eta(s)\delta$ is the inhomogenious part of the trajectory due to dispersion.
- Total horizontal trajectory component: $x(s) = x_\beta(s) + x_p(s)$.
  - x_β is contribution from betatron oscillation.
- Momentum spread σ_p smears beam out with $\sigma_{x_p} = \eta \frac{\sigma_p}{p_0}$.
- Total horizontal spread:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \left(\eta \frac{\sigma_p}{p_0}\right)^2} = \sqrt{\beta_x \epsilon_{\text{rms}} + \left(\eta \frac{\sigma_p}{p_0}\right)^2}.$$

  - Must add spreads from independent variable in quadrature.

Momentum compaction

- Momentum compation: $\alpha_p = \frac{\Delta L}{L} \Big/ \frac{\Delta p}{p}$

  with the circumference being given by

$$L = \oint ds, \quad \text{and} \quad L + \Delta L = \oint d\sigma, \tag{1}$$

  where $d\sigma$ is an infinitesimal arc of trajectory referred to an off–momentum particle, while ds is referring to a particle with the design momentum.

- Since both ds and $d\sigma$ cover the same infinitesimal azimuthal angle

$$d\theta = \frac{d\sigma}{\rho + x_p} = \frac{ds}{\rho}, \qquad \text{we have} \qquad d\sigma = \left(1 + \frac{x_p}{\rho}\right) ds.$$

- So $\quad \Delta L = \oint \frac{x_p}{\rho}\, ds.$

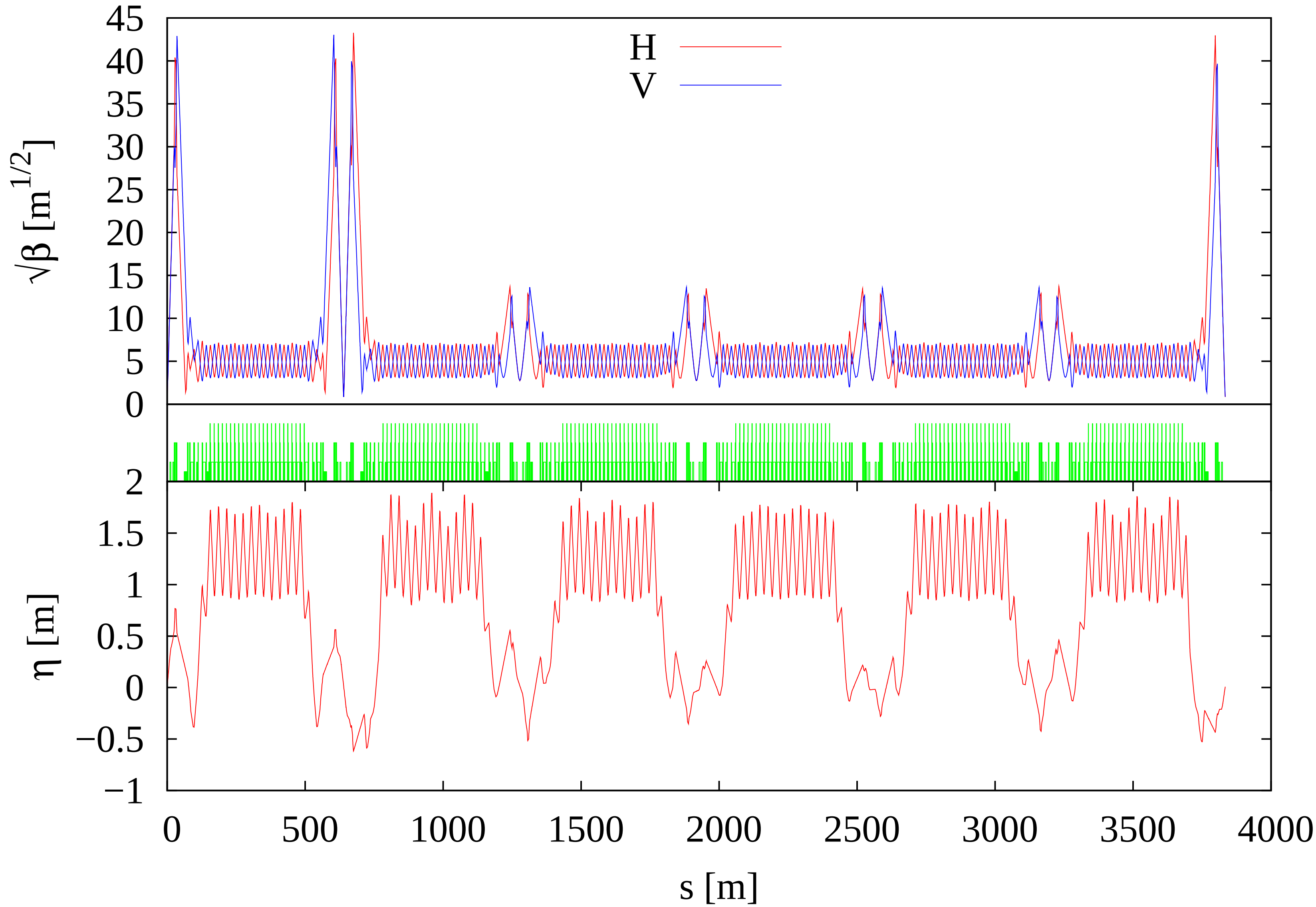
Using the relation, $x_p = \eta\,\delta$, produces

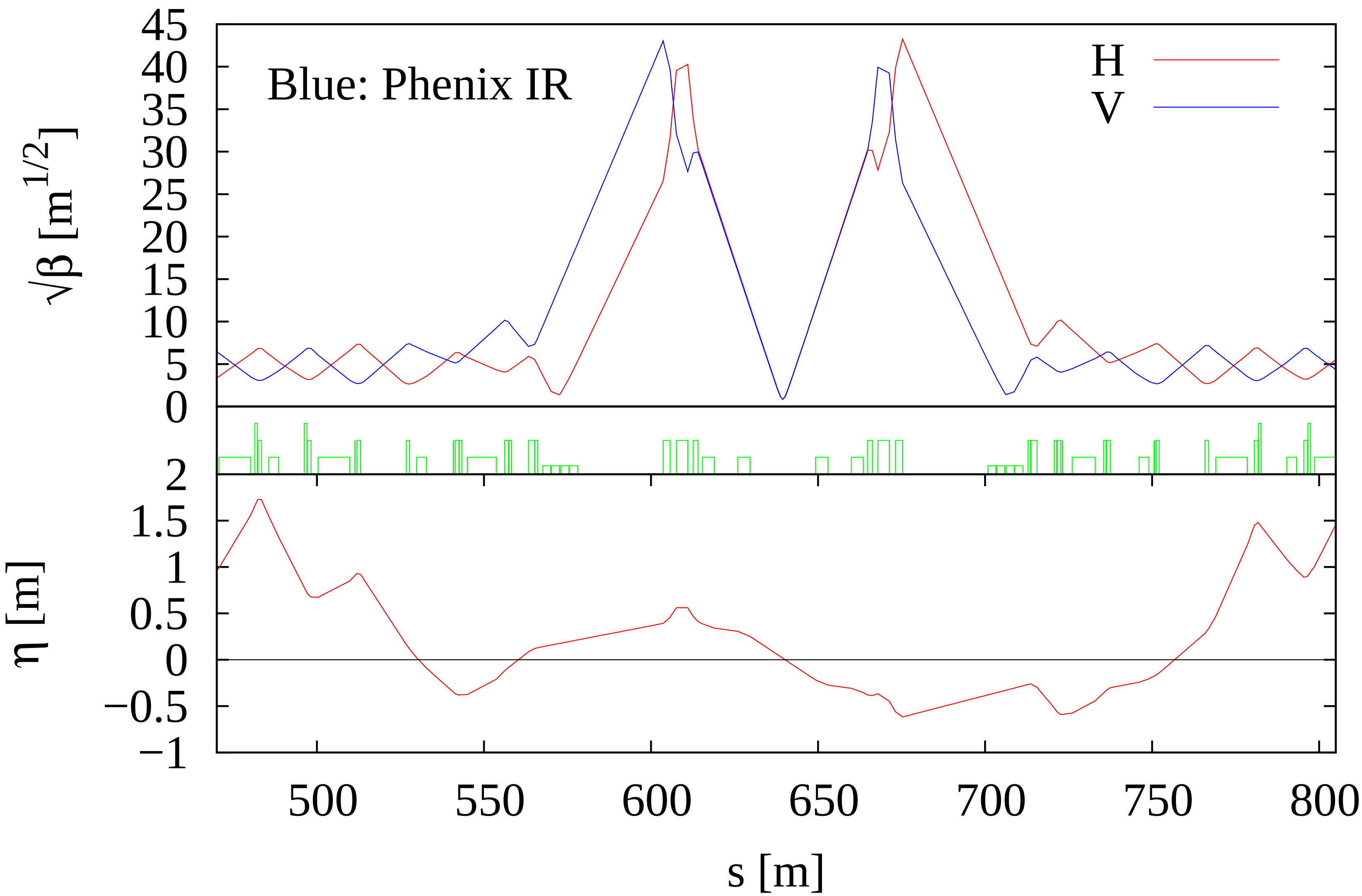
$$\Delta L = \frac{\Delta p}{p} \oint \frac{\eta(s)}{\rho(s)}\,ds,$$

and gives the momentum compaction as

$$\alpha_p = \frac{1}{L} \oint \frac{\eta(s)}{\rho(s)}\,ds.$$

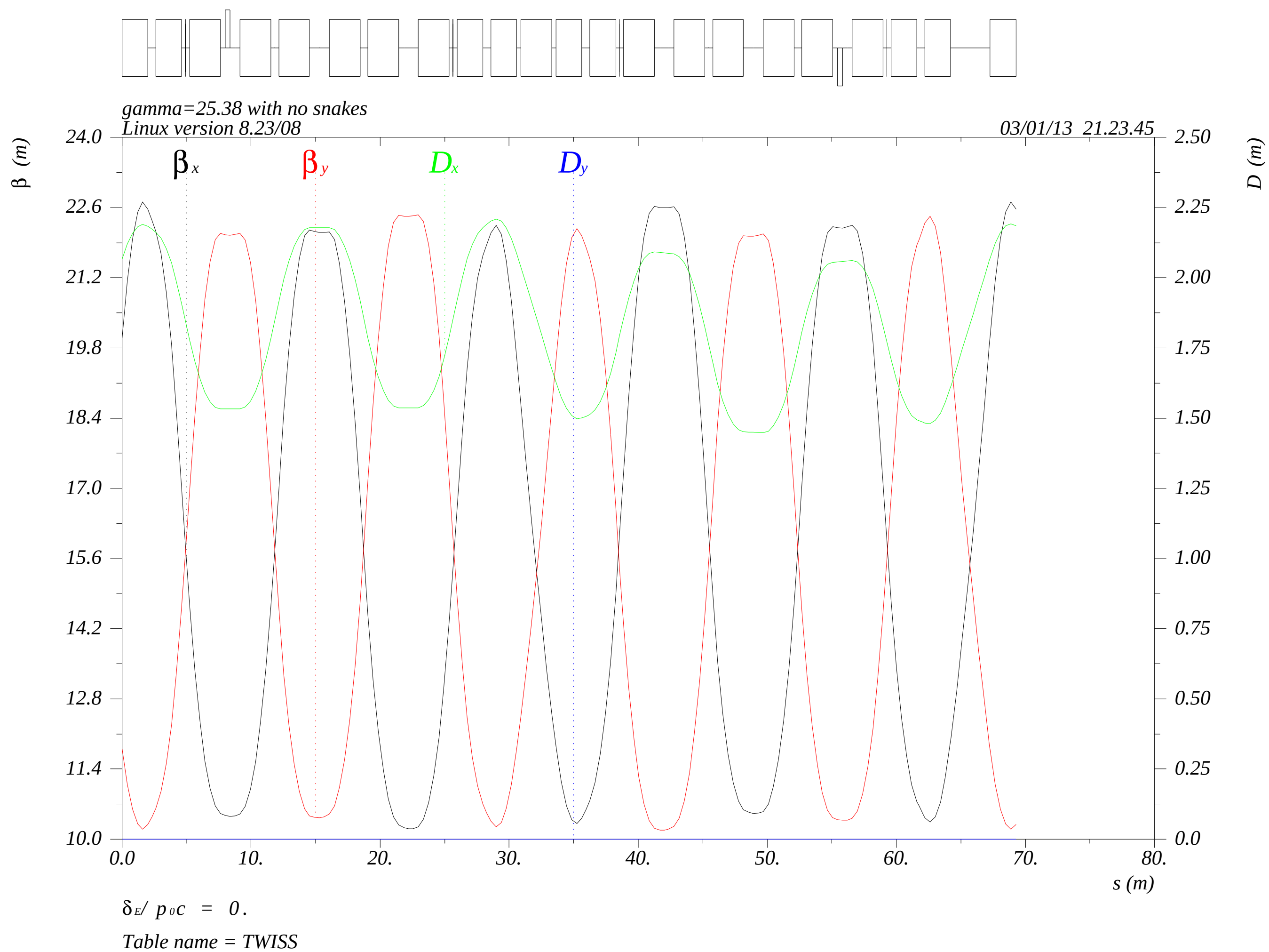
RHIC optics functions





$$Q_{\rm H} = 28.695, \quad Q_{\rm V} = 29.685, \quad \beta_{\rm H,V}{}^* = 0.7\text{m}.$$

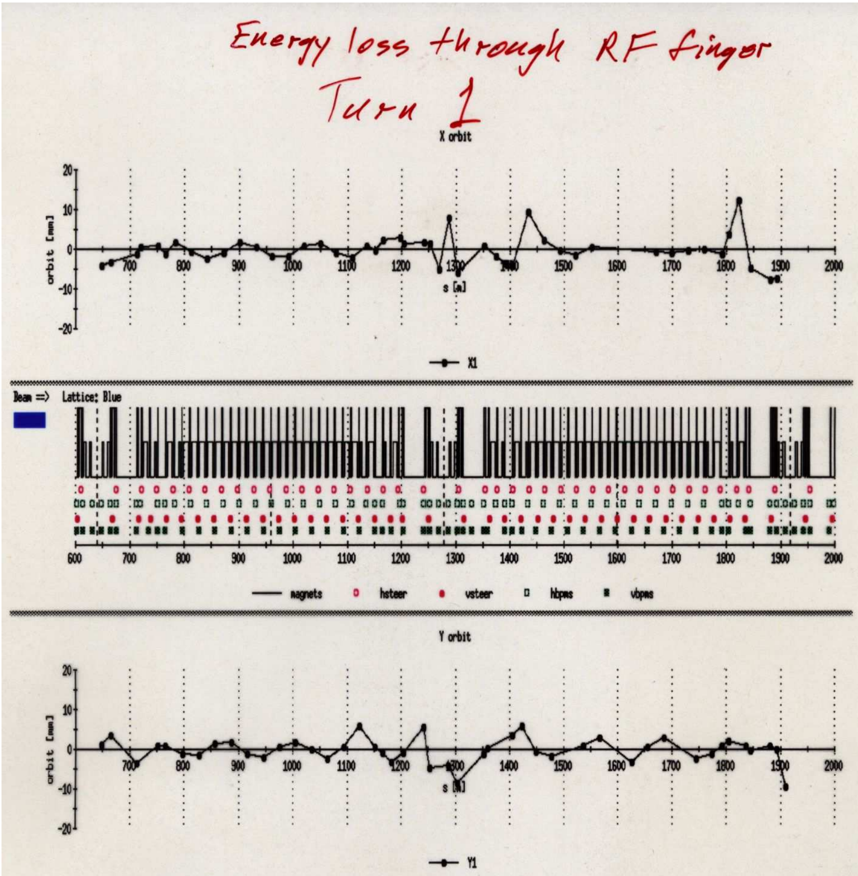
AGS optics functions

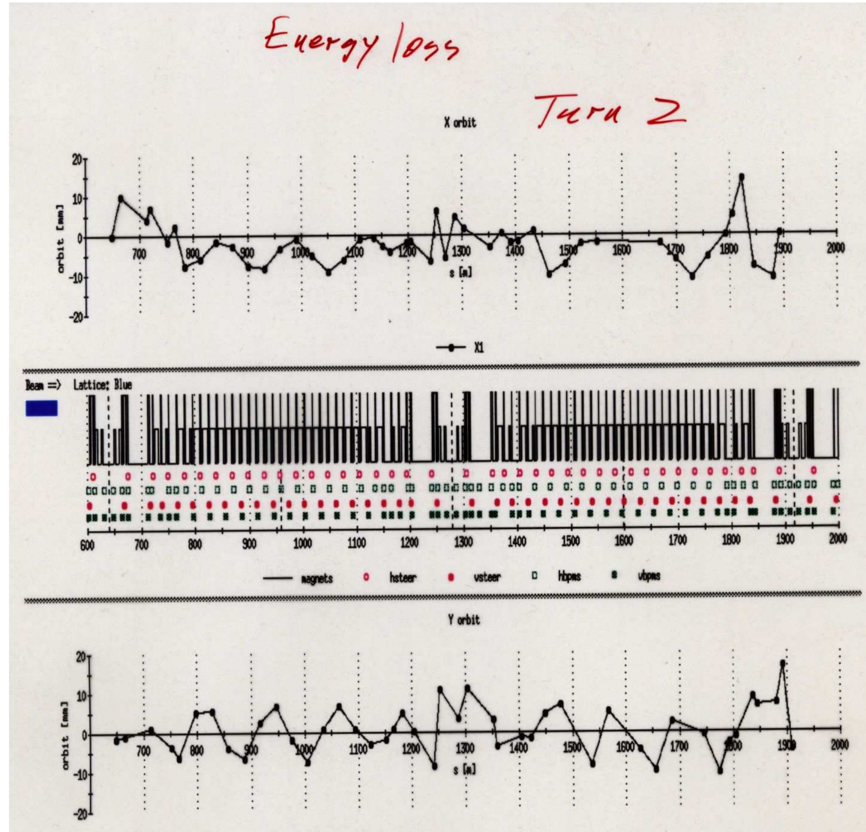


$$L = 807.076 \text{ m},$$
$$Q_{\mathrm{H}} = 8.6648,$$
$$Q_{\mathrm{V}} = 8.7288$$
$$\alpha_p = 0.0142,$$
$$\gamma_{\mathrm{tr}} = 8.407,$$
$$\xi_{\mathrm{H}} = -23.3,$$
$$\xi_{\mathrm{V}} = 4.099.$$

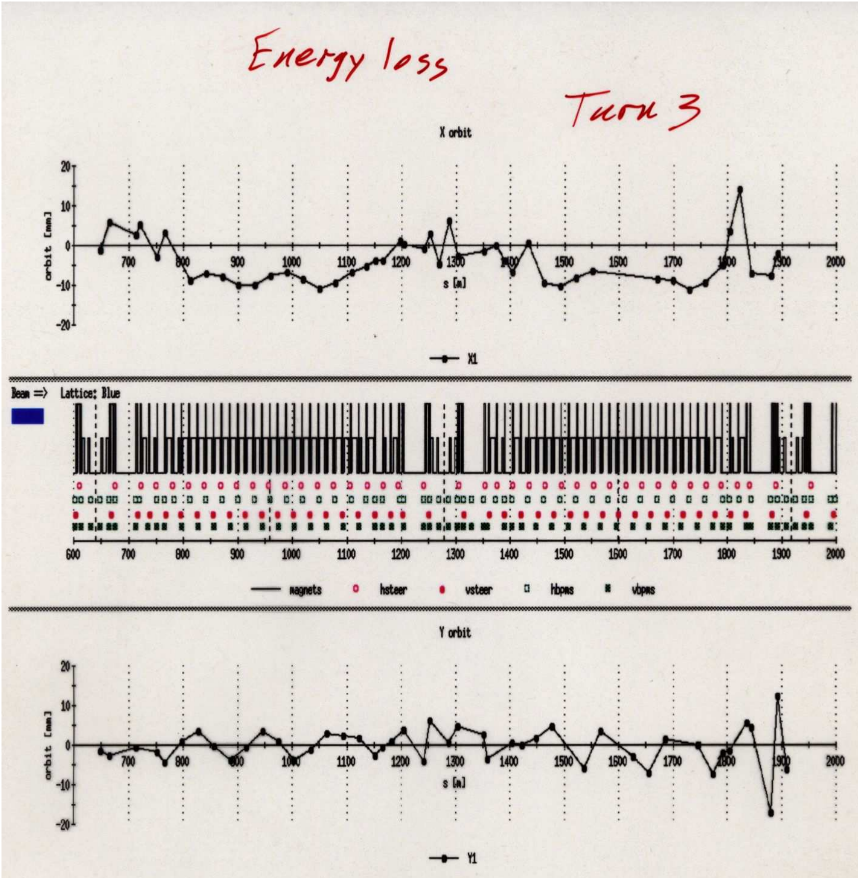
- Superperiodicity: $P = 12$. (1st superperiod + 1 dipole plotted)

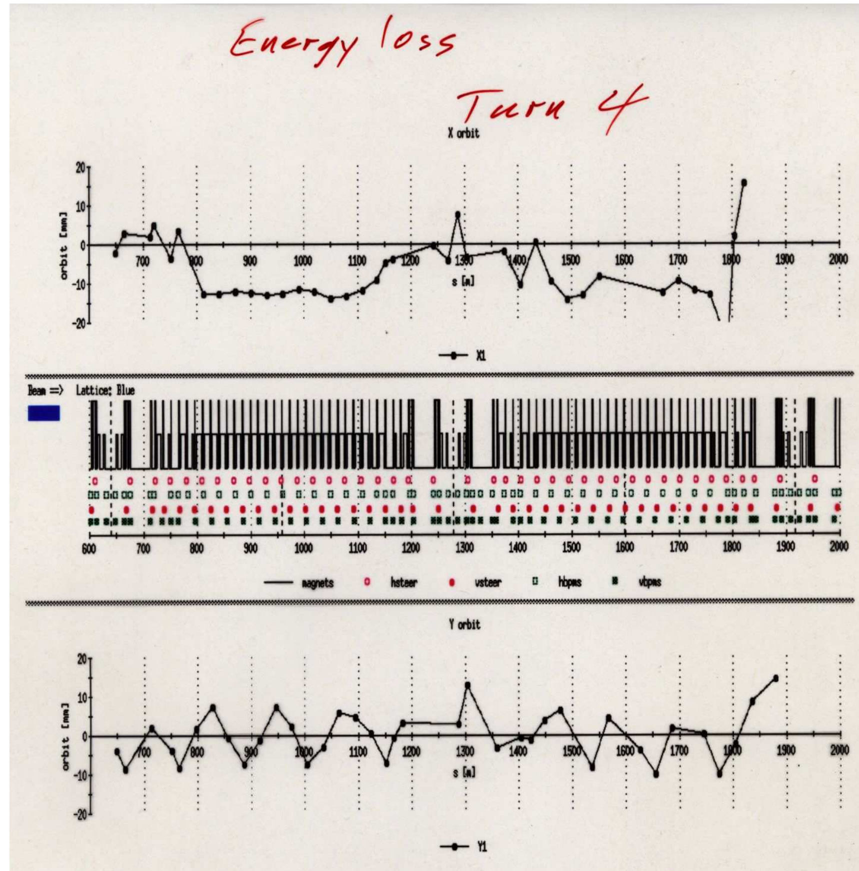
RHIC commissioning



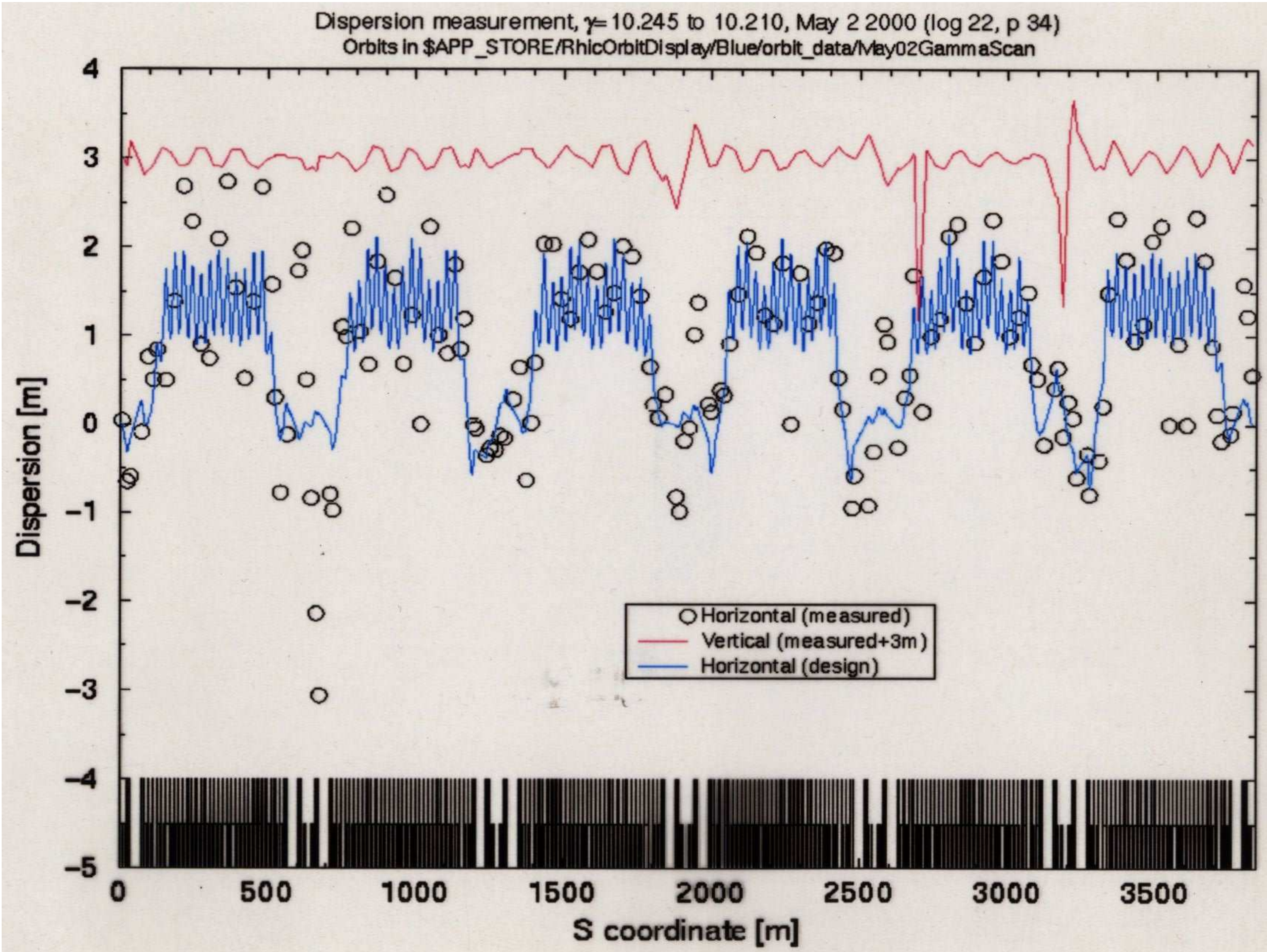


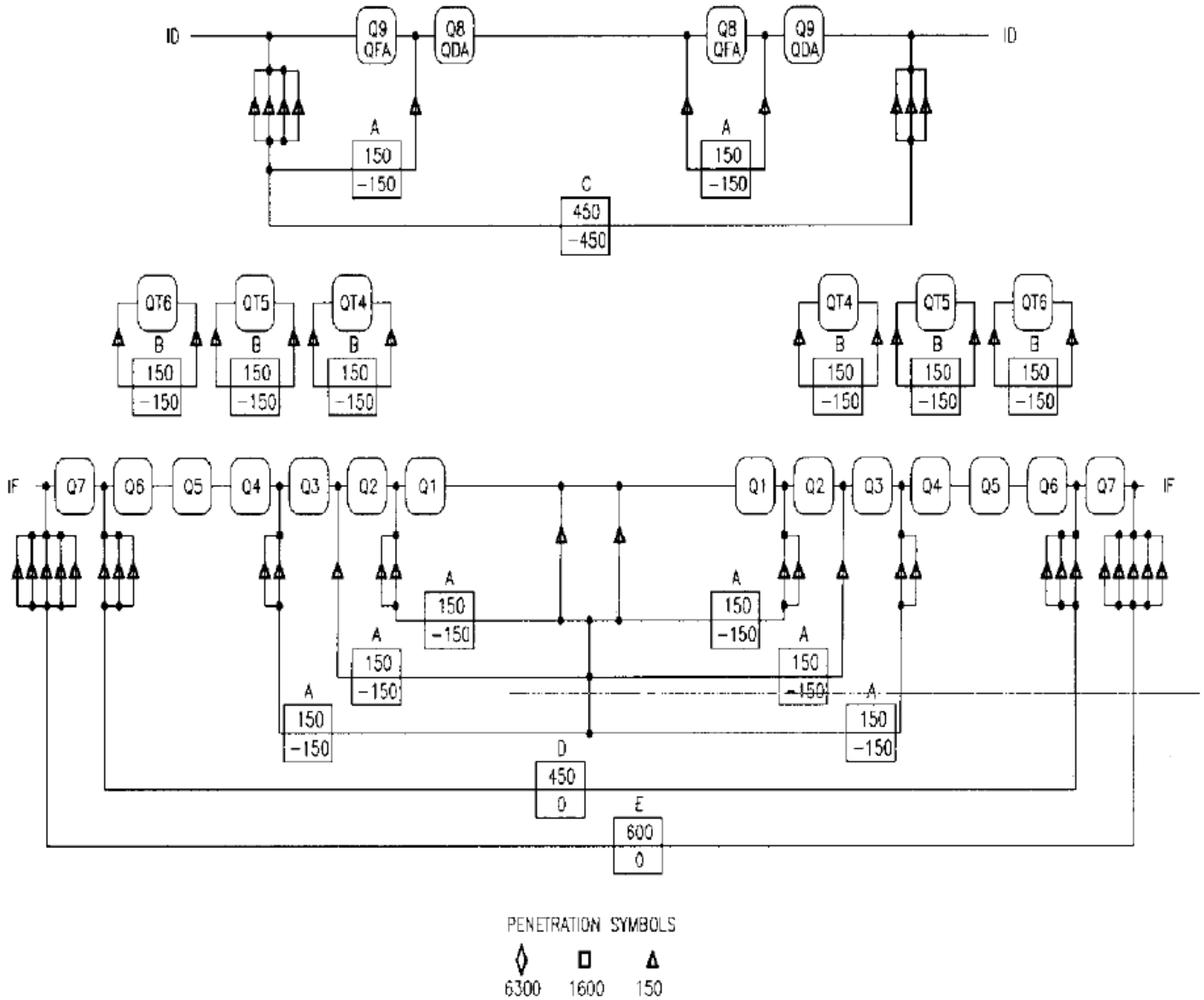
- Yellow ring: rf finger bent inward.
- Energy loss ($\frac{dE}{dx}$) seen each turn.

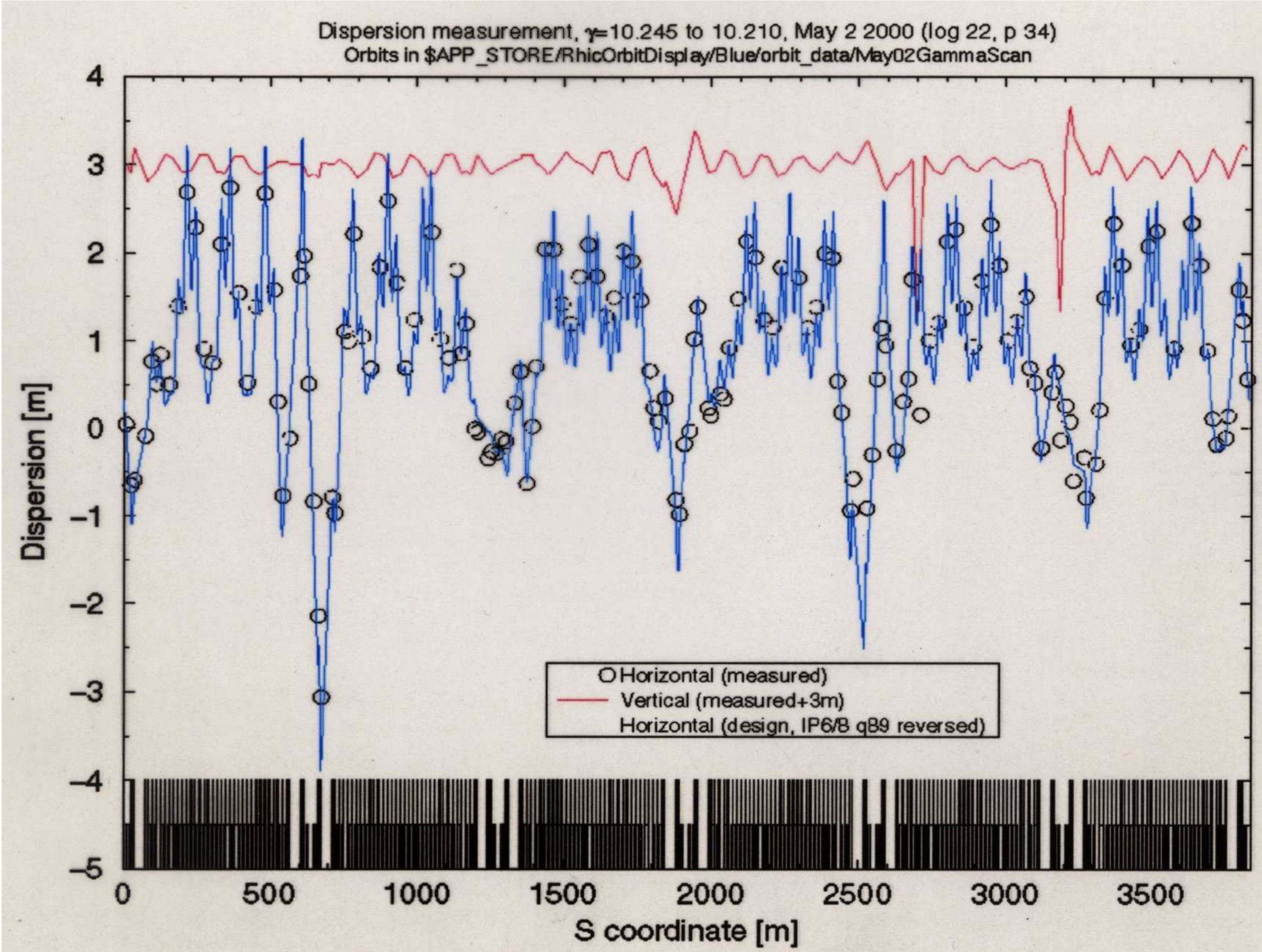

Energy loss
Turn 3
X orbit
orbit [mm]
s [m]
X1
Beam =>
Lattice: Blue
magnets
hsteer
vsteer
hbpms
vbpms
Y orbit
Y1


Energy loss
Turn 4
X orbit
orbit [mm]
s [m]
X1
Beam =>
Lattice: Blue
magnets
hsteer
vsteer
hbpms
vbpms
Y orbit
Y1

RHIC Commissioning: bad dispersion

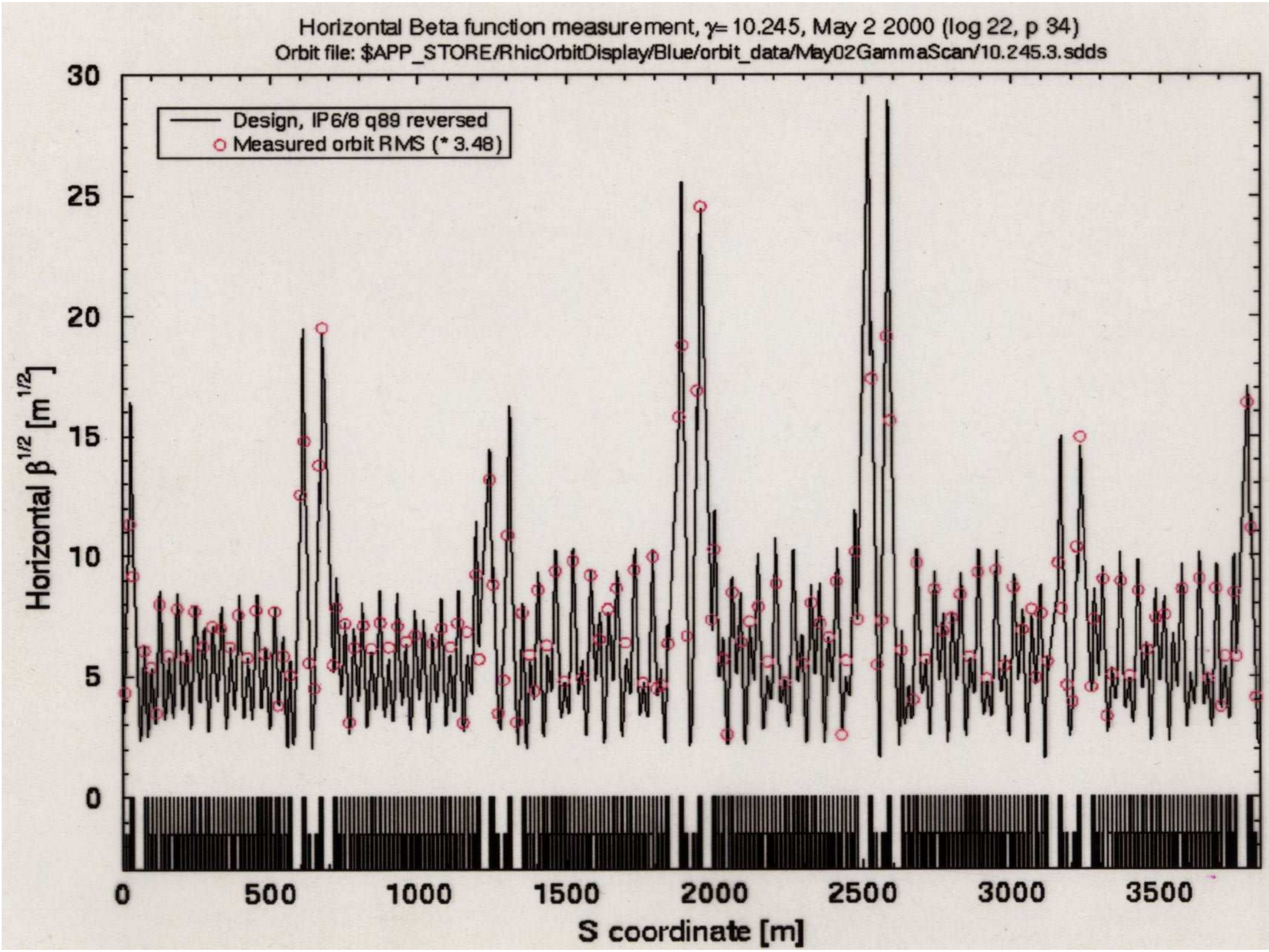



ID
Q9 QFA
Q8 QDA
Q8 QFA
Q9 QDA
ID
A
150
-150
A
150
-150
C
450
-450
QT6
QT5
QT4
B
150
-150
QT4
QT5
QT6
IF
Q7
Q6
Q5
Q4
Q3
Q2
Q1
Q1
Q2
Q3
Q4
Q5
Q6
Q7
IF
D
450
0
E
600
0
PENETRATION SYMBOLS
6300
1600
150



- Modelled with Q8-Q9 trim supplies reversed at IR's 6 and 2.

Horizontal beta measuerments after PS fix



Vertical beta measuerments after PS fix