USPAS Accelerator Physics 2019
Northern Illinois University and UT-Batelle

Strong Focusing Transverse Optics

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Happy Birthday to Guy Fieri, Francis Bacon, Wilbur Scoville, Lord Byron, and Lev Landau!
Happy National Hot Sauce Day, Answer Your Cat's Questions Day, and International Sweatpants Day!

Refresher and Review: Section 3.1

LINEAR STABILITY

ONE-TURN MATRIX

$$M = M_{m,m-1} \cdots M_{21} M_{10}$$

Matrices M_{ij} are block diagonal, so horizontal stability becomes a 2 × 2 problem

(A) $$\boxed{\bar{X}_n = M^n \bar{X}_0}$$ $$\bar{X}_0 = \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

M has 2 complex eigenvectors, $\bar{v}_1$ and $\bar{v}_2$ (B)

that solve the equation $$\boxed{M\bar{v} = \lambda \bar{v}}$$

EIGENVALUE complex scalar

Peggs & Satogata Introduction to Accelerator Dynamics 11

Refresher and Review: Section 3.1

If motion IS stable, then the ONE-TURN matrix IS written in general as

$$M(s) = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}$$

where β, α & γ are "Twiss" or "Courant-Snyder" parameters satisfy

$$\gamma \equiv \frac{1+\alpha^2}{\beta}$$

$$\mu_x \equiv 2\pi Q_x$$

α, β, γ are functions of s.... but NOT μ

Interesting Observations

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Describes **single particle** and **centroid** linear motion

Twiss parameters should have x subscripts, e.g. $\mu_{(x)} = 2\pi Q_x$

- μ is independent of s: this is the **betatron phase advance** of this periodic system
- Determinant of matrix M is 1!
- Looks like a rotation and some scaling
- M can be written down in a **beautiful** and **deep** way

$$M = I\cos\mu + J\sin\mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu} \qquad M^n = e^{J(s)\,(\mu n)}$$

Convenient Calculations

- If we know the transport matrix M, we can find the lattice parameters (periodic in C)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance per cell} \quad \cos\mu = \frac{1}{2}\,\mathrm{Tr}\,M$$

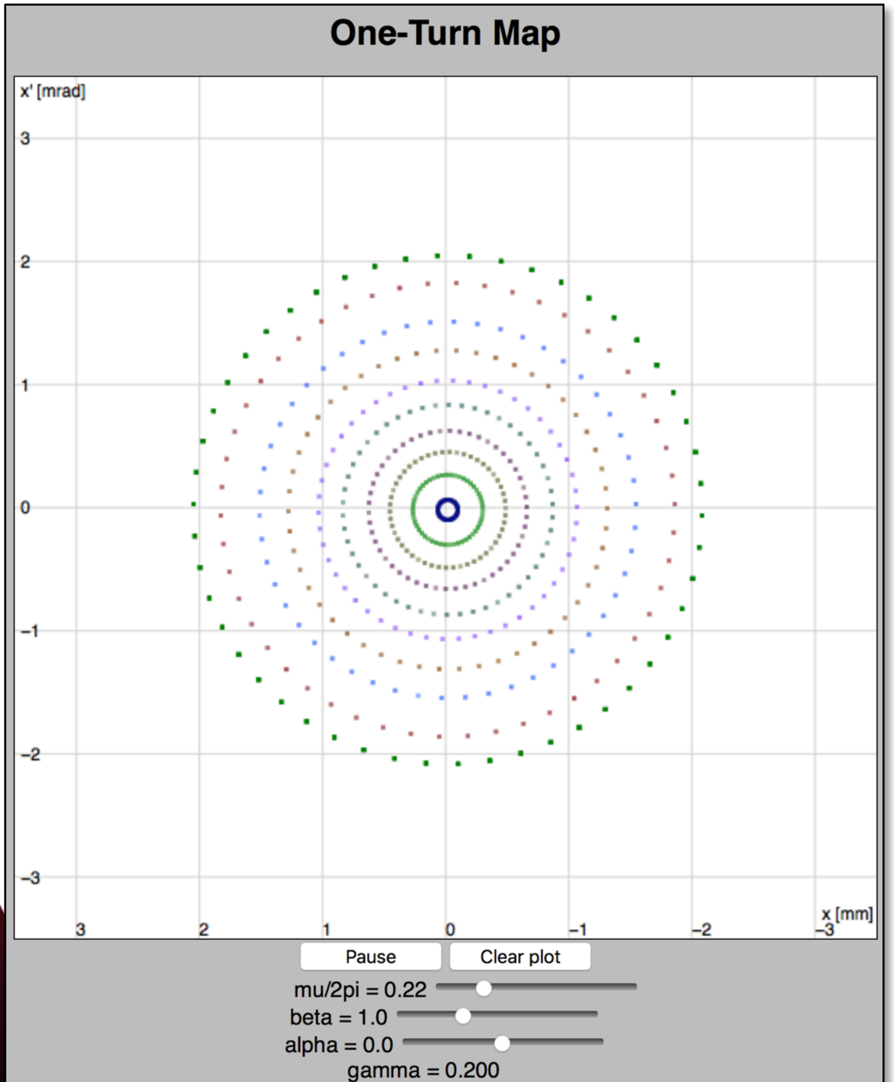
$$\beta(0) = \beta(C) = \frac{m_{12}}{\sin\mu}$$

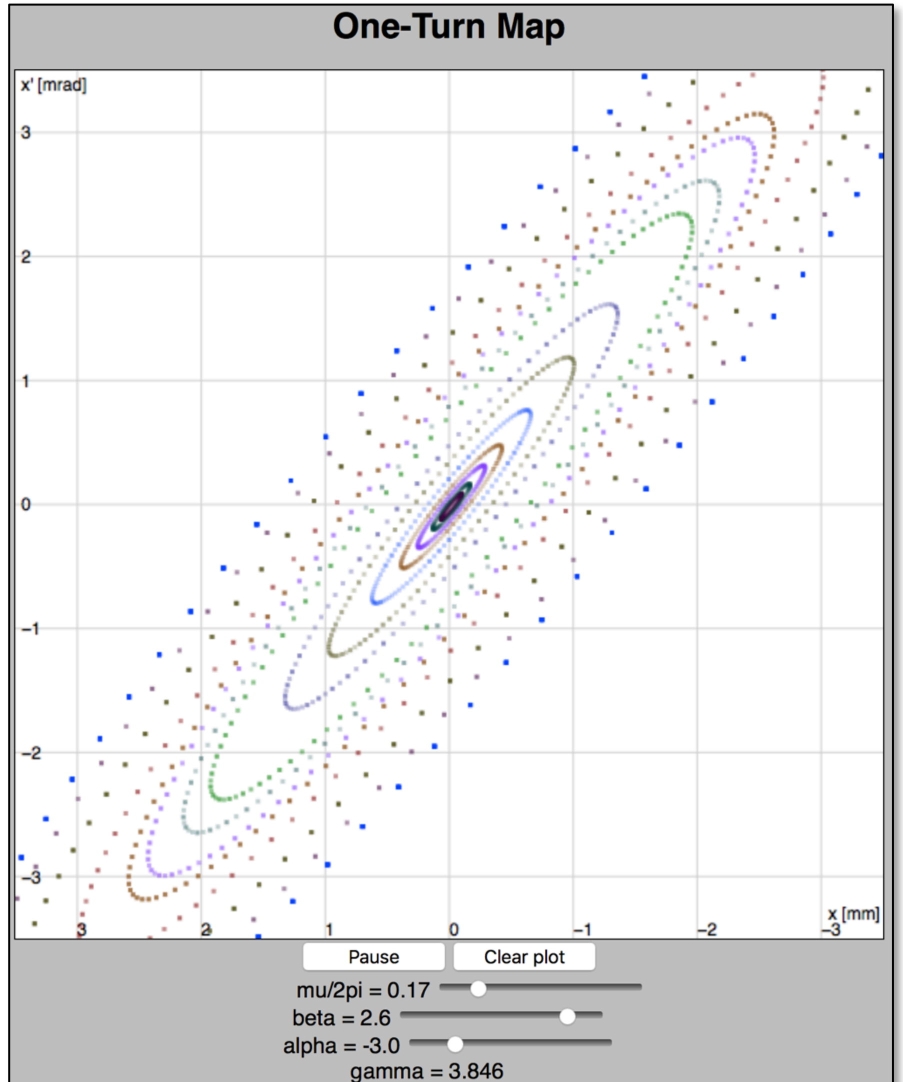
$$\alpha(0) = \alpha(C) = \frac{m_{11} - \cos\mu}{\sin\mu}$$

$$\gamma(0) \equiv \frac{1+\alpha^2(0)}{\beta(0)}$$

One Turn Map

http://www.toddsatogata.net/2019-USPAS/OneTurnMap.html





3.2 Floquet Transformation

- All these one-turn (or one-period) maps are points around concentric ellipses
- There must be a **transformation** that separates out the **ellipse parameters** and the **circular rotation**
- This is called the **Floquet Transformation**

$$M = T^{-1} R T$$

$$R(\mu) = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

$$T = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix}$$

3.2: Normalized Phase Space

$$M = T^{-1} R T$$

$$\vec{x}_1 = M \vec{x}_0 = T^{-1} R T \vec{x}_0$$

$$(T\,\vec{x}_1) = R(T\,\vec{x}_0)$$

- If we define new coordinates $\vec{x}_N \equiv (T\,\vec{x})$ then

$$\vec{x}_{N,1} = R \vec{x}_{N,0}$$

and motion is just circles with the trivial solution

$$x_{N,1} = a\;\sin(\mu + \phi_0)$$
$$x'_{N,1} = a\;\cos(\mu + \phi_0)$$

3.2 Back to General Physical Solution

$$x_{N,1} = a\ \sin(\mu + \phi_0)$$
$$x'_{N,1} = a\ \cos(\mu + \phi_0)$$

$$\vec{x}_N \equiv (T\,\vec{x}) \quad \Rightarrow \quad \vec{x} = T^{-1}\vec{x}_N$$

$$T^{-1} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix}$$

$$x_1 = a\sqrt{\beta}\sin(\mu + \phi_0)$$
$$x'_1 = \frac{a}{\sqrt{\beta}}\left[\cos(\mu + \phi_0) - \alpha\sin(\mu + \phi_0)\right]$$

Eqn 3.28

- Particle position motion scales like $\sqrt{\beta}$
- For $\alpha = 0$ ellipse aspect ratio $x/x' = \beta$

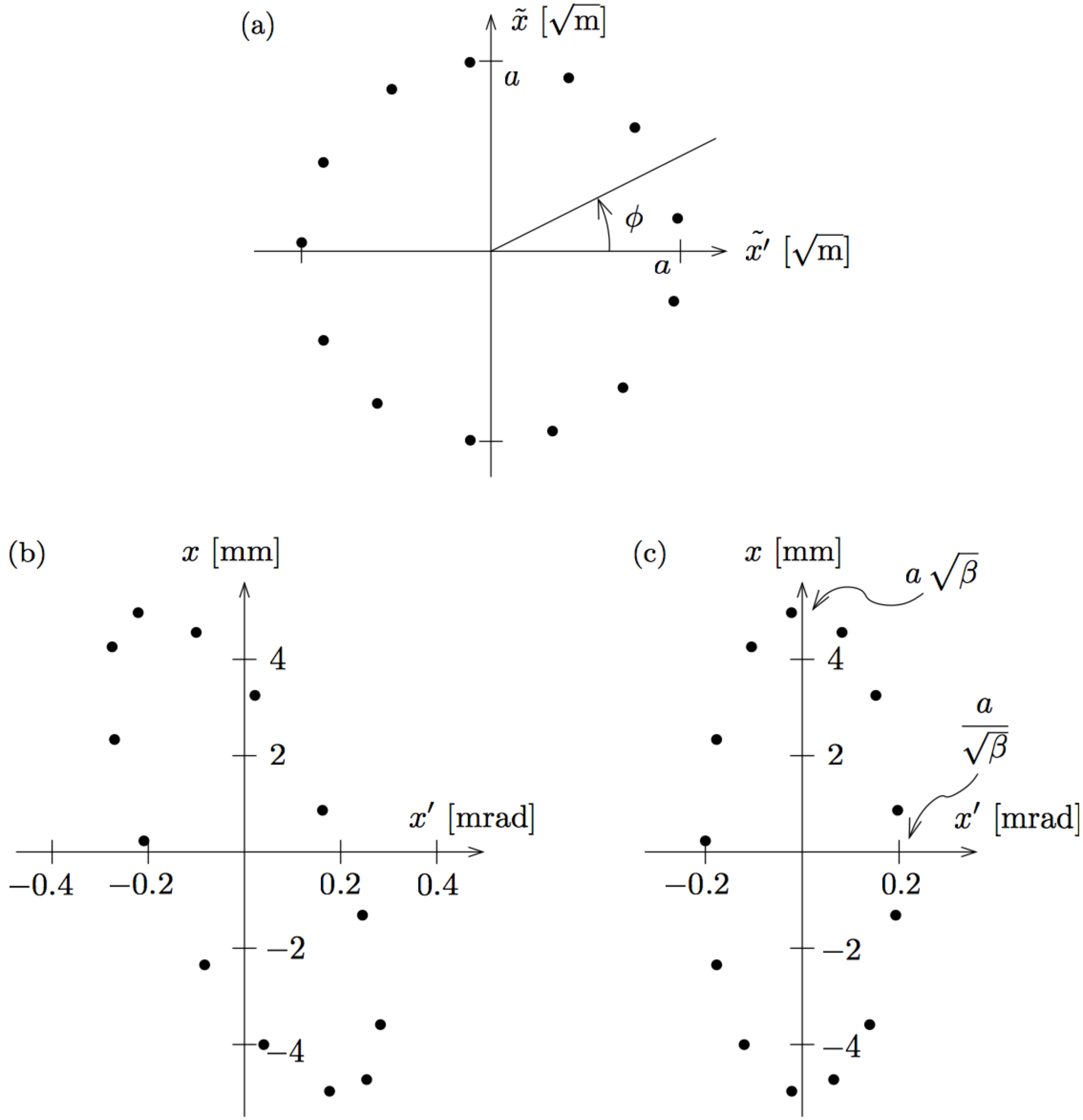


Figure 3.3 Linear motion for 12 turns with a fractional tune of mod $(Q_x, 1) = 0.155$. The motion in normalised phase space, shown in (a), is on a circle of radius a, advancing by an angle of $\Delta\phi = 2\pi Q_x$ on every turn.

3.3: General Linear Beam Transport

- The normalized phase space transformation gives us a clever path to general linear beam transport
  - From one location (s_1) to another (s_2)
  - Rotation is just a portion of the **phase advance**, not the full tune
  - Floquet transformations can use **different Twiss Parameters** (reminder: they depend on s coordinate!)

$$\vec{x}(s_2) = M_{21}\vec{x}(s_1)$$

$$M_{21} = T(s_2)^{-1}\; R(\phi_2 - \phi_1)\; T(s_1)$$

$$T^{-1}(s_2) = \begin{pmatrix} \sqrt{\beta(s_2)} & 0 \\ -\alpha(s_2)/\sqrt{\beta(s_2)} & 1/\sqrt{\beta(s_2)} \end{pmatrix} \qquad T(s_1) = \begin{pmatrix} 1/\sqrt{\beta(s_1)} & 0 \\ \alpha(s_1)/\sqrt{\beta(s_1)} & \sqrt{\beta(s_1)} \end{pmatrix}$$

General Linear Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using lattice parameters and $\Delta\phi \equiv \phi(s_2) - \phi(s_1)$

$$M_{s_1 \to s_2} = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1}}[\cos\Delta\phi + \alpha(s_1)\sin\Delta\phi & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\phi+[1+\alpha(s_1)\alpha(s_2)]\sin\Delta\phi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi] \end{pmatrix}$$

- This does not have a pretty form like the periodic matrix

  However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

  where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

  A common use of this matrix is the m_{12} term:

$$\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\phi)\,\Delta x'(s_1)$$

Effect of angle kick
on downstream position

General Linear Transport Matrix

- Equation 3.31, a very general form from all this math:

$$M_{21} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta 1}}\,(c_{21} + \alpha_1 s_{21}) & \sqrt{\beta_2 \beta_1}\, s_{21} \\ \frac{-(1+\alpha_1\alpha_2)\, s_{21} + (\alpha_2 - \alpha_1)\, c_{21}}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}\,(c_{21} - \alpha_2 s_{21}) \end{pmatrix}$$

with $s_{21} \equiv \sin(\phi_2 - \phi_1), c_{21} \equiv \cos(\phi_2 - \phi_1)$

- This is how one can parameterize uncoupled x or y transport from one location s_1 to another location s_2 where the Twiss parameters of those locations and their relative phase advance is known
  - Such as from a table from a modeling code like madx
- Most modeling codes can also output matrix elements between specific locations at starts or ends of magnets

Twiss Parameter Propagation

- Labeling the transport matrix elements as m_{12} etc:

$$M_{s_1 \to s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

then we (okay, you) can show

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{21}m_{11} & 1+2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

and the phase advance between two locations is also related to the transport matrix elements and initial Twiss parameters by

$$\tan(\phi_2 - \phi_1) = \frac{m_{12}}{m_{11}\beta_1 - m_{12}\alpha_1}$$

3.4: (very small) General Linear Transport Matrix

- To this point we have religiously avoided **differential equations**
  - But the general linear transport matrix suggests a path
- Look at motion over very short distances Δs and very small phase advances $\Delta\phi$

$$s_{21} \equiv \sin(\Delta\phi) \approx \Delta\phi \qquad c_{21} \equiv \cos(\Delta\phi) \approx 1$$

$$\beta_2 = \beta_1 + \Delta\beta \qquad \Delta\beta \ll \beta_{1,2}$$

Keep only leading terms in general Twiss matrix:

$$M_{21} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta 1}}\,(c_{21} + \alpha_1 s_{21}) & \sqrt{\beta_2\beta_1}\, s_{21} \\ \frac{-(1+\alpha_1\alpha_2)\, s_{21} + (\alpha_2 - \alpha_1)\, c_{21}}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}\,(c_{21} - \alpha_2 s_{21}) \end{pmatrix}$$

3.4: (very small) General Linear Transport Matrix

- We find equation 3.37:

$$M_{21} \approx \begin{pmatrix} \left(1+\frac{1}{2}\frac{\Delta\beta}{\beta}\right)(1+\alpha\Delta\phi) & \beta\,\Delta\phi \\ \sim & \left(1-\frac{1}{2}\frac{\Delta\beta}{\beta}\right)(1-\alpha\Delta\phi) \end{pmatrix}$$

But this **must** be close to a drift matrix for reasonable strength magnets and very small $\Delta s, \Delta\phi$

$$M_{21} \approx \begin{pmatrix} 1 & \Delta s \\ \sim & 1 \end{pmatrix}$$

$$m_{12} \text{ terms} \quad \Rightarrow \quad \frac{d\phi}{ds} = \frac{1}{\beta}$$

$$m_{11} \text{ terms} \quad \Rightarrow \quad \alpha = -\frac{1}{2}\frac{d\beta}{ds}$$

3.4: Differential and Integral Twiss Relations

$$\frac{d\phi}{ds} = \frac{1}{\beta} \quad \Rightarrow \quad \Delta\phi = \phi_2 - \phi_1 = \int_1^2 \frac{ds}{\beta(s)}$$ Generally

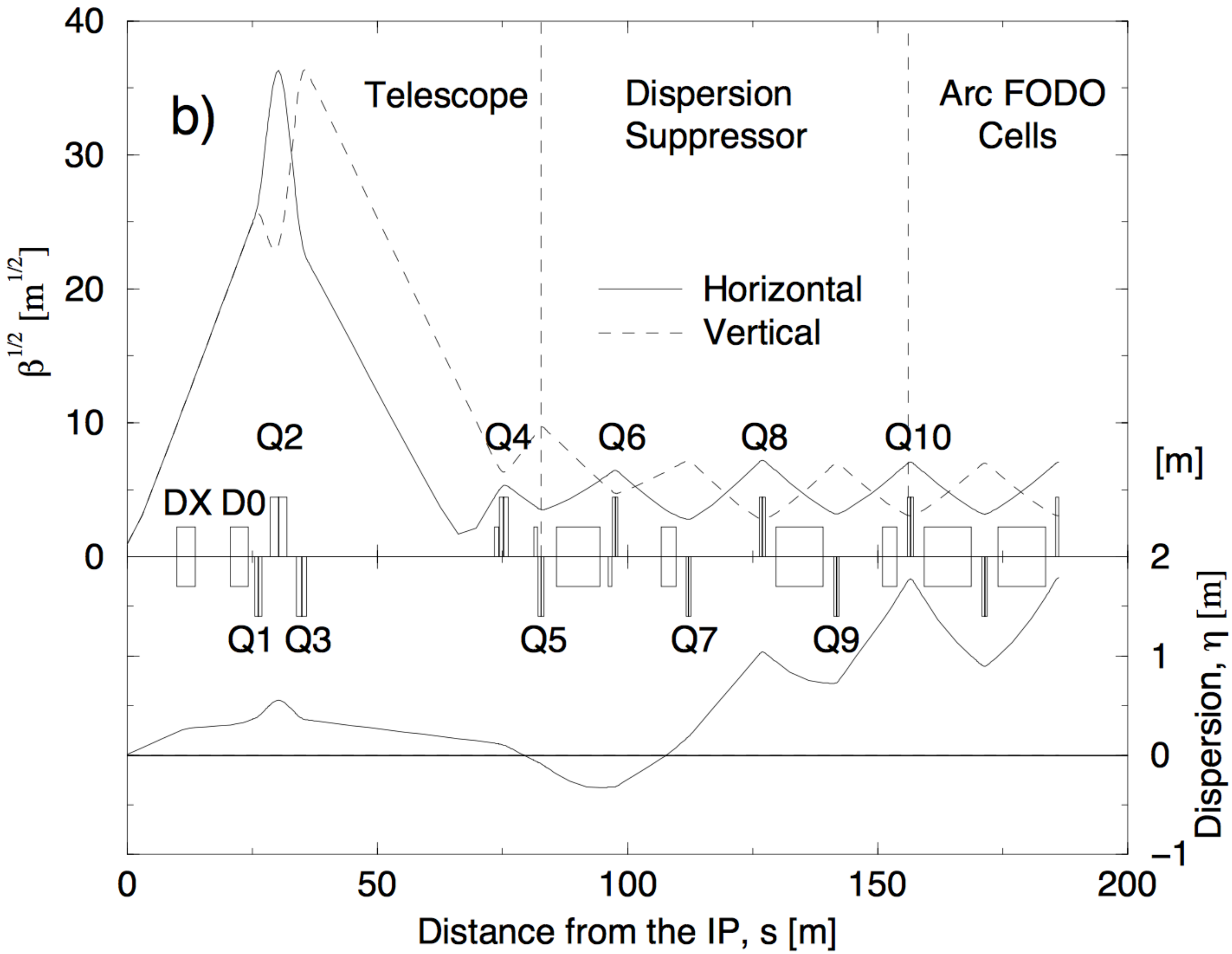
$$2\pi Q = \oint \frac{ds}{\beta}$$ Ring

$$x_1 = a\sqrt{\beta}\sin(\mu + \phi_0)$$

$$x_1' = \frac{a}{\sqrt{\beta}}\left[\cos(\mu + \phi_0) - \alpha\sin(\mu + \phi_0)\right]$$

Differentiation x by s and applying differential relations gives same equation for x'

Continuous Twiss Parameter Plot: RHIC IR



Review

Particle position motion $\quad \mathrm{x(s) = a\sqrt{\beta(s)}\sin(\Delta\phi)}$

$$\beta(s) = \beta(s+C) \qquad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \Delta\phi = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

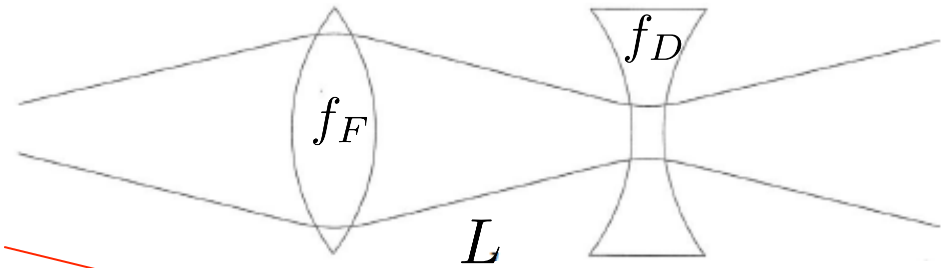
betatron phase advance $\quad \mu = 2\pi Q = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \quad \mathrm{Tr}\, M = 2\cos\mu$

$$M = I\cos\mu + J\sin\mu \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}$$

Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift L



Thin quadrupole matrices

$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$f_F, f_D > 0$

$$\frac{1}{f_{\text{doublet}}} = \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D}$$

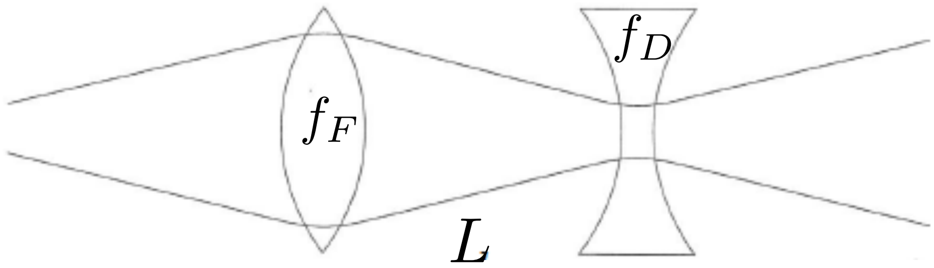
(C&M 5.1 with $f_F = -f_D$)

$$f_D = f_F = f \quad \Rightarrow \quad \frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2}$$

There is **net focusing** given by this **alternating gradient** system

A fundamental point of optics, and of accelerator **strong focusing**

Strong Focusing: Another View



$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$$\text{incoming paraxial ray} \qquad \begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \end{pmatrix} x_0$$

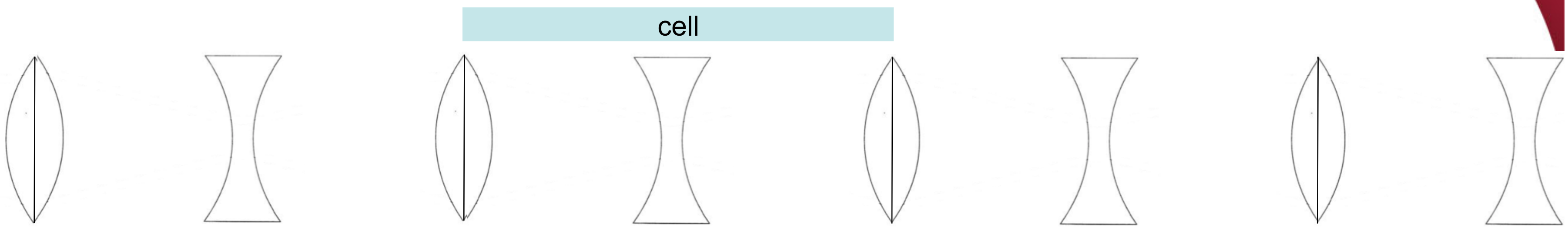
For this to be focusing, x' must have opposite sign of x where these are coordinates of transformation of incoming paraxial ray

$$f_F = f_D \quad x' < 0 \quad \textbf{BUT} \quad x > 0 \text{ iff } f_F > L$$

Equal strength doublet is net focusing under condition that each lens' focal length is greater than distance between them

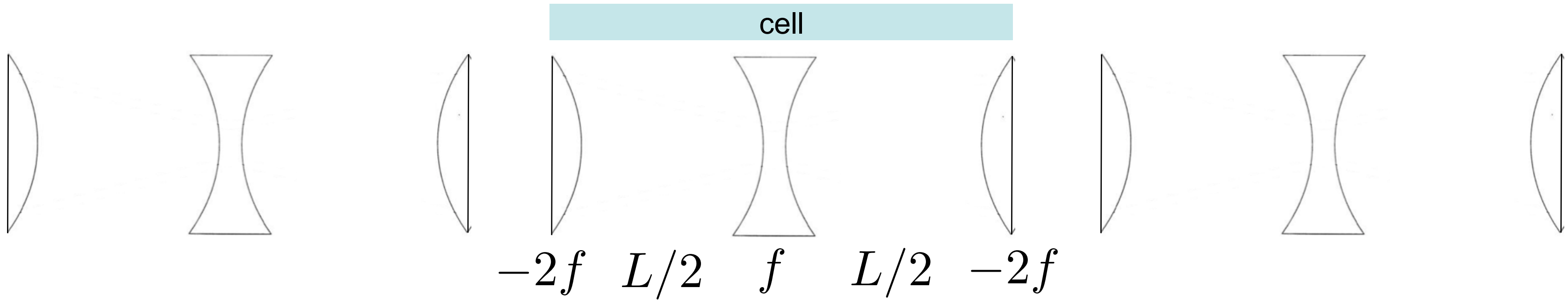
3.5 (kinda): Lattice Optics: FODO Lives!

(Be very careful in comparisons to book!)



- Most accelerator lattices are designed in modular ways
  - Design and operational clarity, separation of functions
- One of the most common modules is a FODO module
  - Alternating focusing and defocusing “strong” quadrupoles
  - Spaces between are combinations of drifts and dipoles
  - Strong quadrupoles dominate the focusing
  - Periodicity is one FODO “cell” so we’ll investigate that motion
  - Horizontal beam size largest at centers of focusing quads
  - Vertical beam size largest at centers of defocusing quads

Periodic Example: FODO Cell Phase Advance



- Select periodicity between centers of **focusing** quads
  - A natural periodicity if we want to calculate maximum b(s)

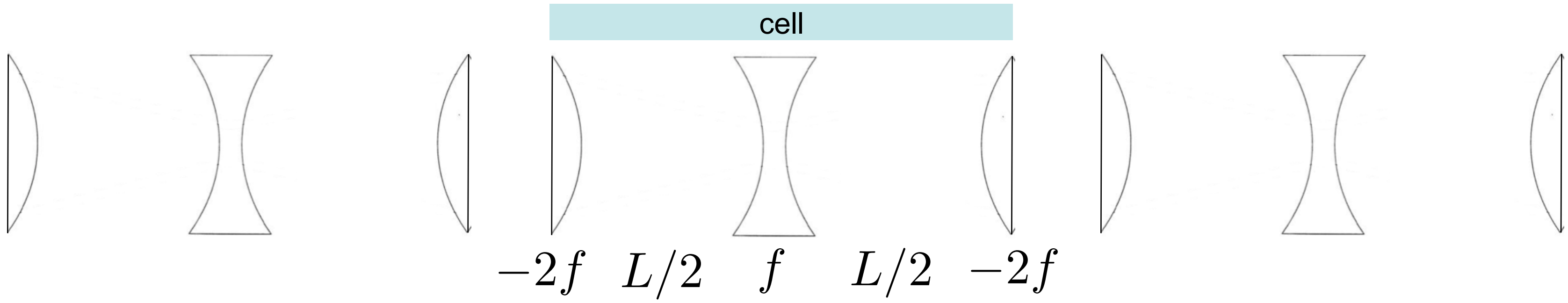
$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\mu = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\mu = 1 - 2\sin^2\frac{\mu}{2} \quad \Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f}$$

  - μ only has real solutions (stability) if $\frac{L}{4} < f$

Periodic Example: FODO Cell Beta Max/Min



- What is the maximum beta function, $\hat{\beta}$?
  - A natural periodicity if we want to calculate maximum b(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad m_{12} = \beta \sin\mu$$

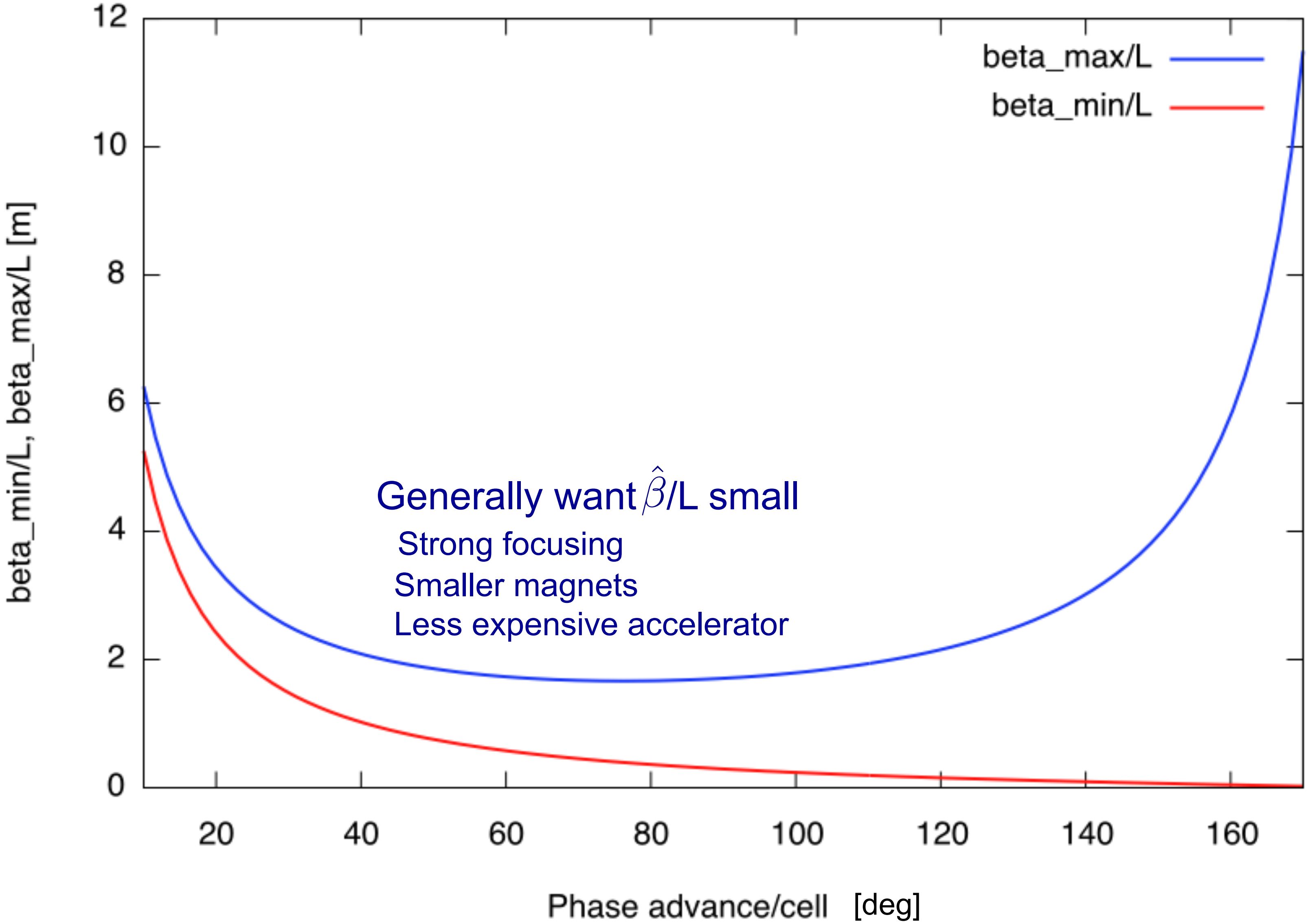
$$\hat{\beta}\sin\mu = \frac{L^2}{4f} + L = L\left(1 + \sin\frac{\mu}{2}\right)$$

$$\hat{\beta} = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right)$$

  - Follow a similar strategy reversing F/D quadrupoles to find the minimum b(s) within a FODO cell (center of D quad)

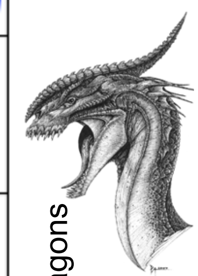
$$\check{\beta} = \frac{L}{\sin\mu}\left(1 - \sin\frac{\mu}{2}\right)$$

FODO Betatron Functions vs Phase Advance



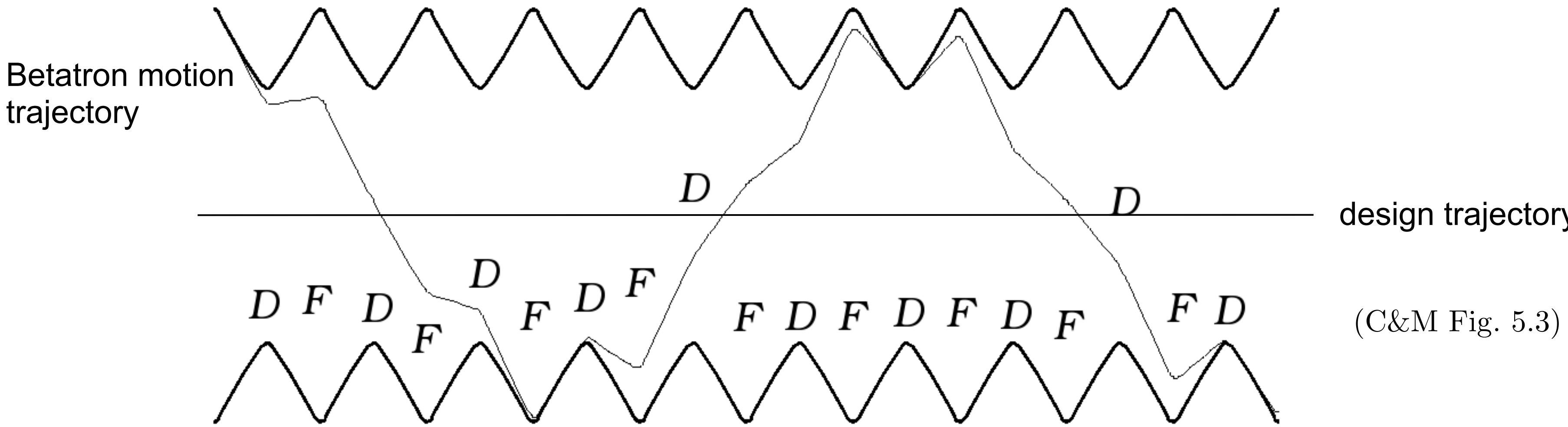
Generally want $\hat{\beta}$/L small

- Strong focusing
- Smaller magnets
- Less expensive accelerator



Here there be dragons

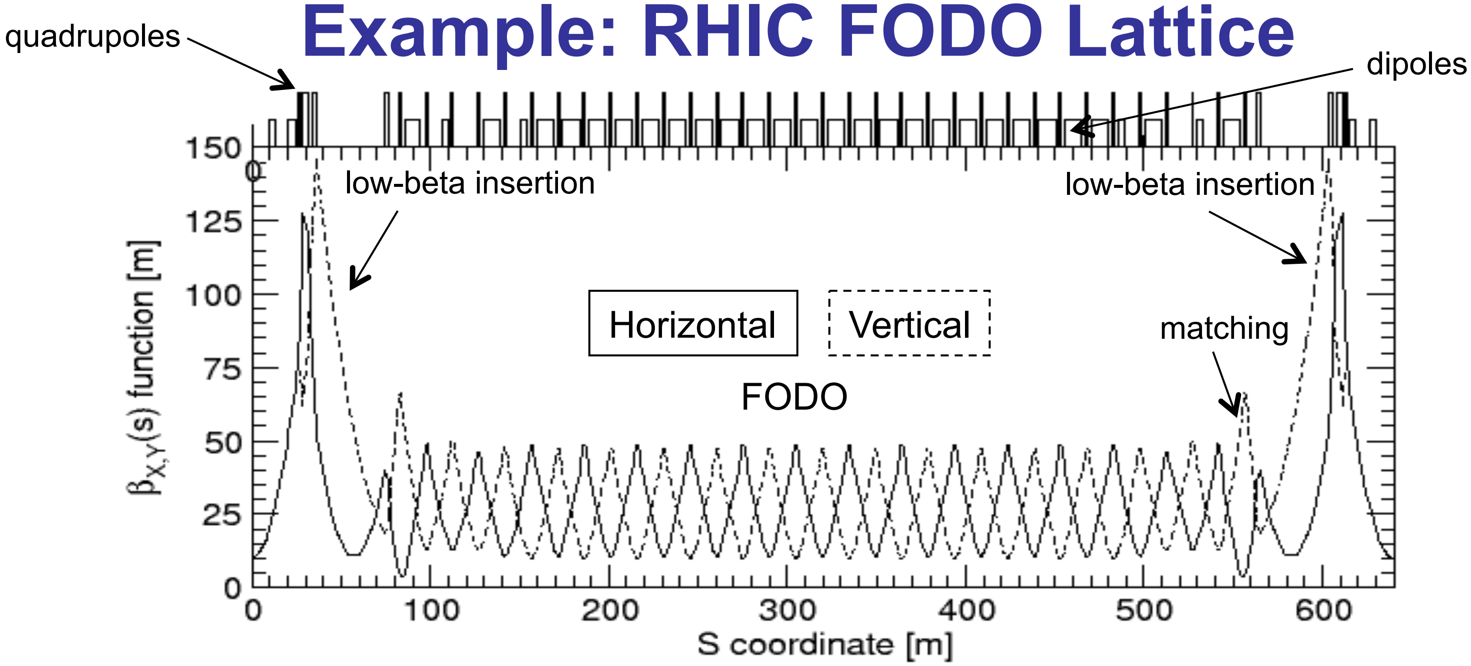
FODO Beta Function, Betatron Motion



- This is a picture of a FODO lattice, showing contours of $\pm\sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$
  - This also shows a particle oscillating through the lattice
  - Note that $\sqrt{\beta(s)}$ provides an "envelope" for particle oscillations
    - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
  - Min beta is at defocusing quads, max beta is at focusing quads
  - 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \quad \mu = 360^\circ/6.5 \approx 55^\circ$$

Example: RHIC FODO Lattice



- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about L=30 m
  - Phase advance per FODO cell is about $\mu = 77^\circ = 1.344 \text{ rad}$

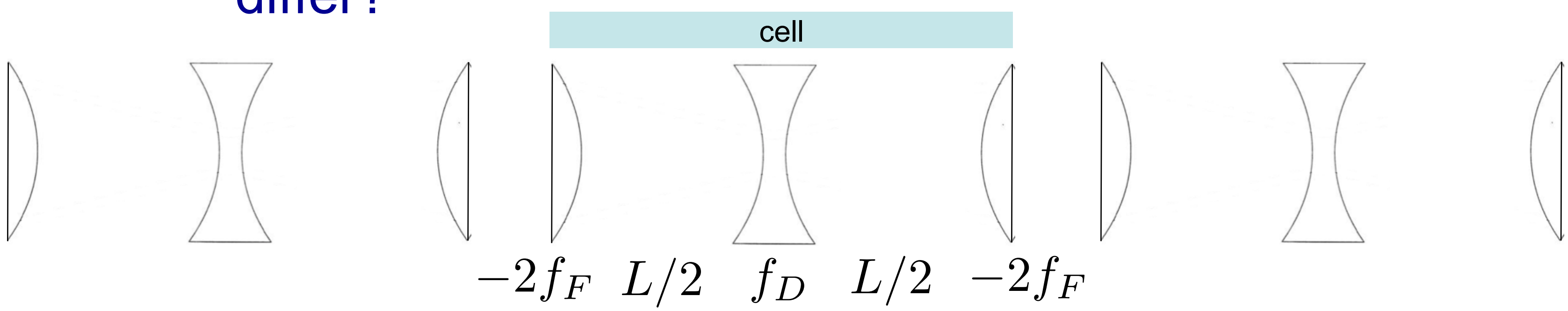
$$\hat{\beta} = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right) \approx 53 \text{ m}$$

$$\check{\beta} = \frac{L}{\sin\mu}\left(1 - \sin\frac{\mu}{2}\right) \approx 8.7 \text{ m}$$

☺

Stability Diagrams

- Designers often want or need to change the focusing of the two transverse planes in a FODO structure
  - What happens if the focusing/defocusing strengths differ?



  - Recalculate the M matrix and use dimensionless quantities

$$F \equiv \frac{L}{2f_F} \qquad D \equiv \frac{L}{2f_D}$$

then take the trace for stability conditions to find

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

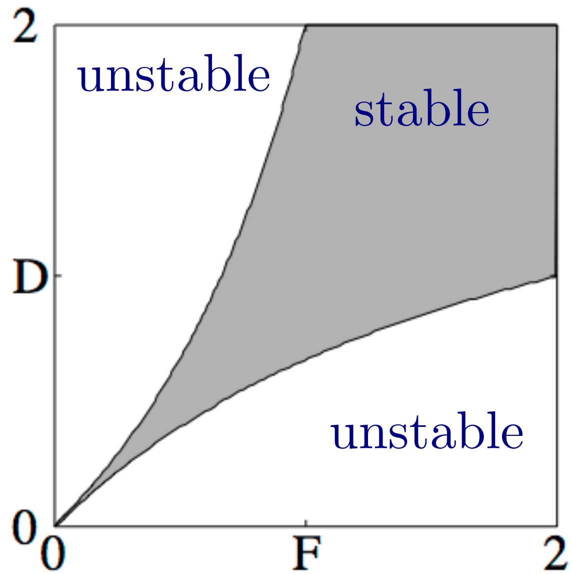
Stability Diagrams II

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2}$$

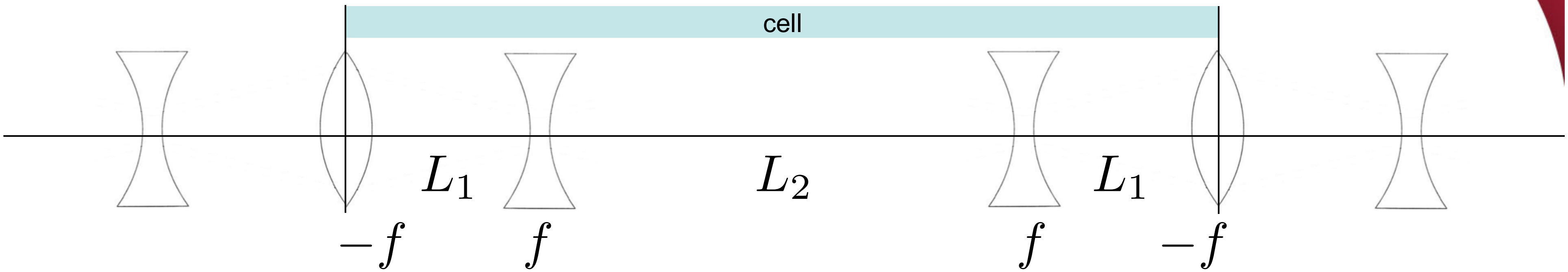
- For stability, we must have $-1 < \cos\mu < 1$
- Using $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$, stability limits are where

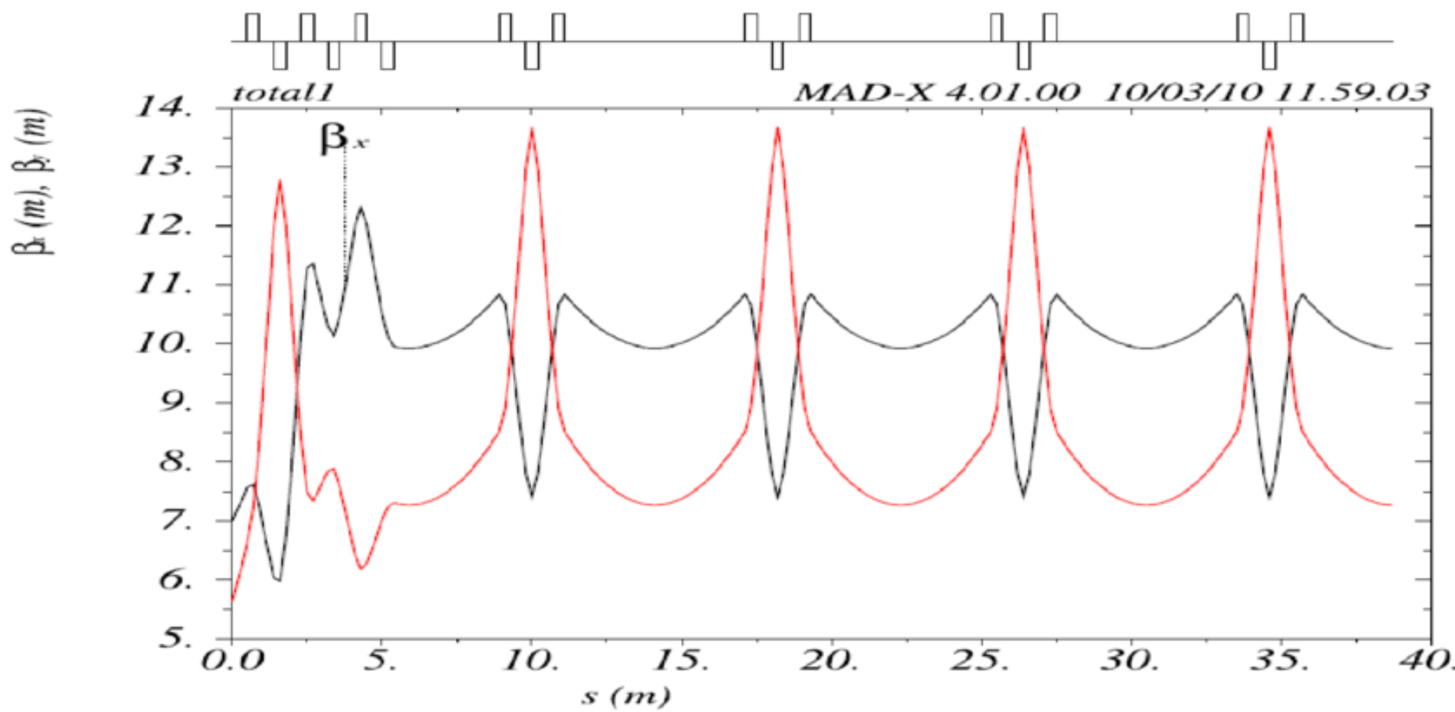
$$\sin^2\frac{\mu}{2} = 0 \qquad \sin^2\frac{\mu}{2} = 1$$

- These translate to a FODO "necktie" stability diagram



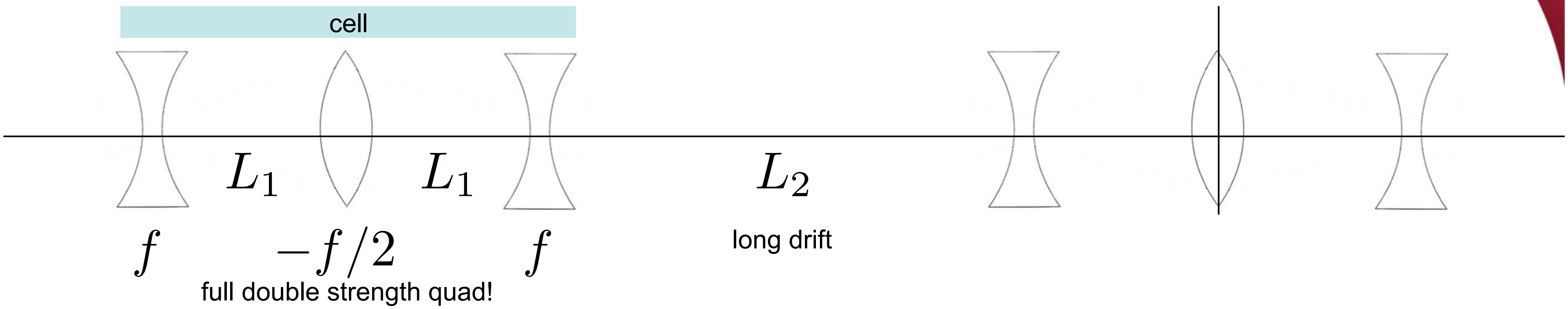
Triplet Optics: Extra Straight Space

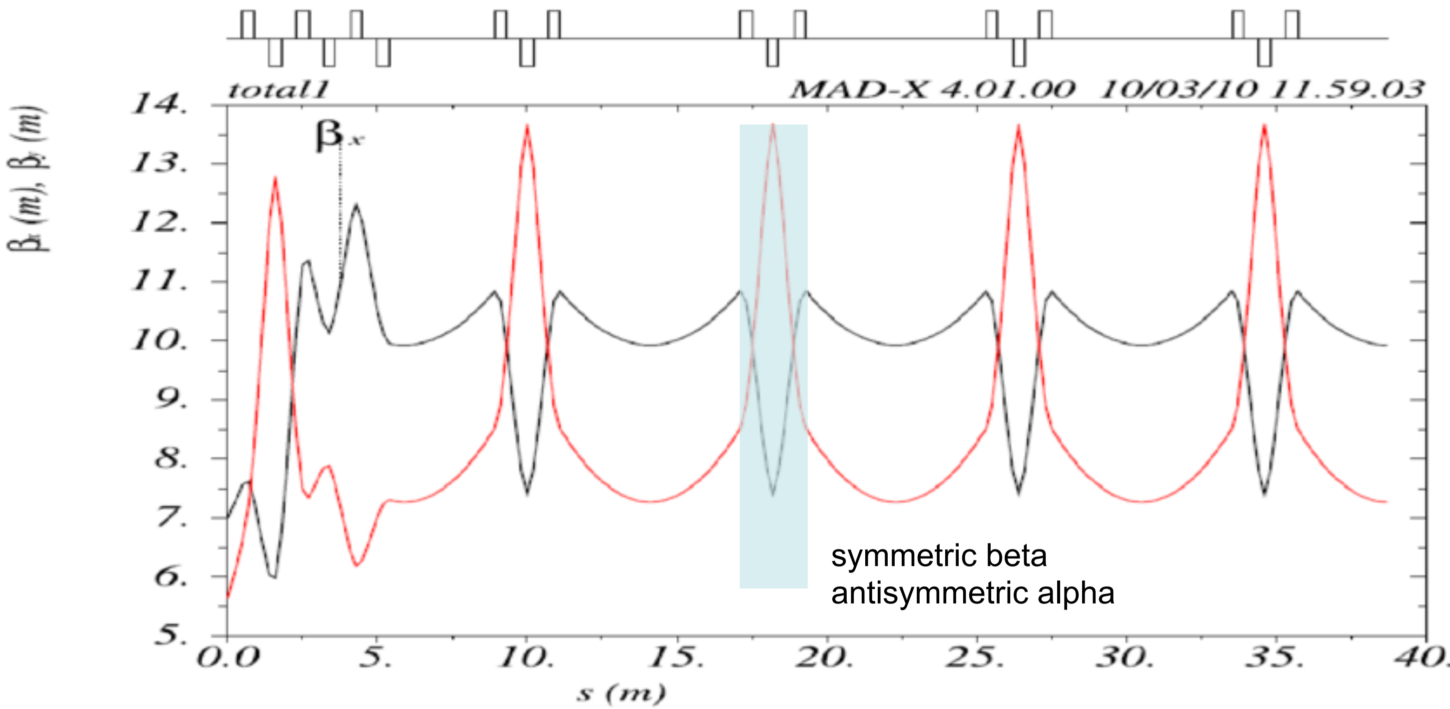




From R. Chehab et al., "The CLIC Positron Capture and Acceleration in the Injector Linac", 2010.

Triplet Optics: Extra Straight Space



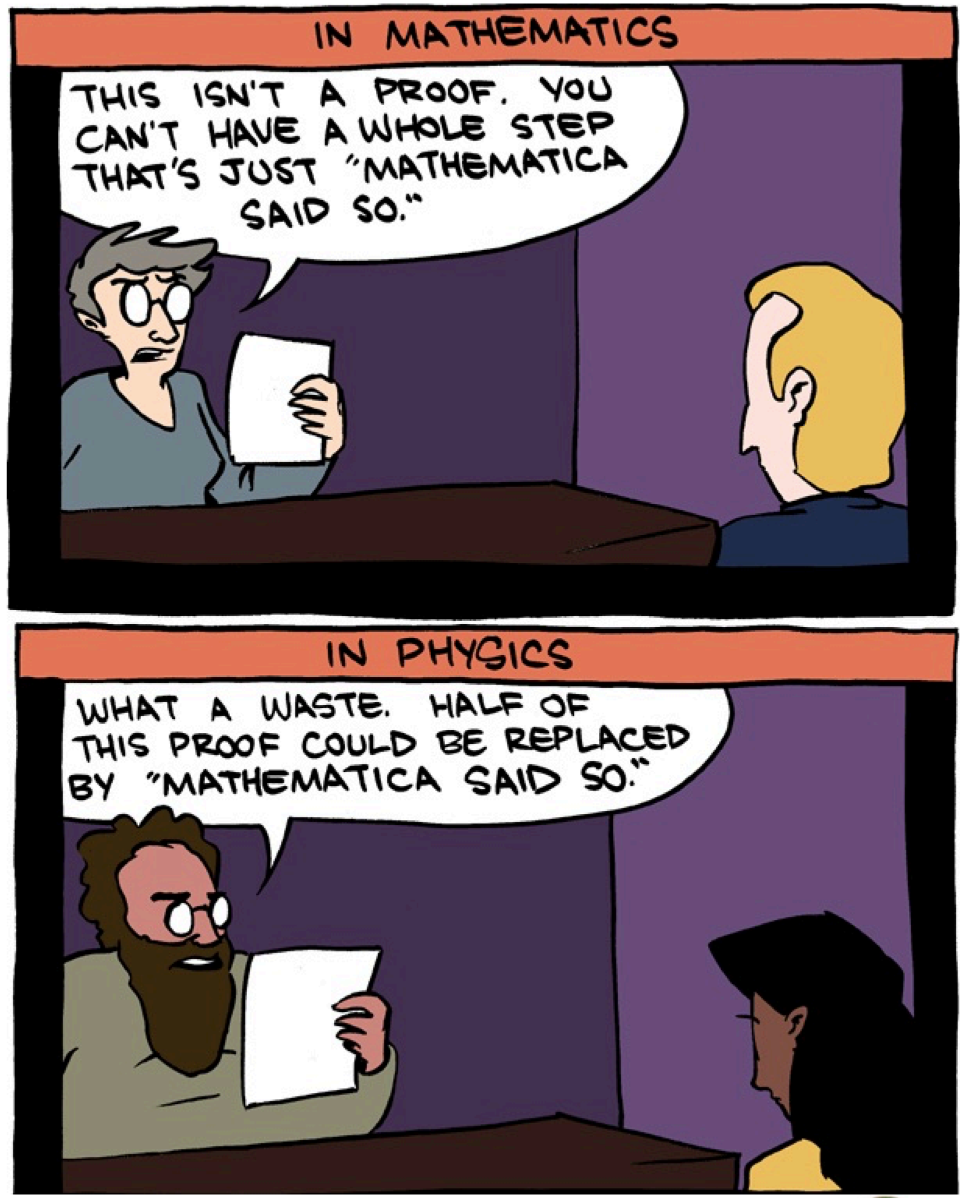


More on Friday

From R. Chehab et al., "The CLIC Positron Capture and Acceleration in the Injector Linac", 2010.

Warning: Professoral Egoistic Indulgence

- Differential Equations Ahead
- The following is extraneous to the course and text...
- ... but it illuminates a connection between 2nd order linear differential equations and matrices that your lecturer finds beautiful so you will hear about it (if we have time)



More Math: Hill's Equation

- Let's go back to our quadrupole equations of motion for $R \to \infty$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)$$

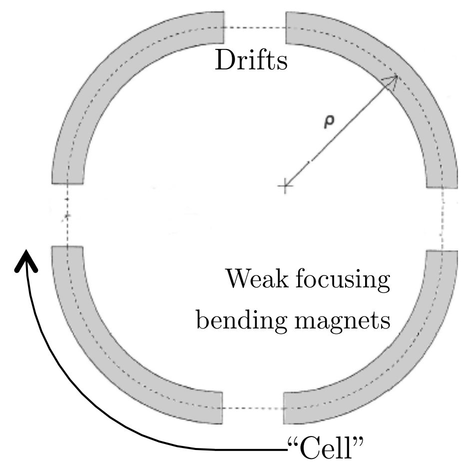
What happens when we let the focusing K vary with s?

Also assume K is **periodic** in s with some periodicity C

$$x'' + K(s)x = 0 \qquad K(s) \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)(s) \qquad K(s+C) = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated "cell" of the layout

The simple harmonic oscillator equation with a **periodically** varying spring constant K(s) is known as **Hill's Equation**



Hill's Equation Solution Ansatz

$$x'' + K(s)x = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)(s)$$

- Solution is a quasi-periodic harmonic oscillator

$$x(s) = A\, w(s)\, \cos[\Psi(s) + \Psi_0]$$

where w(s) is periodic in C but the phase Y(s) is not!!

Substitute this educated guess ("ansatz") to find

$$x' = Aw' \cos[\Psi + \Psi_0] - Aw\Psi' \sin[\Psi + \Psi_0]$$

$$x'' = A(w'' - w\Psi'^2) \cos[\Psi + \Psi_0] - A(2w'\Psi' + w\Psi'') \sin[\Psi + \Psi_0]$$

$$x'' + K(s)x = -A(2w'\Psi' + w\Psi'')\sin(\Psi + \Psi_0) + A(w'' - w\Psi'^2 + Kw)\cos(\Psi + \Psi_0) = 0$$

For $w(s)$ and $\Psi(s)$ to be independent of Ψ_0, coefficients of the sin and cos terms must vanish identically

Courant-Snyder Parameters

$$2ww'\Psi' + w^2\Psi'' = (w^2\Psi')' = 0 \quad \Rightarrow \quad \Psi' = \frac{k}{w(s)^2}$$

$$w'' - (k^2/w^3) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2$$

- Notice that in both equations $w^2 \propto k$ so we can scale this out and define a new set of functions, **Courant-Snyder Parameters** or **Twiss Parameters**

$$\beta(s) \equiv \frac{w^2(s)}{k}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s)$$

$$\gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\Rightarrow$$

$$\Psi' = \frac{1}{\beta(s)} \qquad \Psi(s) = \int \frac{ds}{\beta(s)}$$

$$K\beta = \gamma + \alpha'$$

$\beta(s), \alpha(s), \gamma(s)$ are all periodic in C

$\Psi(s)$ is **not** periodic in C

Towards The Matrix Solution

- What is the matrix for this Hill's Equation solution?

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

Take a derivative with respect to s to get $x' \equiv \frac{dx}{ds}$

$$\Psi' = \frac{1}{\beta(s)} \qquad x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

Now we can solve for A and B in terms of initial conditions $(x(0), x'(0))$

$$x_0 \equiv x(0) = A\sqrt{\beta(0)} \qquad x_0' \equiv x'(0) = \frac{1}{\sqrt{\beta(0)}}[B - \alpha(0)A]$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x_0' + \alpha(0)x_0]$$

And take advantage of the periodicity of β, α to find $x(C), x'(C)$

Hill's Equation Matrix Solution

$$x(s) = A\sqrt{\beta(s)}\cos\Psi(s) + B\sqrt{\beta(s)}\sin\Psi(s)$$

$$x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\Psi(s) - [A + \alpha(s)B]\sin\Psi(s)\}$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x_0' + \alpha(0)x_0]$$

$$x(C) = [\cos\Psi(C) + \alpha(0)\sin\Psi(C)]x_0 + \beta(0)\sin\Psi(C)x_0'$$
$$x'(C) = -\gamma(0)\sin\Psi(C)x_0 + [\cos\Psi(C) - \alpha(0)\sin\Psi(C)]x_0'$$

We can write this down in a matrix form where $\mu = \Psi(C) - \Psi(0)$ is the betatron phase advance through one period C

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

phase advance per cell