USPAS Accelerator Physics 2019
Northern Illinois University and UT-Batelle

16: Routes to Chaos
(perhaps yet another self-referential lecture)

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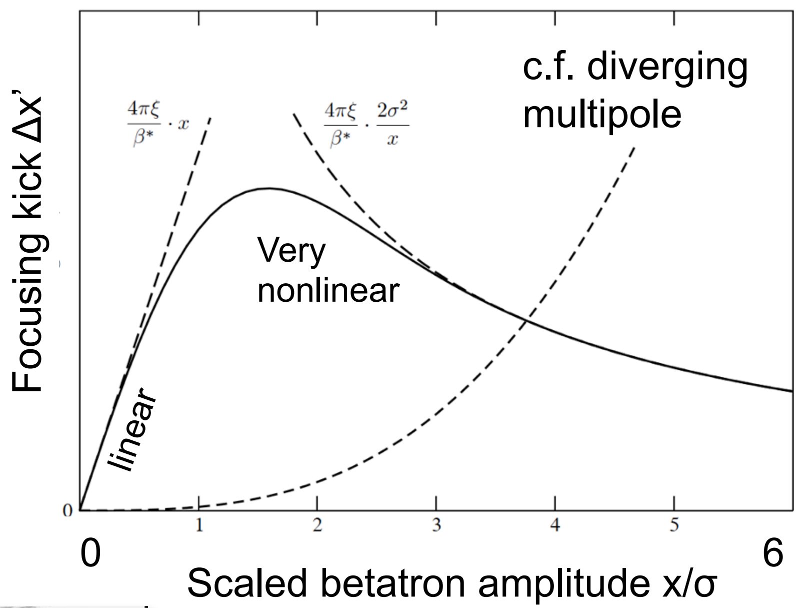
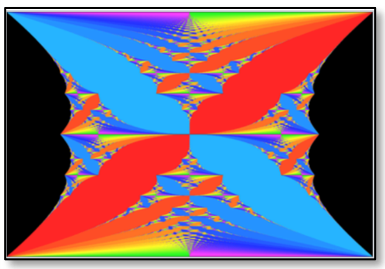
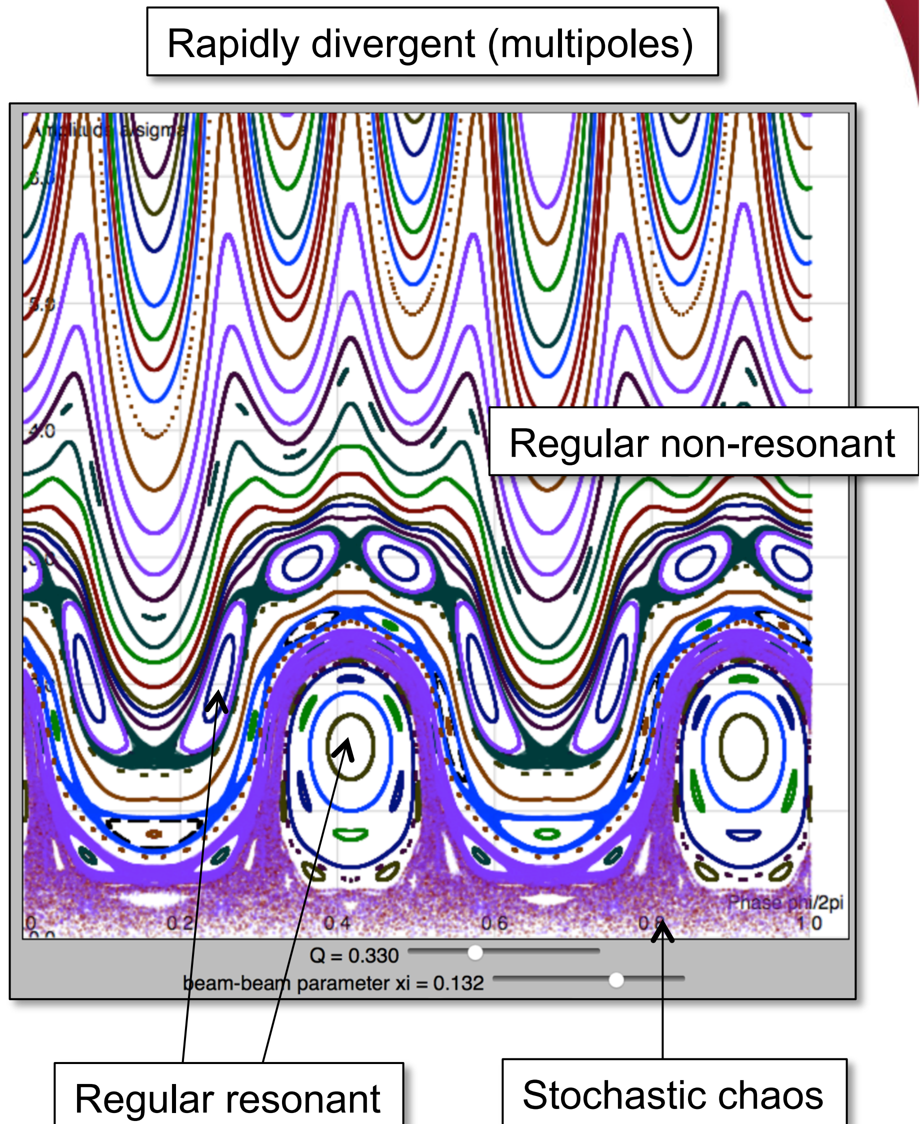
Happy birthday to Irving Langmuir (Nobel 1932) and Rudolf Mössbauer (Nobel 1961)!
Happy National Hot Chocolate Day and Hell Is Freezing Over Day!

Review: 1D Beam-Beam (Steve, yesterday)



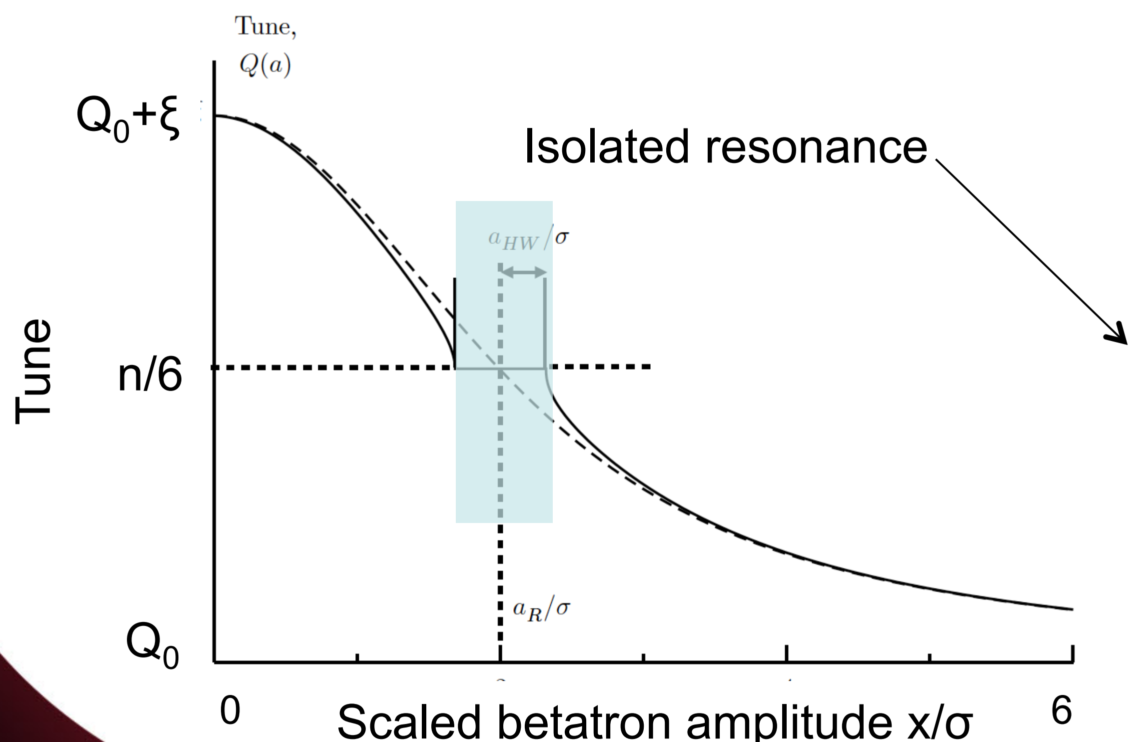


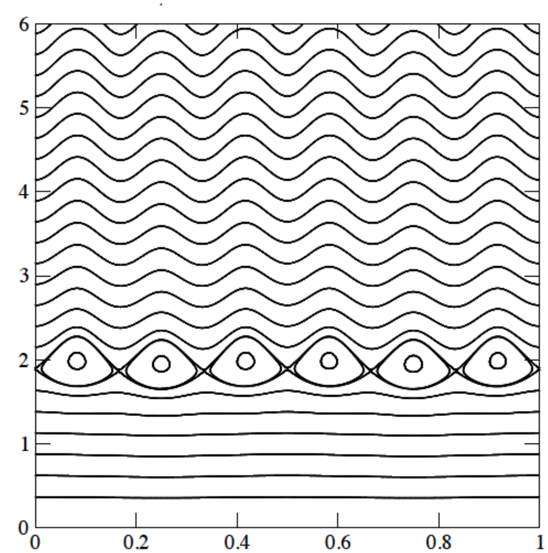


D. Hofstadter: "*... an eerie type of chaos ... just behind a facade of order – and yet, deep inside the chaos lurks an even eerier type of order.*"

Review: 1D Beam-Beam

- 1D beam-beam dynamics are surprisingly tractable
  - Can predict where **isolated** resonances are located
    - Lecture yesterday and homework
  - Can predict **other isolated resonance properties**
    - N-turn Hamiltonians give "island tunes", resonance widths





Can we predict the onset of chaos, even roughly?

16.1: Resonance overlap

- We have evaluated isolated resonances using the n-turn action-angle Kobayashi Hamiltonian where $Q_0 - \frac{p}{n} \ll 1$

$$H_n = \underbrace{2\pi \left(Q_0 - \frac{p}{n}\right) J}_{\text{Small-amplitude frequency}} + \underbrace{2\pi\xi\, U(J)}_{\text{U''(J): detuning}} - \underbrace{2\pi\xi\, V_n(J)\cos(n\phi)}_{V_n\text{(J): resonance driving}}$$

- Here we had assumed that all other resonances either
  - have small V_n (so their widths are small enough to ignore) or
  - their Hamiltonian terms phase average to near zero over many ierations of this map (over many turns)

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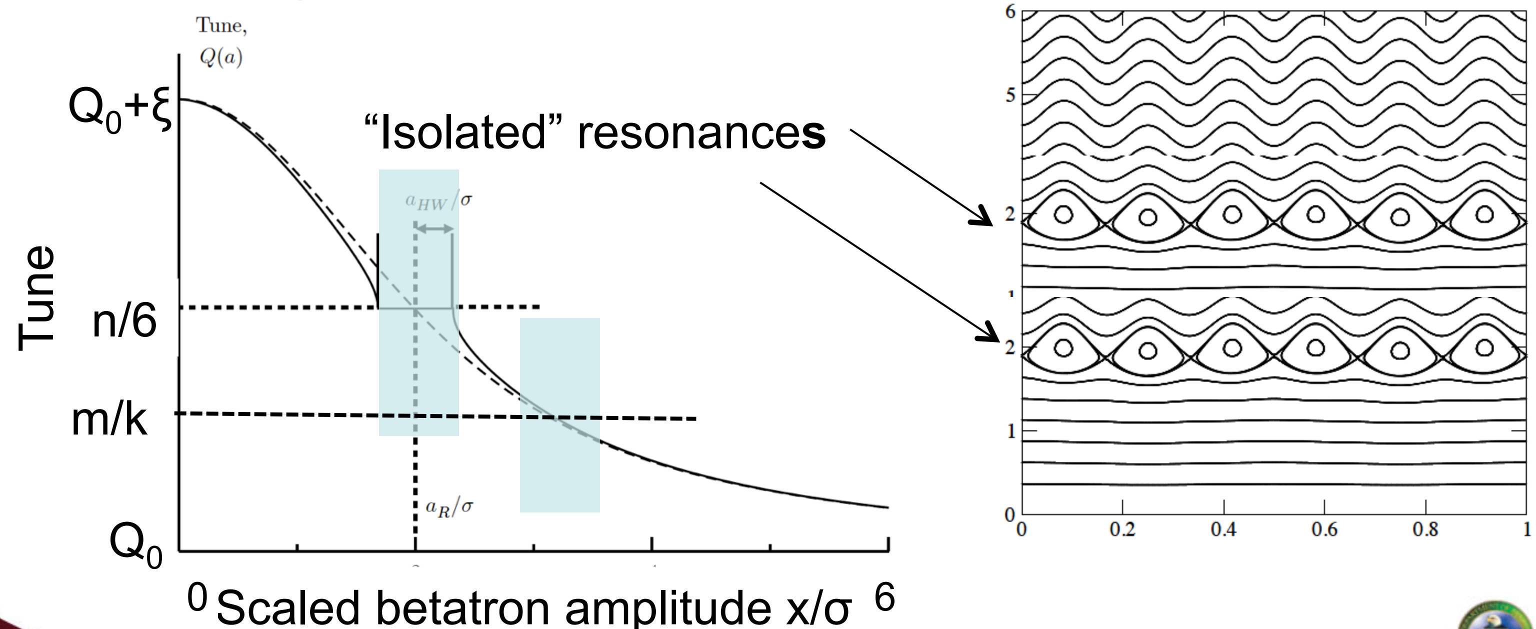
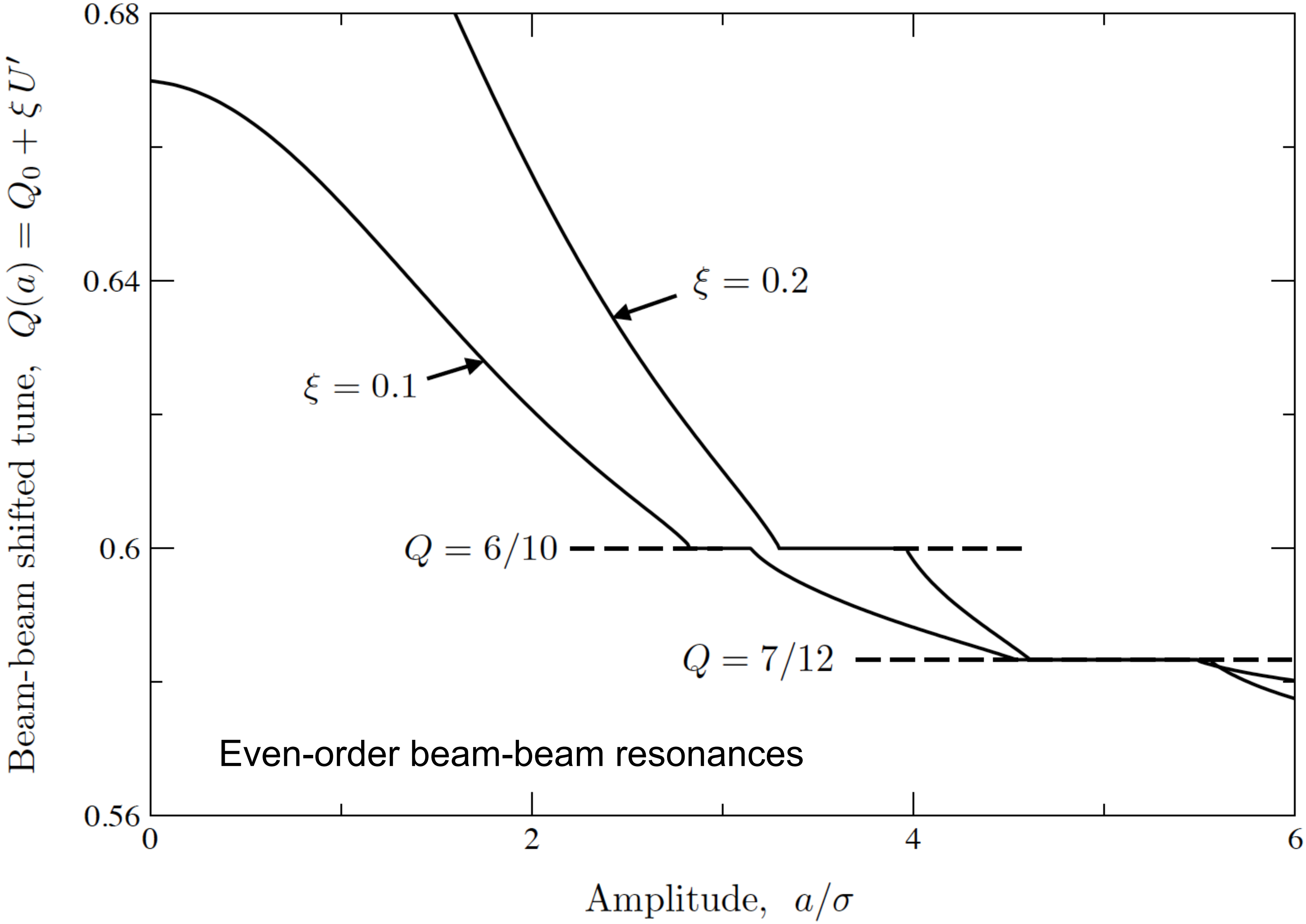


Fig 16.1: Resonance Overlap



16.1: Resonance overlap: Chirikov

- What happens when this assumption breaks down?
  - Resonances approach the point of overlapping
  - Separatrices are the first to interact
    - But separatrices have infinite period and therefore are infinitely vulnerable to perturbation

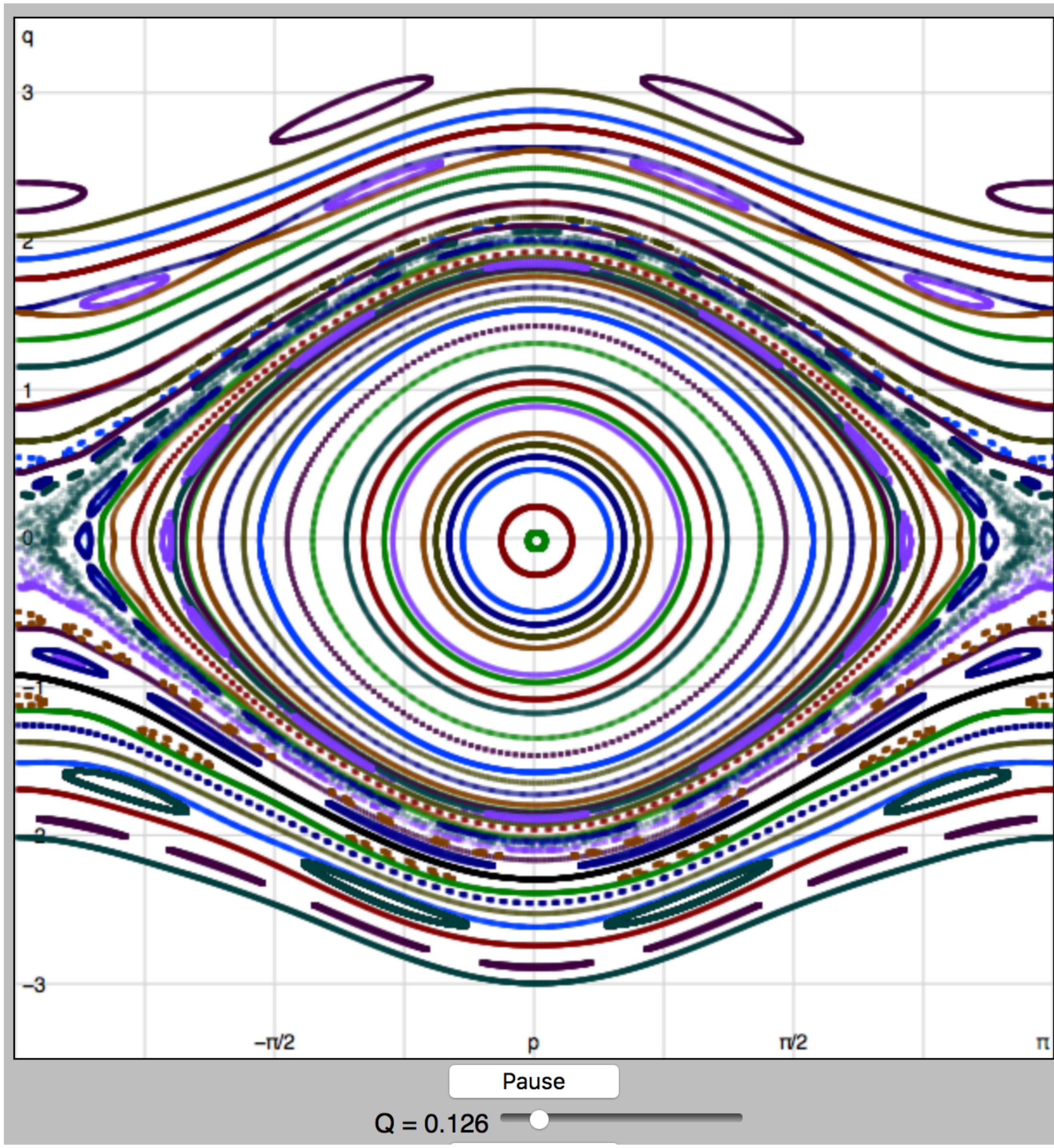
- Chirikov hypothesized that chaos emerges when nearby resonance widths are large enough that they overlap

  - This is calculable and verifiable
  - "A Universal Instability of Many-Dimensional Oscillator Systems" – Physics Reports **52** 5, May 1979, pp. 263-379.
  - 5000+ citations: a "famous" paper and surprisingly readable

16.1: Chirikov Overlap and the Standard Map

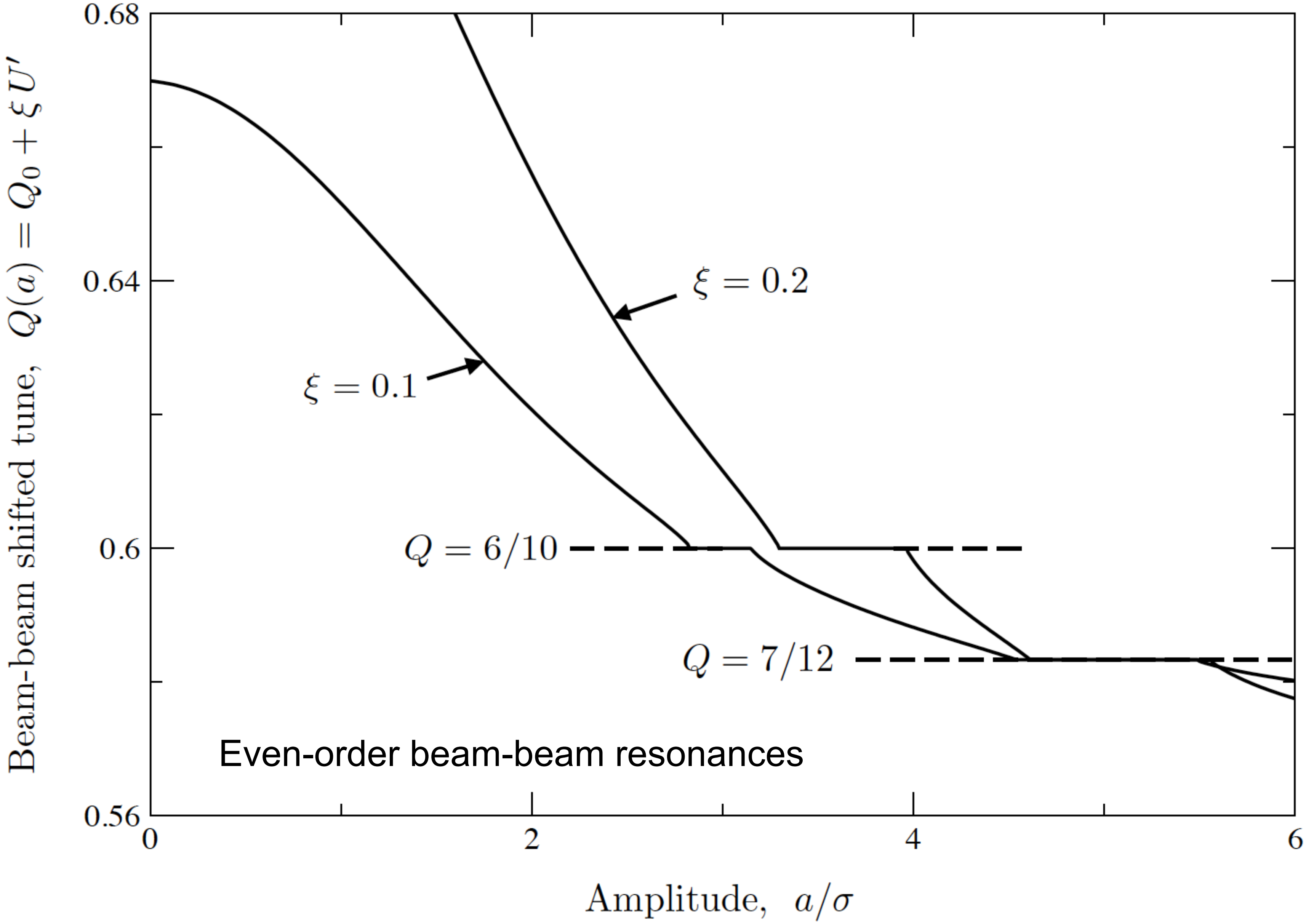
Chaotic resonance overlap: separatrix becomes chaotic

RF motion with large Qs!



"Isolated" resonances

Whoa, Wait a sec: ξ=0.2 could be stable?



EIC Beam-Beam Parameters (European Stragegy Submission)

design parameter	eRHIC proton	eRHIC electron	JLEIC proton	JLEIC electron
center-of-mass energy [GeV]	105		44.7	
energy [GeV]	275	10	100	5
number of bunches	1320		3228	
particles per bunch [10^{10}]	6.0	15.1	0.98	3.7
beam current [A]	1.0	2.5	0.75	2.8
horizontal emittance [nm]	9.2	20.0	4.7	5.5
vertical emittance [nm]	1.3	1.0	0.94	1.1
β_x^* [cm]	90	42	6	5.1
β_y^* [cm]	4.0	5.0	1.2	1
tunes (Q_x, Q_y)	.315/.305	.08/.06	.081/.132	.53/.567
hor. beam-beam parameter	0.013	0.064	0.015	0.068
vert. beam-beam parameter	0.007	0.1	0.015	0.068
IBS growth time hor./long. [min]	126/120	n/a	0.7/2.3	n/a
synchrotron radiation power [MW]	n/a	9.2	n/a	2.7
bunch length [cm]	5	1.9	1	1
hourglass and crab reduction factor	0.87		0.87	
peak luminosity [10^{34} cm^{-2}s^{-1}]	1.05		2.1	
integrated luminosity/week [fb^{-1}]	4.51		9.0	

16.2: 6D Motion and Tune Modulation

- We have been (understandably) rather naive
  - This is a perfect 1D uncoupled nonlinear model

$$\Delta x' = \frac{B' L}{(B\rho)} x$$

- Reality (aside from noise)
  - Dispersion couples longitudinal and transverse motion
    - Almost always have to bend the beam somewhere
    - Off-center motion in quadrupoles also gives dipole "feed-down"
  - Coupling couples transverse motion
    - Quadrupoles have random rotations relative to design plane
  - Sextupoles are necessary (mostly)
    - Chromaticity correction in large accelerators

- Any coupling adds different frequencies to our system
  - Tune modulation, e.g. $Q_0 = Q_{00} + q \sin(2\pi Q_s t)$

16.2: Tune modulation

- The isolated resonance Kobayashi Hamiltonian was

$$H_n = 2\pi\left(Q_0 - \frac{p}{n}\right)J + 2\pi\xi\, U(J) - 2\pi\xi\, V_n(J)\cos(n\phi)$$

$$Q_0 = Q_{00} + q\,\sin(2\pi Q_s t)$$

- Modulation of the tune looks like a time-dependent driving term
  - Poincare and periodicity asides, large-N-turn maps
- To first order, the phase modulation also appears in the resonance driving term!
  - Phase-modulated pendulum: Mathieu equation
  - Todd's dissertation: http://www.toddsatogata.net/Thesis/
  - Sidebands and sideband overlap leading to chaotic motion

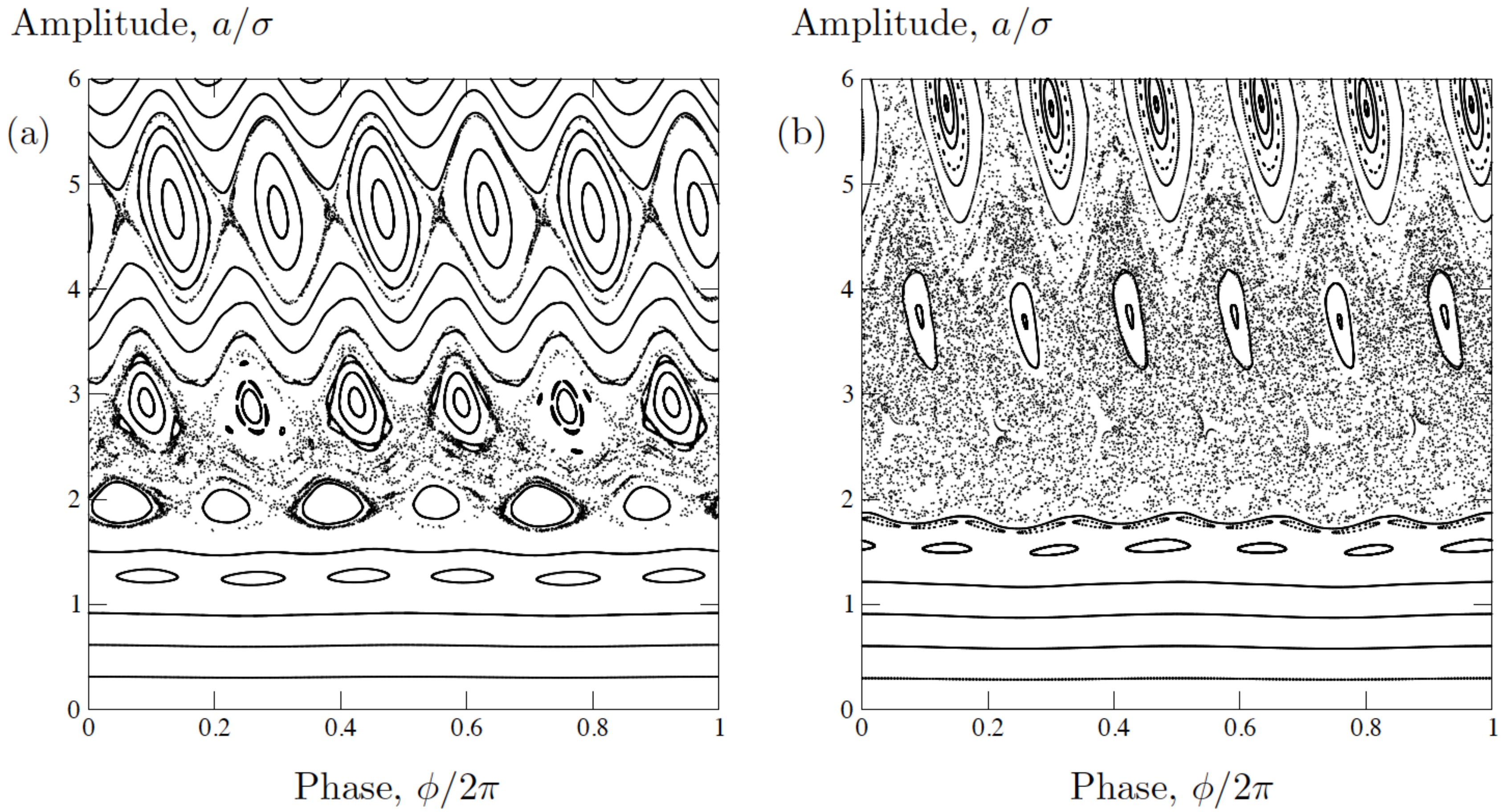


Figure 16.2 Simulated phase space structure due to one round beam-beam kick of strength $\xi = 0.0042$ (a), and $\xi = 0.006$ (b), with the parameters of Equation 16.19. The modest increase in ξ moves the tune modulation side-bands closer together, and dramatically broadens the chaotic sea, allowing

$$Q_0 = Q_{00} + q \sin(2\pi Q_s t)$$ q = 0.001 Qs=0.00515

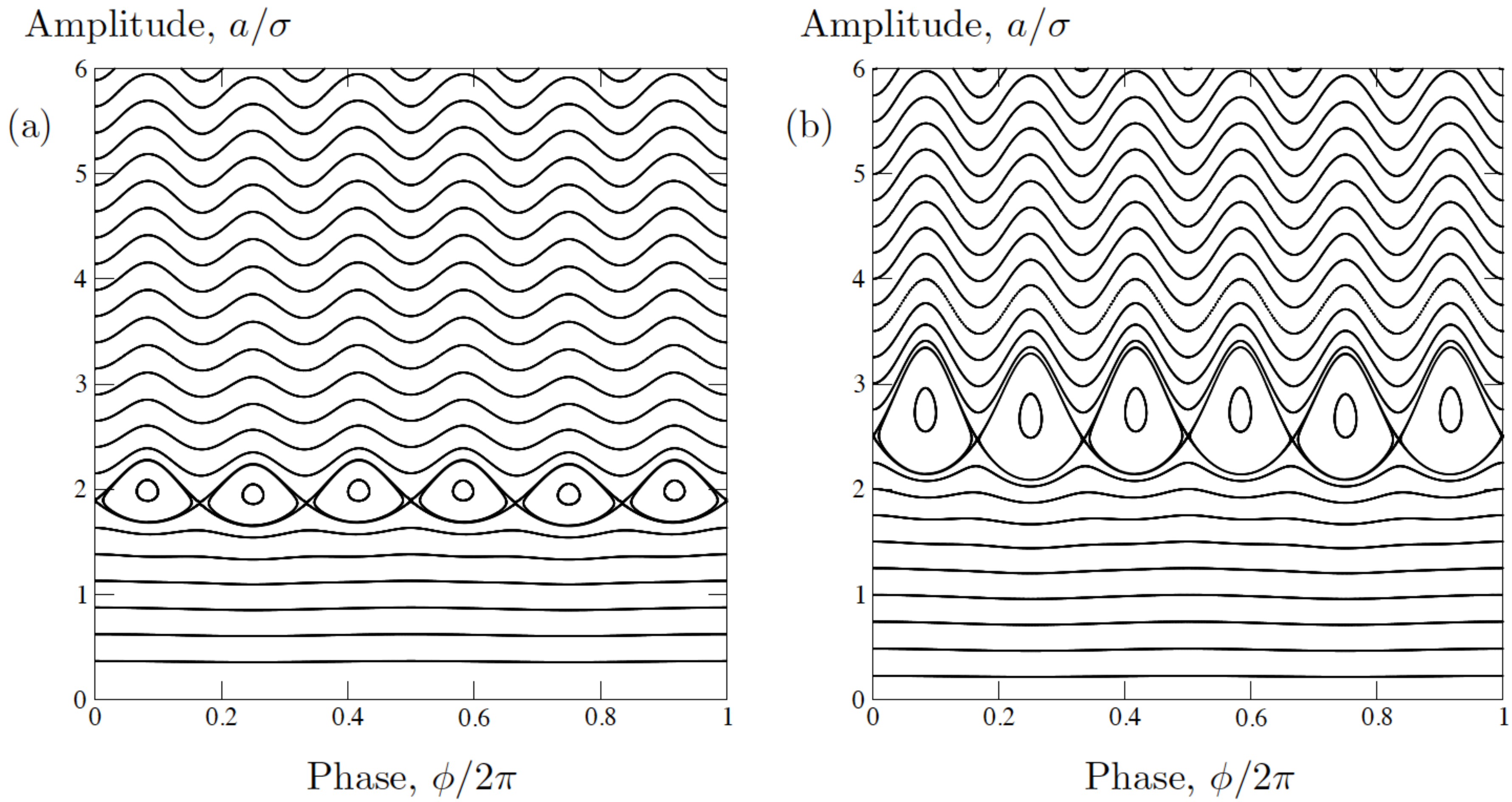
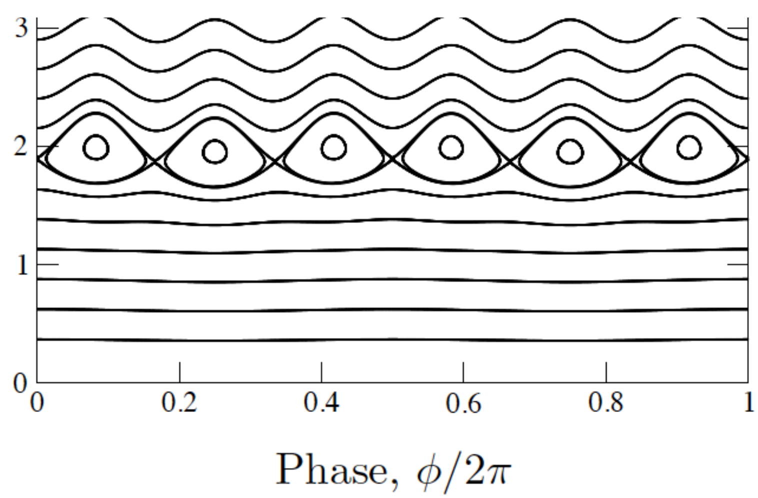
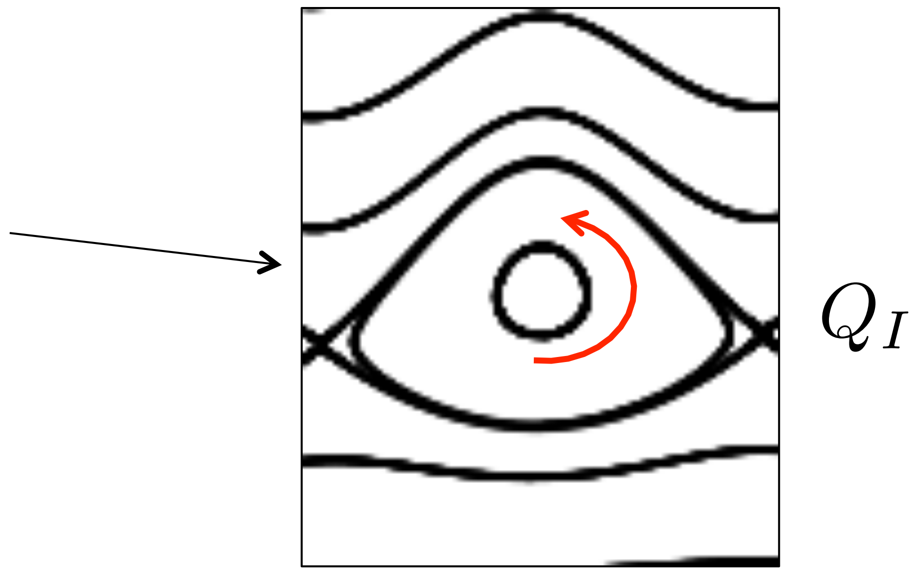


Figure 15.4 Six island chains from the simulation of a single round beam-beam interaction of strength $\xi = 0.0042$ (a), and $\xi = 0.006$ (b), with a base tune of $Q_0 = 0.331$ [40]. The amplitude width of the islands increases as the chain moves to a larger resonance amplitude when ξ is increased. (See also Figure 16.2.)

Tune Modulation Frequency Scale

- Remember, tune modulation is really "just" modulating a pendulum
  - We know kicking around a pendulum near its natural frequency produces excitement
  - What is the natural frequency of resonant motion?
  - Remember these are topologically equivalent to pendula
  - "Island" tune: $Q_I \ll Q_{00}$





Slow Tune Modulation: Amplitude Modulation

$$H_n = 2\pi\left(Q_0 - \frac{p}{n}\right) J + 2\pi\xi\, U(J) - 2\pi\xi\, V_n(J)\cos(n\phi)$$

$$Q_0 = Q_{00} + q\ \sin(2\pi Q_s t)$$

- If modulation frequency is much lower than island tune

$$Q_s \ll Q_I$$

- then the modulation really looks like a slow variation of Q_0
  - Nearly adiabatic with respect to the resonance “island” motion
  - Amplitudes of resonances “breathe” up and down
  - Island widths also vary because their amplitudes are changing
- Conversely, modulation frequency >> island tune...

Tune Modulation Diagram

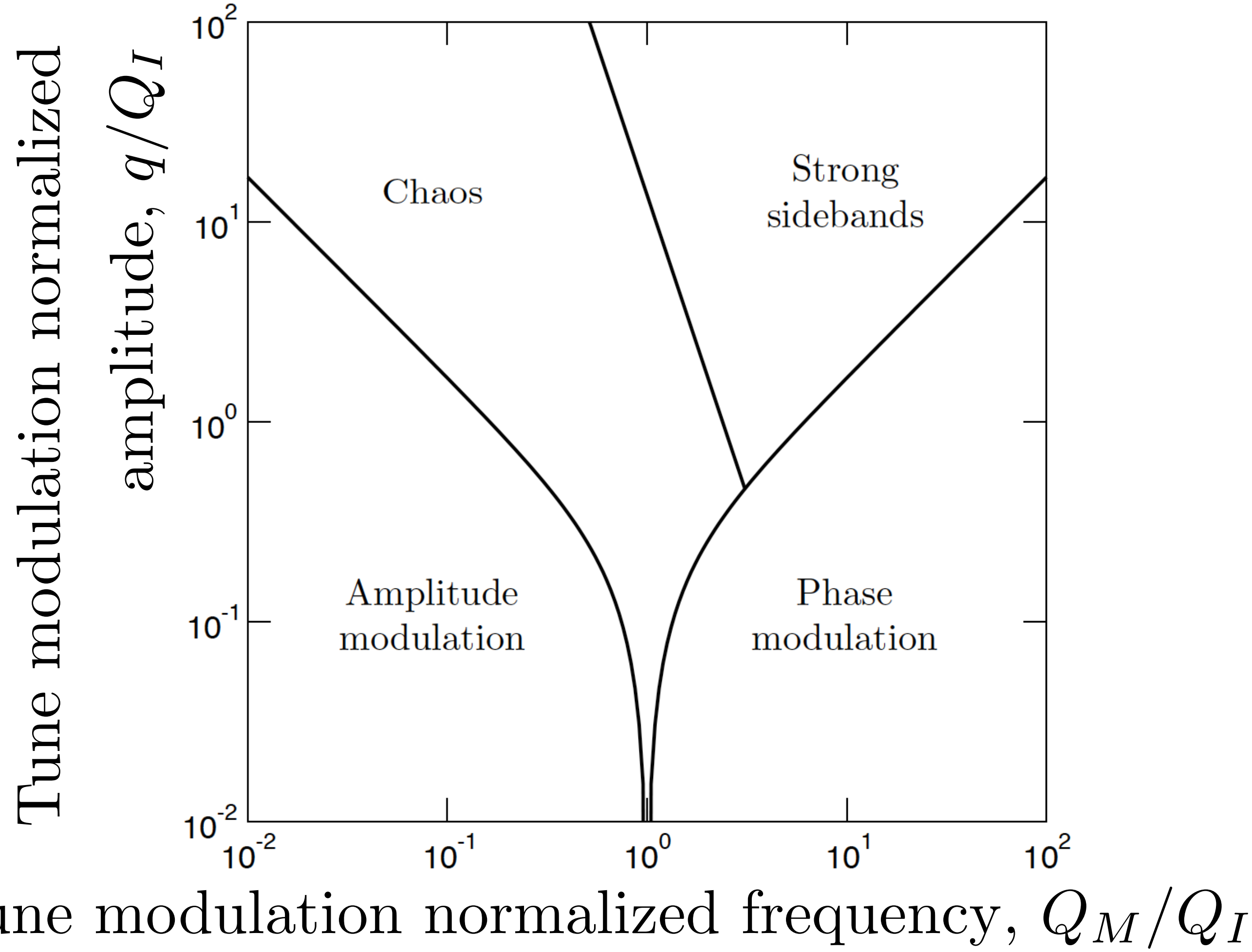


Figure 16.3 Dynamical zones universally predicted in normalised tune modulation space ($q/Q_I, Q_M/Q_I$) for n = 6, with the boundaries defined in Equation 16.23. The island tune Q_I, a scale factor on both axes, is a parameter of central importance.

E778 Persistant Signal

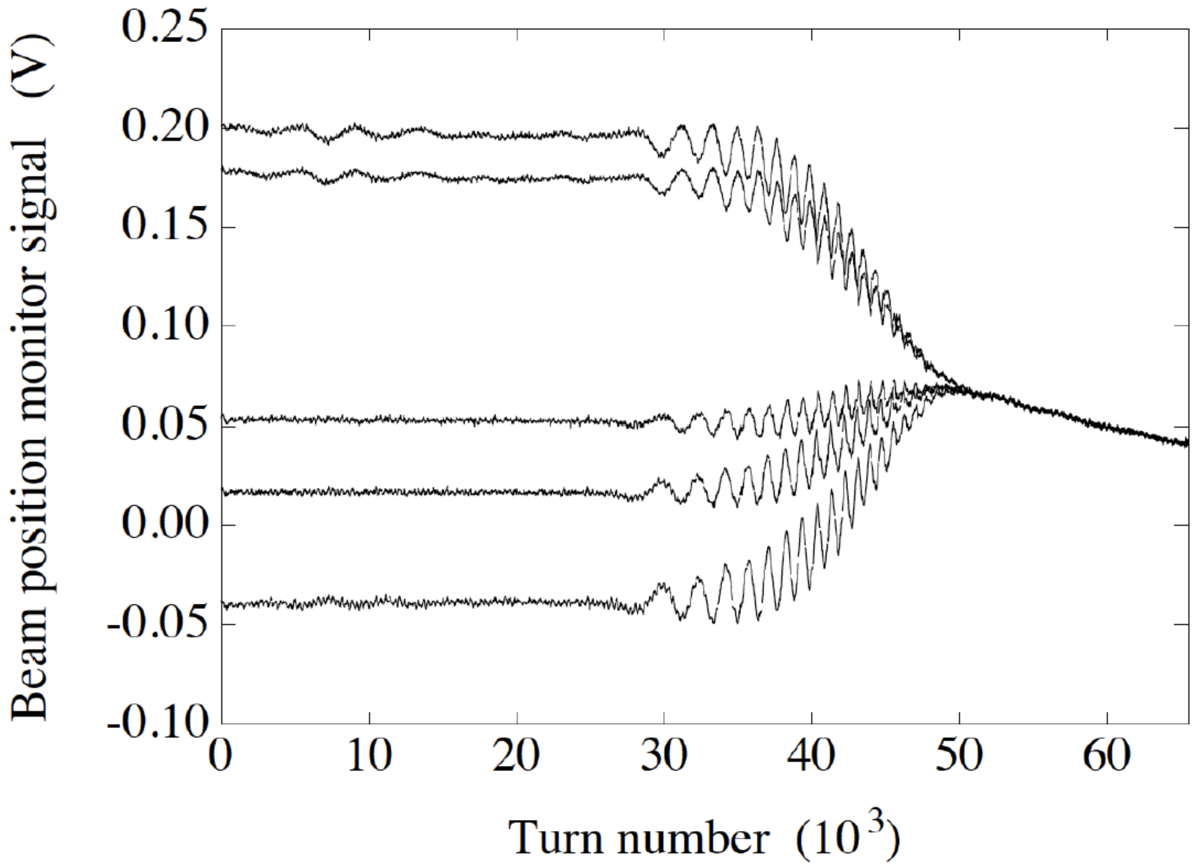
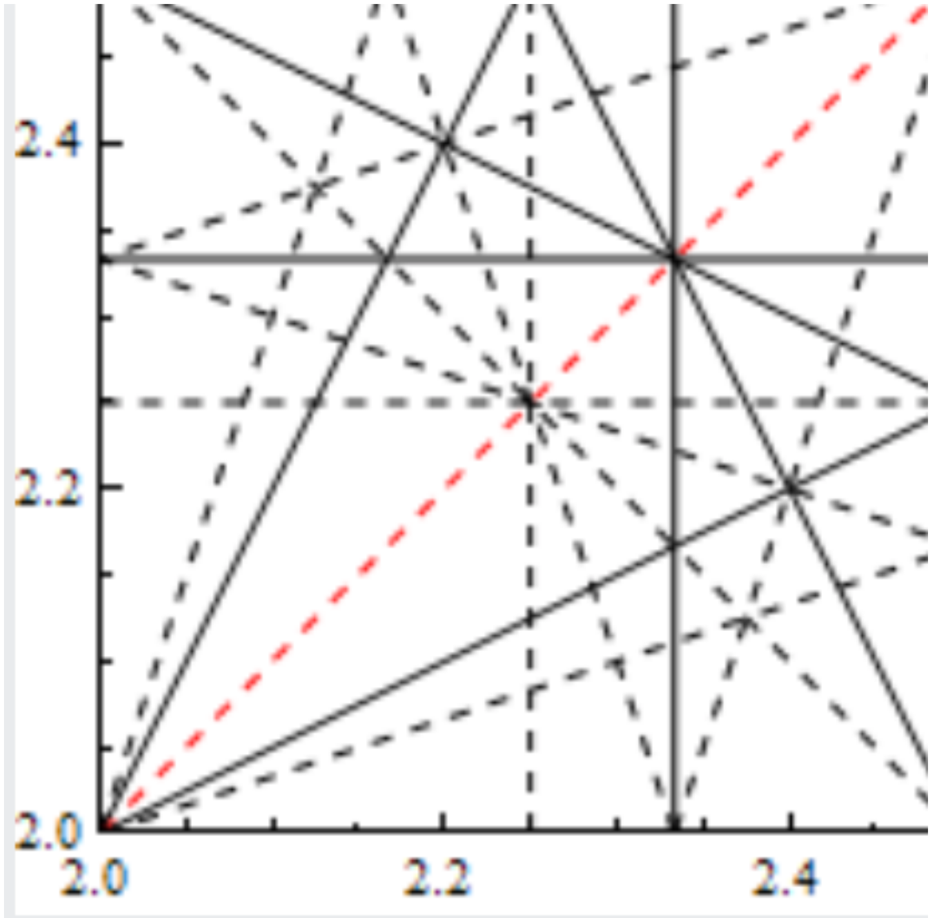


Figure 16.4 Turn-by-turn persistent signal data due to beam trapped in an $n = 5$ resonance island in the nonlinear dynamics Tevatron experiment E778 [47, 48]. The resonance is driven by sextupoles. Chirping the operating point from the *amplitude modulation* zone into the *chaos* zone in Figure 16.3 destroys the resonance, and the persistent signal.

Arnold Diffusion and Integrability

- This is still just 1.5 dimensions!!
  - One dimension plus time
- In more dimensions, particles can move along resonance lines and diffuse in tune space
- A mechanism for long-term amplitude growth in tracking and perhaps even reality
- Visualization makes my brain hurt
- Integrability class next door



Lichtenberg and Lieberman, Jose and Saletan