USPAS Accelerator Physics 2019
Northern Illinois University and UT-Batelle

It's Friday!

Space Charge/Emit Compensation (Todd)
Emittance Measurements/Quad Scans (Todd)
Proton Radiotherapy Medical Accelerators (Steve)
Frequency Map Analysis (Todd, if time)

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http://www.toddsatogata.net/2019-USPAS

Happy birthday to Emilio Segre (1959 Nobel)!

Happy National Baked Alaska Day, National Freedom Day, and Hula in the Coola Day!

(Also Car Insurance Day; please be careful out there)

Transverse Space Charge

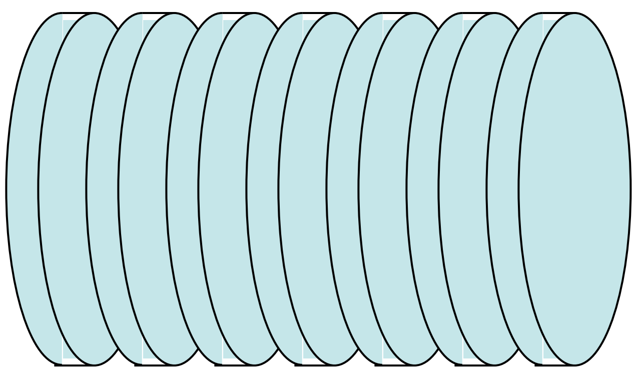
- Space charge is fundamentally Coulomb repulsion of particles within the beam boosted into the lab frame
  - Zero force at center of bunch; grows linearly with transverse r
  - Looks like a quadrupole to first order (just like beam-beam)
  - Force scales as γ^{-2} from boosts
    - electric/magnetic terms cancel at very high boosters/energies

Gaussian

$$F(r) = \frac{Nq^2}{2\pi\epsilon_0 L}\,\frac{1}{\gamma^2}\left[\frac{1-\exp\left(-r^2/2\sigma^2\right)}{r}\right] \approx \frac{q}{4\pi\epsilon_0}\,\frac{r}{\gamma^2}\,\frac{(Nq/L)}{\sigma^2}$$

Linearized/ linear distro

  - Force dependent almost entirely on local current density I/σ^2
  - Assume constant current density in r, slices in z

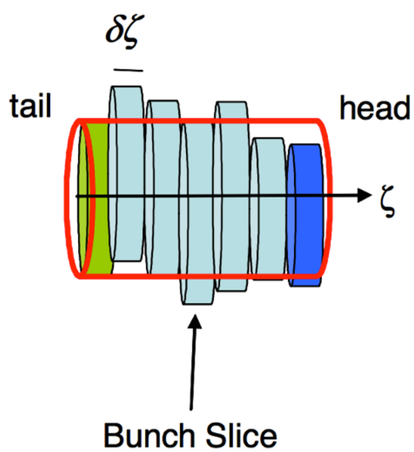


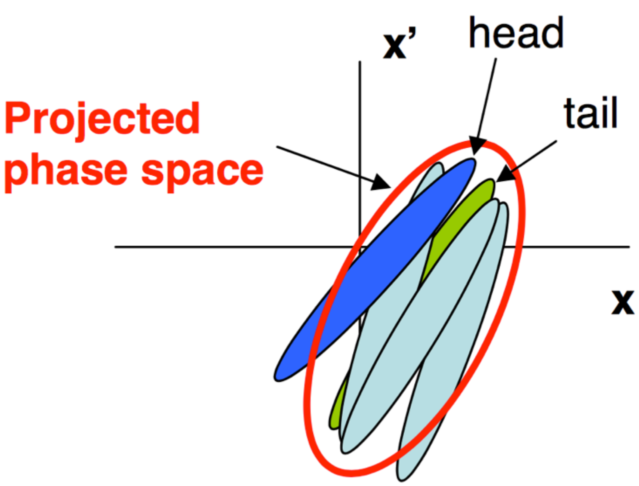
"Beer can distribution"

Current dependence on z!

Emittances

- Slice betatron oscillations dependent on local current I(z)
- Thermal emittance vs projected emittance
  - Thermal emittance: intrinsic emittance of beam from gun
  - Projected emittance: effective emittance combining all slices
- (x,y) rotationally symmetric
  - Focusing at low energies: solenoids
  - Zeroth order: solenoidal field balances space charge
    - Brillouin flow





http://uspas.fnal.gov/materials/10MIT/Lecture5_BeamDynamicsWithSpaceCharge_text.pdf
(David Dowell, 2010 USPAS)

Envelope Equation of Motion

- Equation of motion of beam size oscillations in each slice

$$\sigma'' + K_r\sigma = \frac{I(z)}{2I_A(\beta\gamma)^3\sigma} + \frac{\epsilon_{n,thermal}}{(\beta\gamma)^2\sigma^3} \qquad I_A = \frac{4\pi\varepsilon_0 mc^3}{e} \approx 17\ \text{kA}$$

(the thermal emittance term is struck through on the slide)

solenoid focusing

- Provide solenoid focusing to balance average space charge
  - This is a local equilibrium: expand $\sigma = \sigma_{eq} + \delta\sigma$

$$K_{eq} = \frac{I}{2I_A(\beta\gamma)^3)\sigma_{eq}^2}$$

  - Each slice oscillates around this equilibrium depending on I(z)
- Expand equation of motion and linearize to find

$$\delta\sigma'' + 2K_{eq}\delta\sigma = 0$$

http://uspas.fnal.gov/materials/10MIT/Lecture5_BeamDynamicsWithSpaceCharge_text.pdf
(David Dowell, 2010 USPAS)

Space Charge Compensation

$$\delta\sigma'' + 2K_{eq}\delta\sigma = 0$$

- Beam distribution starts small at photoinjector
  - Projected emittance spoiled by I-dependent space charge
  - But slice emittances individually are still conserved
- Emittance spoiling is an "incoherent" effect created by projection of current variation into projected emittance
  - Slice emittances oscillate with different amplitudes and phases
  - But also move with same **frequency** in linearized approximation
- Therefore there are locations where they realign to locally minimize projected emittance

Some Math

Consider small perturbation of the equilibrium radius for each slice,

$$\sigma(\zeta)=\sigma_{eq}(\zeta)+\delta\sigma(\zeta)$$

with the envelope equation for these perturbations being,

$$\delta\sigma''(\zeta)+2K_{r,eq}\delta\sigma(\zeta)=0\,.$$

The general solution for deviations from the equilibrium radius is,

$$\delta\sigma(z,\zeta)=\frac{\delta\sigma_0'(\zeta)}{\sqrt{2K_r}}\sin\sqrt{2K_r}\,z+\delta\sigma_0(\zeta)\cos\sqrt{2K_r}\,z\,.$$

Setting the initial slice angle to zero and defining the initial radial deviation as,

$$\delta\sigma_0(\zeta)=\sigma_0-\sigma_{eq}(\zeta) \qquad \text{Eqn. 7}$$

results in an oscillating slice rms size as they propagate along the channel [Rosenzweig&Serafini],

$$\sigma(z,\zeta)=\sigma_{eq}(\zeta)+\left[\sigma_0-\sigma_{eq}(\zeta)\right]\cos\sqrt{2K_r}\,z\,. \qquad \text{Eqn. 8}$$

This solution results in an emittance which oscillates $\pi/2$ out of phase with the beam size,

$$\varepsilon(z)=\frac{1}{2}\sqrt{K_r}\,\sigma_0\sigma_{eq}(I_p)\frac{\delta I_{rms}}{I_p}\left|\sin\sqrt{2K_r}\,z\right| \;, \qquad \text{Eqn. 9}$$

where δI_{rms} is the rms current along the ζ-coordinate.

http://uspas.fnal.gov/materials/10MIT/Lecture5_BeamDynamicsWithSpaceCharge_text.pdf
(David Dowell, 2010 USPAS)

Putting It Together

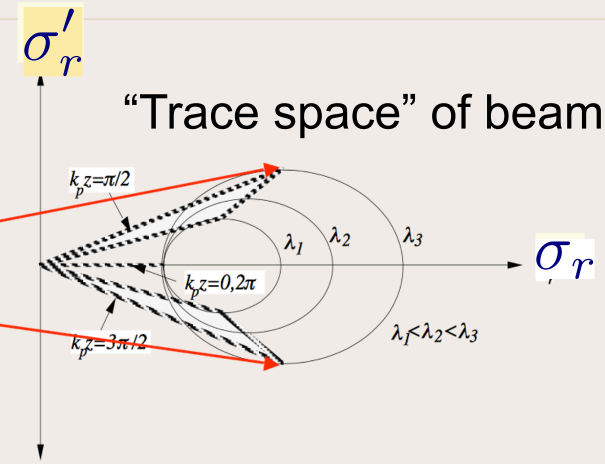
- Emittance (area in phase space) is maximized at

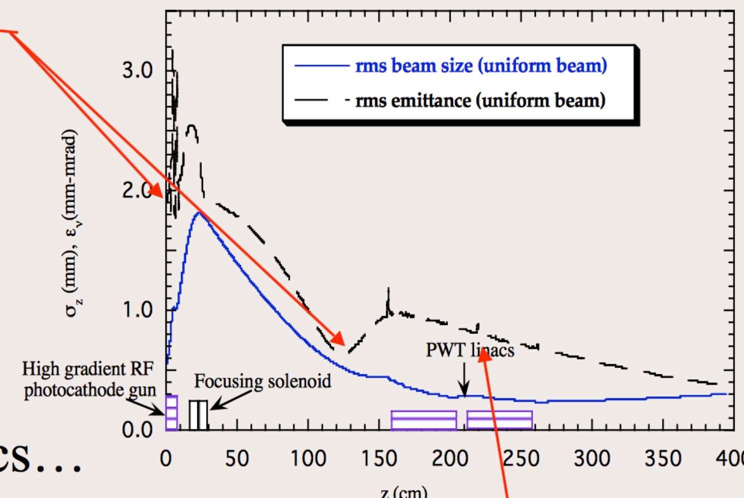
  $k_p z = \pi/2, 3\pi/2$
- Emittance is locally minimized at

  $k_p z = 0, \pi, 2\pi$

- the beam extrema!

- Fairly good agreement of simple model with much more complex beamline
- What about acceleration?
  - In the rf gun, in booster linacs…





Damping of ε…

http://pbpl.physics.ucla.edu/Education/Schools/USPAS_2004/Lecture4.pdf

(Jamie Rosenzweig, 2004 USPAS)

Emittance Measurements: Quad Scans

- We can measure beam profiles and their projections
  - Observables: σ_x, σ_y
  - But $\sigma = \sqrt{\beta\epsilon}$
  - How do we disentangle these to measure beam “beta” and beam emittance?
- One of the most common methods is a **quadrupole scan**
  - Vary focusing K of single quadrupole
  - Measure beam size at point downstream (a “harp”)
  - Matrix from thin quadrupole to harp is M
  - Then total matrix including harp and quadrupole is

$$M_{\mathrm{q}} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} \quad M_{\mathrm{tot}} = M_{\mathrm{q2h}} M_{\mathrm{q}} \models \begin{pmatrix} M_{11} - K M_{12} & M_{12} \\ M_{21} - K M_{22} & M_{22} \end{pmatrix}$$

See Minty and Zimmermann textbook...

Emittance Measurements: Quad Scans

$$M_{\rm tot} = M_{\rm q2h} M_{\rm q} \models \begin{pmatrix} M_{11} - K M_{12} & M_{12} \\ M_{21} - K M_{22} & M_{22} \end{pmatrix}$$

- How does the beam size propagate? Sigma matrix:

$$\Sigma_{\rm harp} = M_{\rm tot} \Sigma_0 M_{\rm tot}^{\rm T}$$

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x x' \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & \alpha \\ -\alpha & -\gamma \end{pmatrix} \qquad \det \Sigma = \epsilon^2$$

- Expand via tedious algebra to find

$$\begin{aligned} \Sigma_{11} = \langle x^2 \rangle \quad &= \quad (M_{11}^2 \Sigma_{11_0} + 2 M_{11} M_{12} \Sigma_{12_0} + M_{12}^2 \Sigma_{22_0}) \\ & \quad\; -2(M_{11} M_{12} \Sigma_{11_0} + M_{12}^2 \Sigma_{12_0}) K + M_{12}^2 \Sigma_{11_0} K^2 \end{aligned}$$

Emittance Measurements

$$\Sigma_{11} = \langle x^2 \rangle = (M_{11}^2 \Sigma_{11_0} + 2M_{11}M_{12}\Sigma_{12_0} + M_{12}^2 \Sigma_{22_0}) - 2(M_{11}M_{12}\Sigma_{11_0} + M_{12}^2 \Sigma_{12_0})K + M_{12}^2 \Sigma_{11_0} K^2$$

- This is a quadratic relating an observable $<x^2>=\sigma^2$ to a control knob K (quadrupole strength)
- Plot measured σ^2 vs set K and use model M_{ij}
- Usually M is a drift matrix and so is known reasonably well

- "Convenient" parameterization of above quadratic

$$\Sigma_{11} = A(K-B)^2 + C = AK^2 - 2ABK + (C + AB^2)$$

$$\begin{aligned}
\Sigma_{11_0} &= \frac{A}{M_{12}^2} \\
\Sigma_{12_0} &= -\frac{A}{M_{12}^2}\left(-B + \frac{M_{11}}{M_{12}}\right) \\
\Sigma_{22_0} &= \frac{1}{M_{12}^2}\left[(AB^2 + C) - 2AB\left(\frac{M_{11}}{M_{12}}\right) + A\left(\frac{M_{11}}{M_{12}}\right)^2\right]
\end{aligned}$$

Some More Math

$$\Sigma_{11_0} = \frac{A}{M_{12}^2}$$
$$\Sigma_{12_0} = -\frac{A}{M_{12}^2}\left(-B + \frac{M_{11}}{M_{12}}\right)$$
$$\Sigma_{22_0} = \frac{1}{M_{12}^2}\left[(AB^2 + C) - 2AB\left(\frac{M_{11}}{M_{12}}\right) + A\left(\frac{M_{11}}{M_{12}}\right)^2\right]$$

- But remember sigma matrix elements are related to beam alphas, betas, and emittances

$$\beta_0 = \frac{\Sigma_{11_0}}{\epsilon_0} = \sqrt{\frac{A}{C}} \qquad \epsilon_0^2 = \det\Sigma_0 = \frac{AC}{M_{12}^4} \qquad \Rightarrow \qquad \epsilon_0 = \frac{\sqrt{AC}}{M_{12}^2}$$
$$\alpha_0 = -\frac{\Sigma_{12_0}}{\epsilon_0} = \sqrt{\frac{A}{C}}\left(-B + \frac{M_{11}}{M_{12}}\right)$$
$$\gamma_0 = \frac{\Sigma_{22_0}}{\epsilon_0} = \frac{1}{\sqrt{AC}}\left[(AB^2 + C) - 2AB\left(\frac{M_{11}}{M_{12}}\right) + A\left(\frac{M_{11}}{M_{12}}\right)^2\right]$$

Systematic Beam Size Scale Errors

4.4.1 Systematic beam size measurement scale errors

What happens when there is a systematic scale error in the beam size measurements at the wire scanner? Say that all beam size measurements are multiplied by a constant multiplicative factor ξ_σ, where $\xi_\sigma = 1$ is equivalent to no scale error. This is equivalent to multiplying both sides of equation 4.16 by ξ_σ^2:

$$\xi_\sigma^2 \Sigma_{11} = \xi_\sigma^2 A(K-B)^2 + \xi_\sigma^2 C = A_{\rm new}(K-B)^2 + C_{\rm new} \qquad (4.24)$$

where $A_{\rm new} = \xi_\sigma^2 A$ and $C_{\rm new} = \xi_\sigma^2 C$.

The new emittance is found by using the new fit parameters $A_{\rm new}$ and $C_{\rm new}$:

$$\epsilon_{0,\rm new} = \frac{\sqrt{A_{\rm new} C_{\rm new}}}{M_{12}^2} = \xi_\sigma^2 \epsilon_0 \qquad (4.25)$$

Note that the Twiss parameters β_0, α_0, and γ_0 do not change since they are all dependent on only ratios of A/C. Thus **systematic scale errors of measured beam size only affect the measured emittance, and not the measured Twiss parameters.** This is a mildly surprising result.

(Tech note I wrote for a summer student... ask for a copy if you want!)

4.4.2 Systematic quadrupole strength scale errors

What happens when there is a systematic scale error in the strength of the quadrupole that is being changed for this measurement? Say that all quadrupole strengths K are multiplied by a constant multiplicative factor ξ_K, where $\xi_K = 1$ is equivalent to no scale error. This is equivalent to changing equation 4.16 to:

$$\begin{aligned}
\Sigma_{11} &= A(\xi_K K - B)^2 + C \\
&= \xi_K^2 A(K - \xi_K^{-1} B)^2 + C \\
&= A_{\text{new}}(K - B_{\text{new}})^2 + C
\end{aligned}$$

where $A_{\text{new}} = \xi_K^2 A$ and $B_{\text{new}} = \xi_K^{-1} B$.

We can then calculate the new emittances and twiss parameters from equations 4.20 to 4.22:

$$\epsilon_{0,\text{new}} = \frac{\sqrt{A_{\text{new}} C}}{M_{12}^2} = \xi_K \epsilon_0 \tag{4.26}$$

$$\beta_{0,\text{new}} = \sqrt{\frac{A_{\text{new}}}{C}} = \xi_K \beta_0 \tag{4.27}$$

$$\alpha_{0,\text{new}} = \sqrt{\frac{A_{\text{new}}}{C}} \left(-B_{\text{new}} + \frac{M_{11}}{M_{12}}\right) = \sqrt{\frac{A}{C}} \left(-B + \frac{\xi_K M_{11}}{M_{12}}\right) \tag{4.28}$$

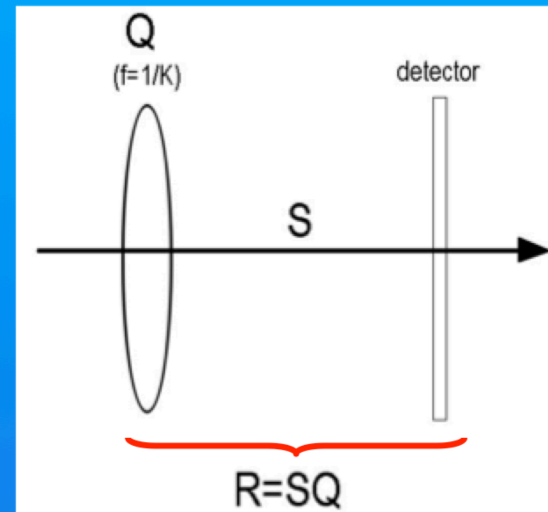
$$= \alpha_0 + \epsilon_0 \left(\frac{M_{11}}{M_{12}}\right)(\xi_K - 1) \tag{4.29}$$

Thus **systematic scale errors of the scanning quadrupole strength are directly translated to scale errors in the measured emittance and beta function**. The α_0 (and $\gamma_0 = (1+\alpha_0^2)/\beta_0$) dependences are a little more complicated.

Measurement of the Transverse Beam Emittance

Method I: quadrupole scan

Principle: with a well-centered beam, measure the beam size as a function of the quadrupole field strength



Here
Q is the transfer matrix of the quadrupole
R is the transfer matrix between the quadrupole and the beam size detector

With $Q = \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$ then $R = \begin{pmatrix} S_{11} + KS_{12} & S_{12} \\ S_{21} + KS_{22} & S_{22} \end{pmatrix}$ with $\Sigma_{\text{beam}} = R\Sigma_{\text{beam},0}R^t$

The (11)-element of the beam transfer matrix is found after algebra to be:

$$\Sigma_{11}(= \langle x^2 \rangle) = ({S_{11}}^2 \Sigma_{11_0} + 2S_{11}S_{12}\Sigma_{12_0} + {S_{12}}^2\Sigma_{22_0}) + (2S_{11}S_{12}\Sigma_{11_0} + 2{S_{12}}^2\Sigma_{12_0})K + {S_{12}}^2\Sigma_{11_0}K^2$$

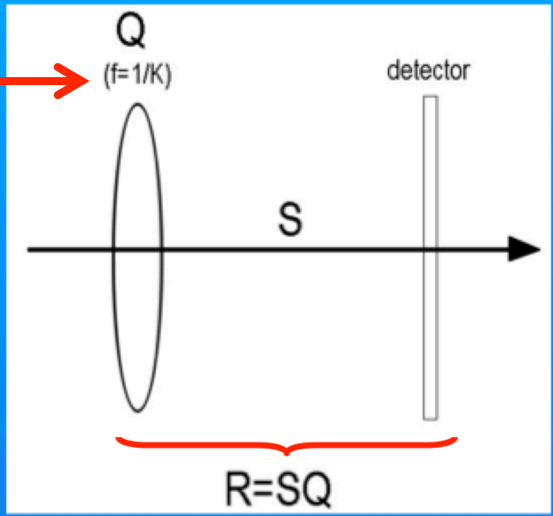
which is quadratic in the field strength, K

M. Minty: Many USPAS's

Measurement of the Transverse Beam Emittance

Method I: quadrupole scan

WARNING! DANGER!!!



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which is quadratic in the field strength, K

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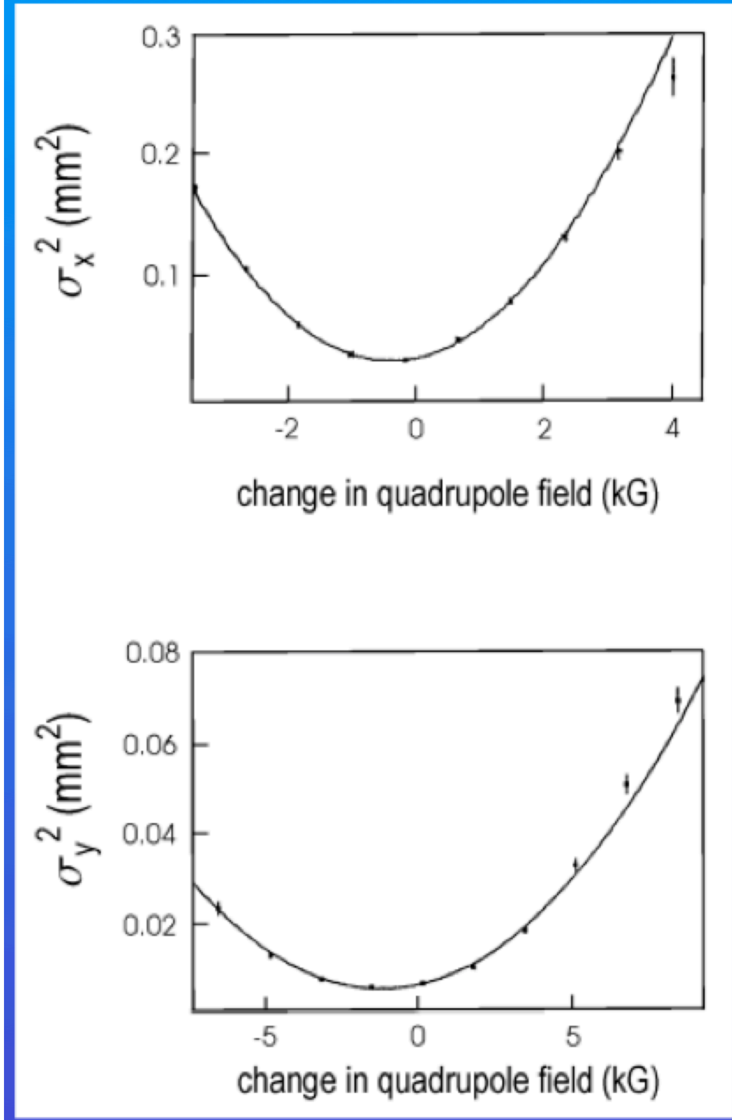
Measurement: measure beam size versus quadrupole field strength

recall:

$$\Sigma_{11}(=\langle x^2\rangle) = ({S_{11}}^2\Sigma_{11_0} + 2S_{11}S_{12}\Sigma_{12_0} + {S_{12}}^2\Sigma_{22_0}) + (2S_{11}S_{12}\Sigma_{11_0} + 2{S_{12}}^2\Sigma_{12_0})K + {S_{12}}^2\Sigma_{11_0}K^2$$

data:



fitting function (parabolic):

$$\Sigma_{11} = A(K-B)^2 + C = AK^2 - 2ABK + (C + AB^2)$$

equating terms (drop subscripts ‘o’),

$$A = S_{12}^2\Sigma_{11}\,,$$

$$-2AB = 2S_{11}S_{12}\Sigma_{11} + 2S_{12}^2\Sigma_{12}\,,$$

$$C + AB^2 = {S_{11}}^2\Sigma_{11} + 2S_{11}S_{12}\Sigma_{12} + {S_{12}}^2\Sigma_{22}$$

solving for the beam matrix elements:

$$\Sigma_{11} = A/{S_{12}}^2\,,$$

$$\Sigma_{12} = -\frac{A}{{S_{12}}^2}\left(B + \frac{S_{11}}{S_{12}}\right)\,,$$

$$\Sigma_{22} = \frac{1}{{S_{12}}^2}\left[(AB^2 + C) + 2AB\left(\frac{S_{11}}{S_{12}}\right) + A\left(\frac{S_{11}}{S_{12}}\right)^2\right]$$

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The emittance is given from the determinant of the beam matrix:

$$\epsilon_x = \sqrt{\det\ \Sigma^x_{\text{beam}}}$$

$$\begin{aligned}\det\ \Sigma^x_{\text{beam}} &= \Sigma_{11}\Sigma_{22} - {\Sigma_{12}}^2 \\ &= AC/S_{12}^4\,,\end{aligned}$$

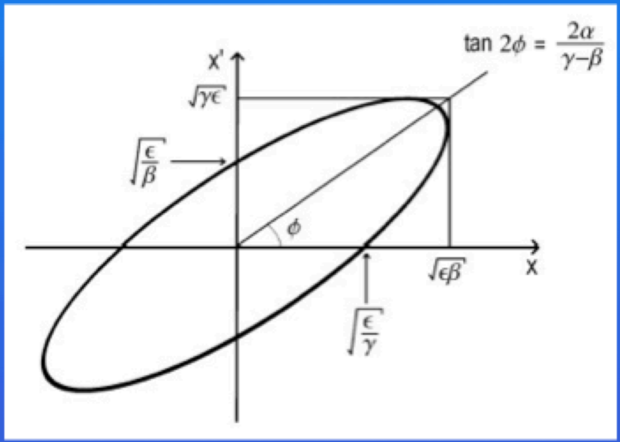
$$\Rightarrow \quad \epsilon_x = \sqrt{AC}/S_{12}^2$$

With these 3 fit parameters (A,B, and C), the 3 Twiss parameters are also known:

$$\beta_x = \frac{\Sigma_{11}}{\epsilon} = \sqrt{\frac{A}{C}}\,,$$

$$\alpha_x = -\frac{\Sigma_{12}}{\epsilon} = \sqrt{\frac{A}{C}}\left(B + \frac{S_{11}}{S_{12}}\right)\,,$$

$$\gamma_x = \frac{{S_{12}}^2}{\sqrt{AC}}\left[(AB^2 + C) + 2AB\left(\frac{S_{11}}{S_{12}}\right) + A\left(\frac{S_{11}}{S_{12}}\right)^2\right]$$



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Method II: fixed optics, measure beam size using multiple measurement devices

Recall: the matrix used to transport the Twiss parameters:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_f = \begin{pmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{pmatrix}_{fi} \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_i$$

with fixed optics and multiple measurements of ? at different locations:

$$\begin{pmatrix} (\sigma_x{}^{(1)})^2 \\ (\sigma_x{}^{(2)})^2 \\ (\sigma_x{}^{(3)})^2 \\ \cdots \\ (\sigma_x{}^{n})^2 \end{pmatrix} = \begin{pmatrix} (R_{11}{}^{(1)})^2 & 2R_{11}^{(1)}R_{12}^{(1)} & (R_{12}{}^{(1)})^2 \\ (R_{11}{}^{(2)})^2 & 2R_{11}^{(2)}R_{12}^{(2)} & (R_{12}{}^{(2)})^2 \\ (R_{11}{}^{(3)})^2 & 2R_{11}^{(3)}R_{12}^{(3)} & (R_{12}{}^{(3)})^2 \\ \cdots & & \\ (R_{11}{}^{(n)})^2 & R_{11}^{(n)}R_{12}^{(n)} & (R_{12}{}^{(n)})^2 \end{pmatrix} \begin{pmatrix} \beta(s_0)\epsilon \\ -\alpha(s_0)\epsilon \\ \gamma(s_0)\epsilon \end{pmatrix}$$

simplify notation: ?$_x$ B o

$$\Sigma_x = \mathbf{B}\cdot\mathbf{o}$$

goal is to determine the vector o by minimizing the sum (least squares fit):

$$\chi^2 = \sum_{l=1}^{n} \frac{1}{\sigma^2_{\Sigma_x^{(l)}}} \left(\Sigma_x^{(l)} - \sum_{i=1}^{3} B_{li} o_i \right)^2$$

with the symmetric n?n covariance matrix, $\mathbf{T} = (\hat{\mathbf{B}}^t \cdot \hat{\mathbf{B}})^{-1}$

➡ the least-squares solution is $\mathbf{o} = \mathbf{T}\cdot\hat{\mathbf{B}}^t\cdot\hat{\Sigma}_x$

(the 'hats' show weighting: $\hat{B}_{li} = \dfrac{B_{li}}{\sigma_{\Sigma_x^{(l)}}}$ $\hat{\Sigma}_x^{(l)} = \dfrac{\Sigma_x^{(l)}}{\sigma_{\Sigma_x^{(l)}}}$)

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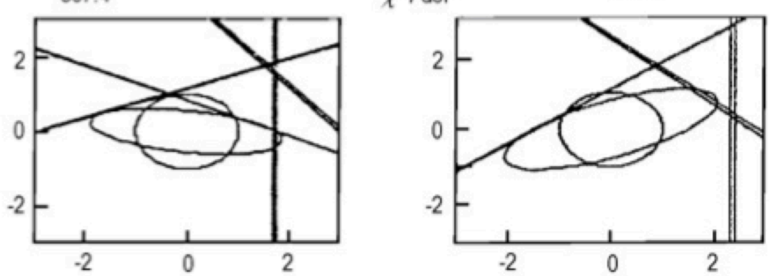
the components of o are known,

$$\epsilon = \sqrt{o_1 o_3 - o_2^2}\,,$$
$$\beta = o_1/\epsilon\,, \text{ and}$$
$$\alpha = -o_2/\epsilon\,.$$

graphical representation of results:

LI01 x-plane elec			LI01 y-plane elec	
20.88 +/- 0.99	(20.00)	$\gamma\epsilon$ (10^{-5} m-r)	32.29 +/- 2.67	(20.00)
39.00 +/- 1.02	(20.00)	$B_{mag}\gamma\epsilon$ (10^{-5} m-r)	54.23 +/- 1.32	(20.00)
1.87 +/- 0.11	(1.00)	B_{mag}	1.68 +/- 0.17	(1.00)
28.01 +/- 1.93	(8.28)	β (m)	15.90 +/- 2.27	(6.00)
-3.05 +/- 0.23	(-1.03)	α	2.06 +/- 0.39	(1.13)
1463.90 +/- 29.29	(843.17)	σ_1 (μm)	1893.29 +/- 33.89	(718.04)
1439.99 +/- 28.80	(620.33)	σ_2 (μm)	1514.39 +/- 30.29	(818.45)
888.05 +/- 13.75	(915.43)	σ_3 (μm)	2052.81 +/- 41.06	(545.70)
778.71 +/- 15.57	(730.80)	σ_4 (μm)	632.62 +/- 12.65	(731.27)
3.68 +/- 0.24		intensity (10^{10} ppb)	3.43 +/- 0.36	
307.4		χ^2 / dof	578.7	



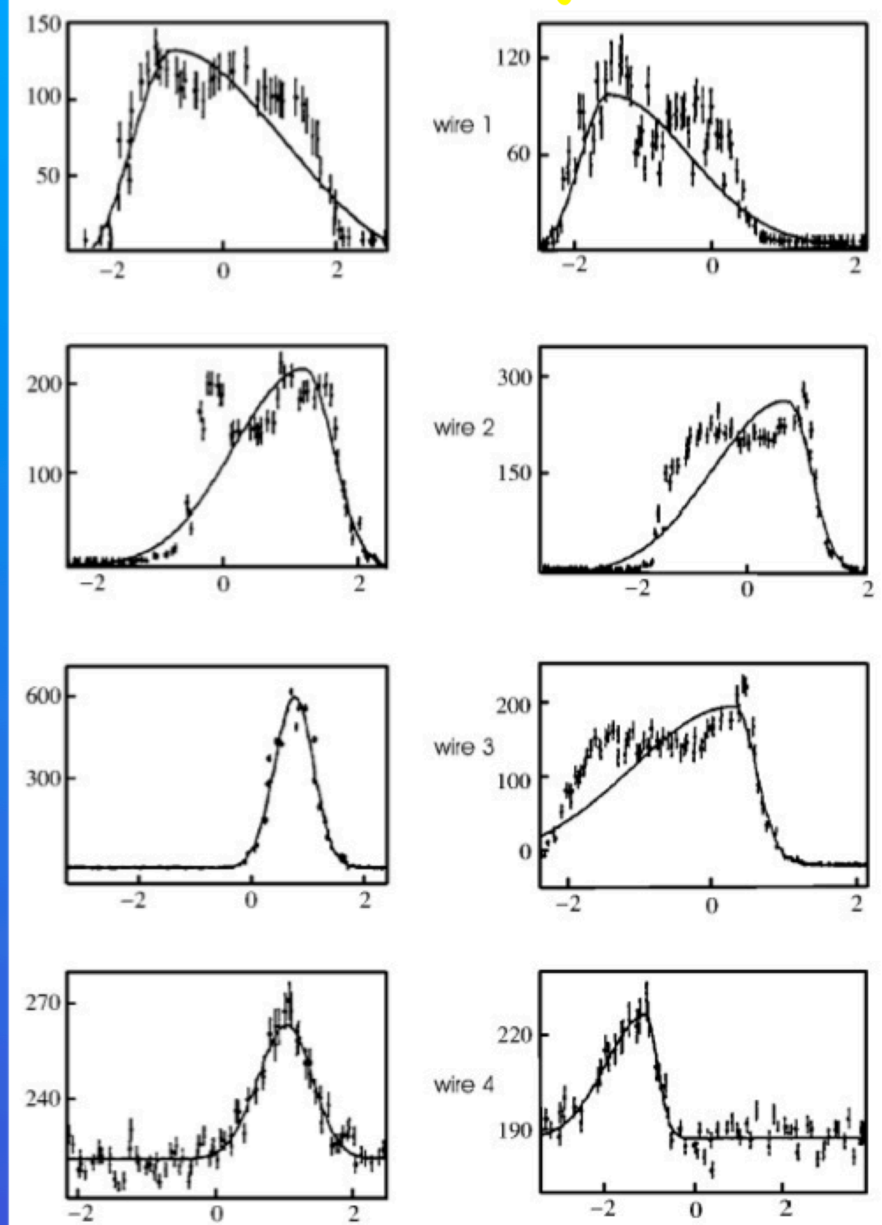
note: coordinate axes are so normalized (design phase ellipse is a circle):

$$\left(\frac{x}{\sqrt{\beta_x}},\ \frac{\alpha_x x + \beta_x x'}{\sqrt{\beta_x}}\right)$$

lines show phase space coverage of wires:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{ref point}} = R^{-1}\begin{pmatrix} \sigma_{x,w} \\ x'_w \end{pmatrix}$$

data: from the SLC injector linac



With methods I & II, the beam sizes may be measured using e.g. screens or wires

(Proton Radiotherapy: Steve)