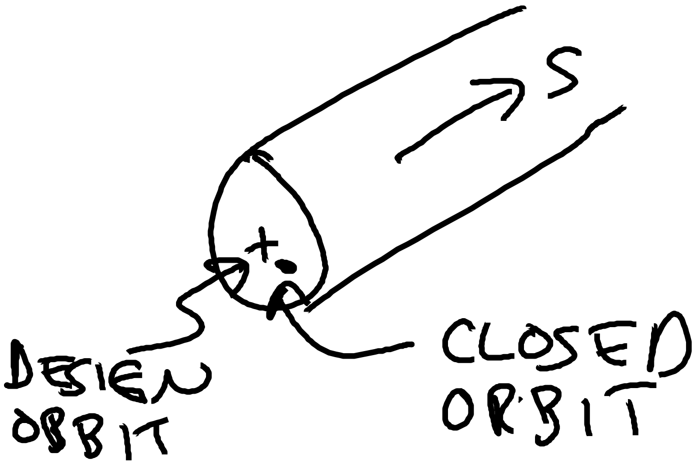
#2: LINEAR MOTION & STABILITY 1/21/19

Mersenne prime # 20 $2^{82589933} - 1$

DESIGN & CLOSED ORBITS



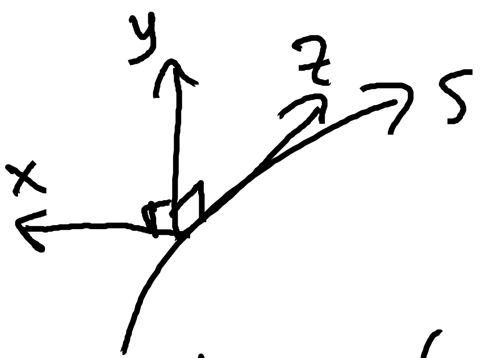
It can be shown that

IF $\frac{dB}{dt} = \frac{dE}{dt} = 0$

there is ONE orbit that exactly repeats itself.

- CLOSED = DESIGN with no errors present
- Measure test particle offsets relative to CLOSED orbit

BETATRON OSCILLATIONS



RIGHT HANDED co-ord system
(x, y, z) rotates tangential to DESIGN orbit

WANT (LINEAR) STABILITY for small betatron oscillations about C.O.:

$$\boxed{x(n) = a \cos(2\pi . Q_x . n + \phi_0)}$$

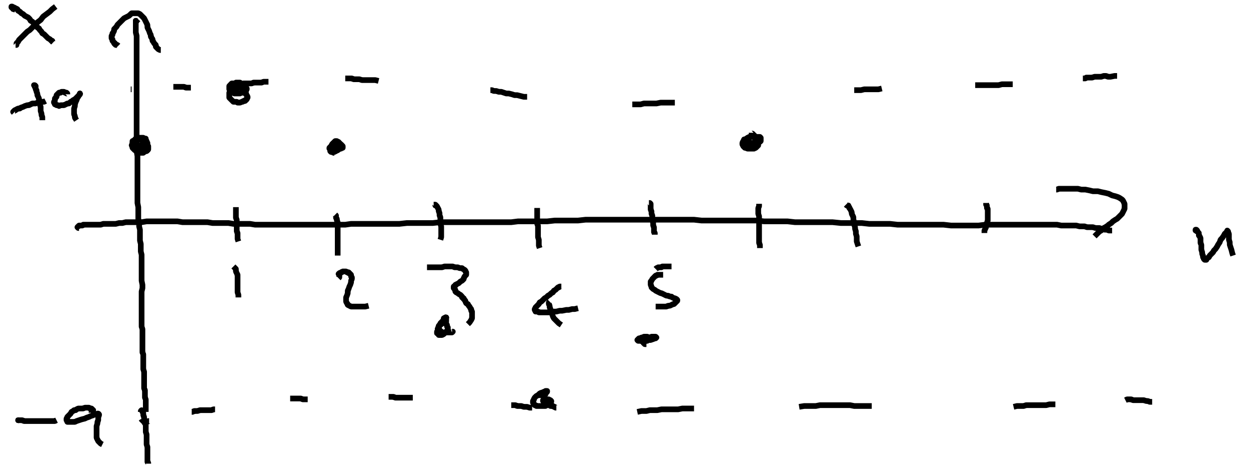
amplitude ~ 1 mm

BETATRON TUNE

turn number integer $n \to \infty$

initial phase

$Q_x \gg 1$, Integer part doesn't matter



$Q_x \gg 1$

$\mathrm{mod}(Q_x, 1) = 0.155$

[Also need longitudinal stability:]

Relative to bunch center

$$\boxed{z(n) = a_z \left(2\pi \cdot Q_s \cdot n\right)}$$

$a_z \sim$ few cm : proton/hadron
few mm electrons ...

MOTION THROUGH A MAGNET

Long magnets : length >> bore diameter :

2-D fields

$$\boxed{B_x = B_x(x, y), \quad B_y = B_y(x, y), \quad B_s = 0}$$

Must solve Maxwell's equs :

$$\bar{\nabla} \times \bar{B} = 0 , \qquad \bar{\nabla} \times \bar{B} = \mu_0 J = 0$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0} , \qquad \boxed{\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial y} = 0}$$

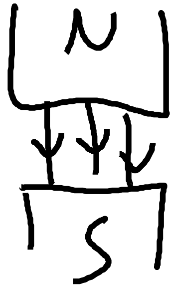
SIMPLEST SOLUTION :

$$B_y = \text{constant}, \quad B_x = 0$$

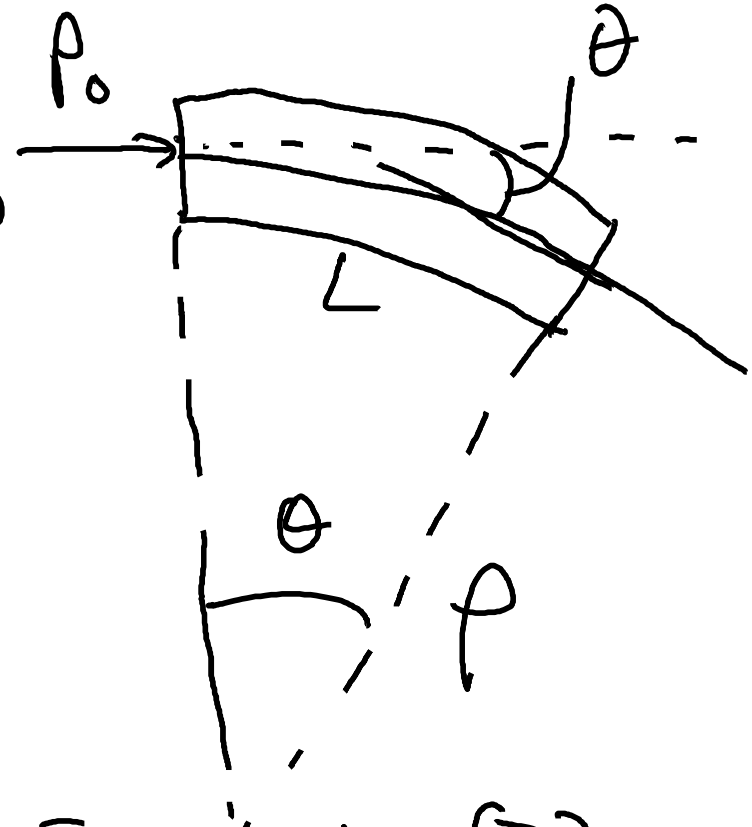
DIPOLE

DIPOLE

END view



TOP VIEW



$$F = \frac{dp_\perp}{dt} = qvB$$

$$\theta = \frac{\Delta p_\perp}{p_0} = \frac{1}{p_0}\frac{dp}{dt}\frac{L}{v} = \frac{L}{\rho}$$

So
$$\frac{1}{\rho} = \frac{B}{(B\rho)}$$

Eg $B = 0.33\,[T]$
$p_0 = 3\,[GeV/c]$
$\rho = 30\,[m]$

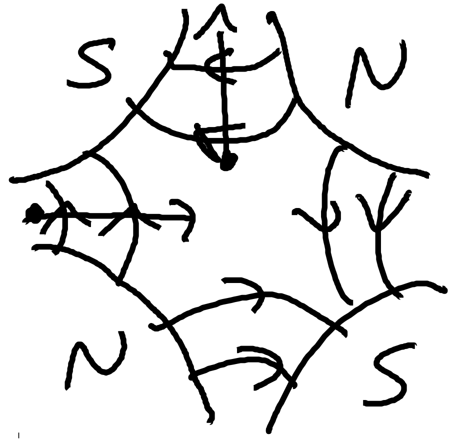
RIGIDITY
$$\boxed{(B\rho) = \frac{p_0}{|q|}}$$
or
$$\boxed{(B\rho)[Tm] = 3.34\, p_0 \left[\frac{GeV}{c}\right]}$$

QUADRUPOLE

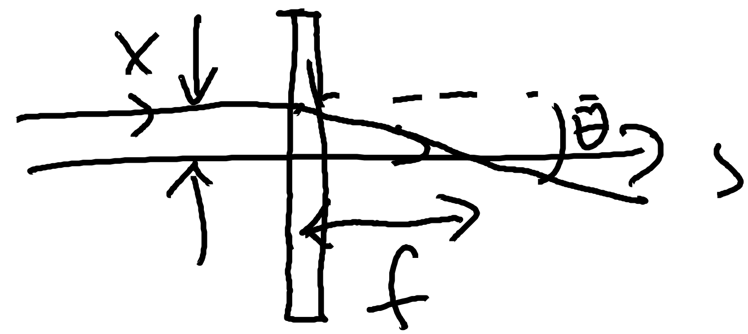
$B_x = B' \cdot y$

$B_y = B' \, x$

constant



TOP VIEW



$$\boxed{\theta = \frac{q B_y}{p_0} \cdot L = \frac{B'}{(B\rho)} L \cdot x}$$

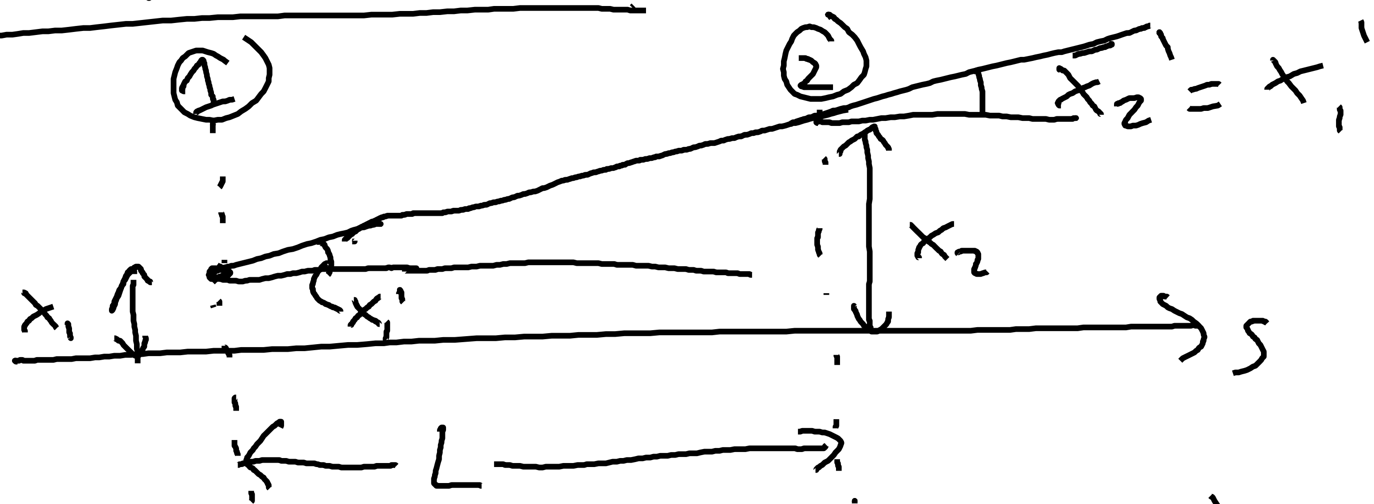
Angle proportional to displacement! A LENS

FOCAL STRENGTH $\boxed{\frac{1}{f} = K L}$

where $\boxed{K = \mathrm{sgn}(q) \, \frac{B'}{(B\rho)}}$

NOTE: QUAD focuses H, DEfocuses V (or vice versa)

MATRICES: DRIFT

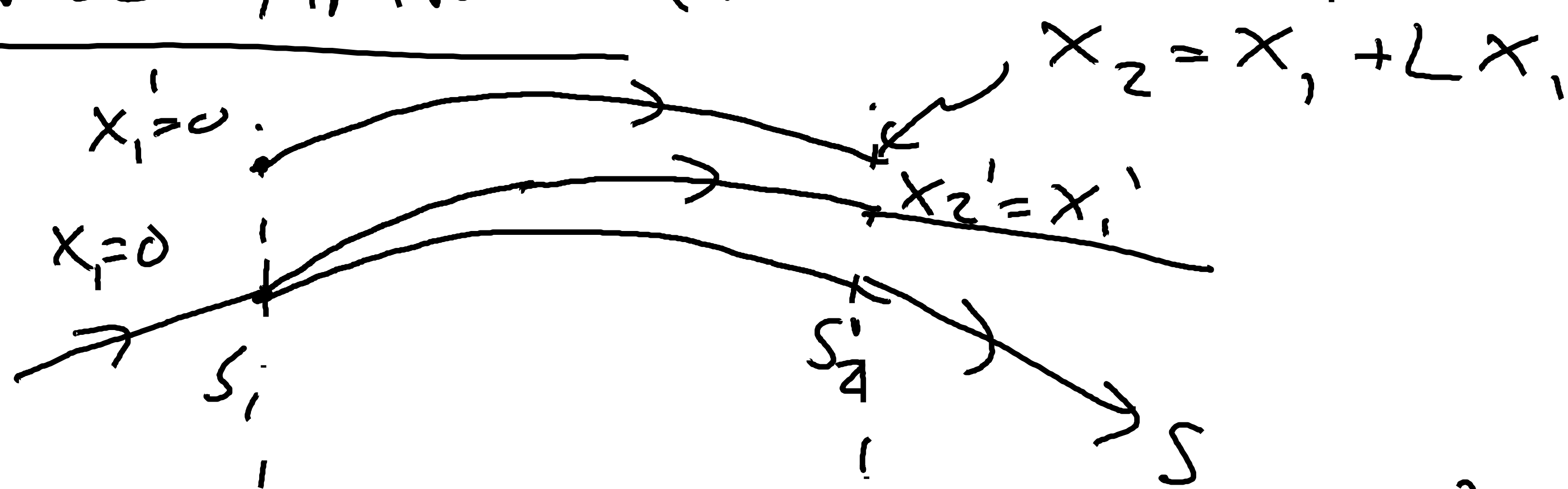


$$x_2 = x_1 + L x_1'$$
$$x_2' = x_1'$$

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = M_{DRIFT} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1$$

$$\boxed{M_{DRIFT} = \left(\begin{array}{cc|cc} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{array}\right)}$$

DIPOLE MATRIX (RECTANGULAR)



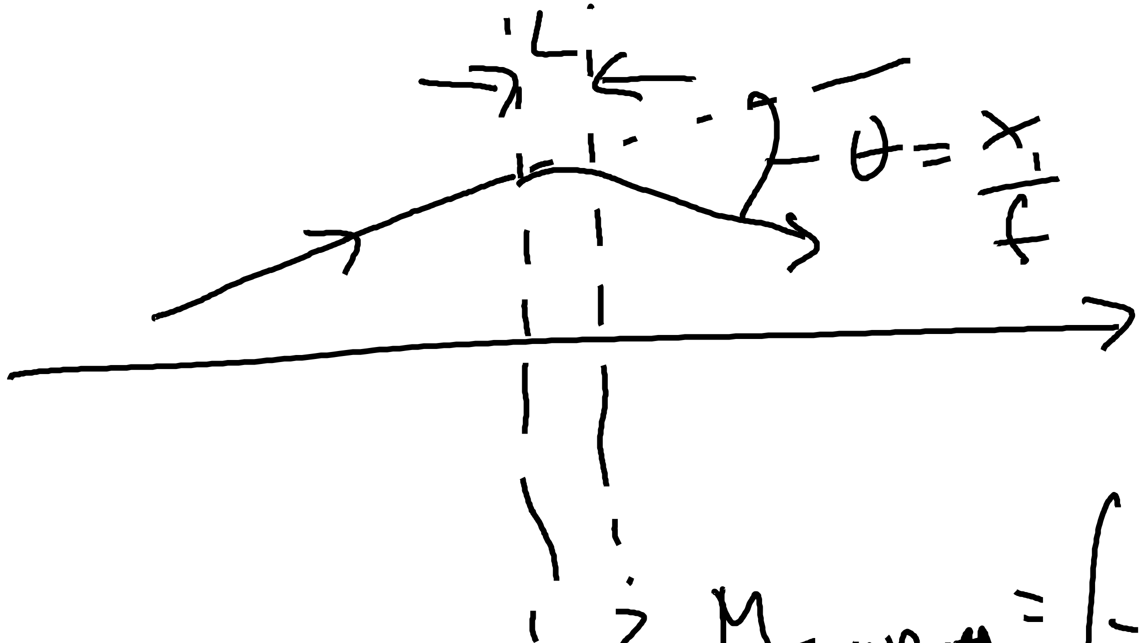
Bend angle is independent of x !

Co-ordinate frame rotates.

$$\boxed{M_{RBEND} = M_{DRIFT}}$$

Rectangular bend !!

QUAD MATRIX



$\theta = \frac{x_1}{f}$

THIN QUAD

$L \ll |f|$

$x_2 = x_1$

$x_2' = x_1' - \frac{x_1}{f}$

$$M_{THINQUAD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & +\frac{1}{f} & 1 \end{pmatrix}$$

$$\boxed{M_{QUAD} \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) & 0 & 0 \\ -k\sin(kL) & \cos(kL) & 0 & 0 \\ 0 & 0 & \cosh(kL) & \sinh(kL) \\ 0 & 0 & k\sinh(kL) & \cosh(kL) \end{pmatrix}}$$

$k = \sqrt{K}$

LEGO PIECES

Make a sequence – LATTICE – of drifts & dipoles & quads together, to bend by 2π

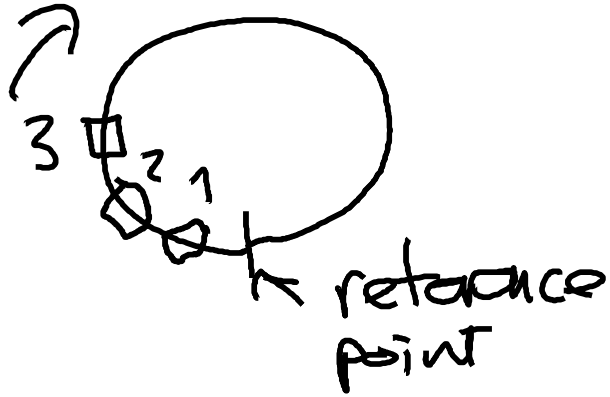
Q1: How to test if linear motion is stable

Q2: How to deal with the focus/defocus problem?

Q3: How to make beam size big/small in different "s" locations?

Q4: Why (& when) must we include nonlinear magnets (especially sextupoles)?

LINEAR STABILITY



ONE-TURN MATRIX

$$M = M_{m,m-1} \cdots M_{21} M_{10}$$

Matrices M_{ij} are block diagonal, so horizontal stability becomes a 2×2 problem

(A) $$\boxed{\bar{X}_n = M^n \bar{X}_0}$$

$$\bar{X}_0 = \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

M has 2 complex eigenvectors, $\bar{v}_1$ and $\bar{v}_2$ (B)

that solve the equation $\boxed{M\bar{v} = \lambda \bar{v}}$ EIGENVALUE complex scalar

Write the initial vector as

$$\bar{x}_0 = A\,\bar{v}_1 + B\,\bar{v}_2$$

(Both sides are REAL, but $A, B, \bar{v}_1$ + $\bar{v}_2$ are complex)

On turn n

$$\bar{x}_n = M^n \bar{x}_0 = A\,\lambda_1^n\,\bar{v}_1 + B\,\lambda_2^n\,\bar{v}_2$$

If $\bar{x}_n$ is bounded for <u>all</u> n, so also must be λ_1^n + λ_2^n

Find EIGENVALUES λ_1 + λ_2 by solving the characteristic EQN

$$\boxed{\det(M - \lambda I) = 0}$$

I is 2×2 unit matrix

Write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then get the quadratic eqn.:

$$\underbrace{(ad - bc)}_{\det(M) = 1} - \underbrace{(a+d)}_{Tr(M)}\lambda + \lambda^2 = 0$$

"unimodular"

$$\boxed{\lambda^{-1} + \lambda = Tr(M)} = a + d$$

By inspection the eigenvalues are reciprocals

$\lambda_1 = 1/\lambda_2$

$$\boxed{\lambda_1 = e^{i\mu}, \quad \lambda_2 = e^{-i\mu}} = \cos(\mu) - i\sin(\mu)$$

If μ is COMPLEX, then either λ_1^n or λ_2^n $\to \infty$ as $n \to \infty$, motion is unstable

THERFFORE MOTION IS STABLE, SINCE

$$\boxed{2\cos(\mu) = Tr(M)}$$

IF

$$\boxed{-1 \le \frac{1}{2} Tr(M) \le 1}$$

If motion IS stable, then the ONE-TURN matrix IS written in general as

$$\boxed{M(s) = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}}$$

where β, α & γ are "Twiss" or "Courant-Snyder" parameters satisfy $\boxed{\gamma \equiv \frac{1+\alpha^2}{\beta}}$ $\boxed{\mu_x \equiv 2\pi Q_x}$

α, β, γ are functions of s .... but NOT μ