# USPAS Accelerator Physics 2021
# (Virtually) Texas A&M University

## Lattice Examples I
### (or starting to put it all together)
#### (or Stupid Lattice Tricks)

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http://www.toddsatogata.net/2021-USPAS
Username test / Password test

# Entire Courses Also Taught On Lattice Design

- https://casa.jlab.org/publications/USPAS_Jan_2018.html

**Practical Lattice Design**

**Alex Bogacz (Jefferson Lab) and Dario Pellegrini (CERN) with Randika Gamage (ODU)**

**January 15 - 19, 2018 Old Dominion University - Norfolk, VA**

**Timeline**

**Course Outline**

| | | |
|---|---|---|
| **Lecture 1:** | Introduction to Transverse Optics | Dario Pellegrini |
| **Lecture 2:** | Introduction to OptiM, FODO Cell | Alex Bogacz |
| **Lecture 3:** | Dispersion Suppressors | Alex Bogacz |
| **Lecture 4:** | Arc-to-Straight Design | Alex Bogacz |
| **Lecture 5:** | Low Beta Optics | Dario Pellegrini |
| **Lecture 6:** | Lattice Imperfections | Dario Pellegrini |
| **Lecture 7:** | Radiation Damping | Alex Bogacz |
| **Lecture 8:** | Low Emittance Lattices, DBA Cell | Dario Pellegrini |

**ASSIGNMENTS**

| | | | |
|---|---|---|---|
| **Day 1:** | Example | Homework | Solutions |
| **Day 2:** | Example 1 Example 2 | Homework | Solutions |
| **Day 3:** | Example 1 Example 2 Example 3 | Homework | Solutions |
| **Day 4:** | Example | Homework | Solutions |
| **January 19, 2018** | Final Exam | Solutions | |

# Today (Fri Feb 5): Mostly 1D and 1D+

- Review: Linear optics, matrices, Twiss parameters
- Easing in: Two-bumps and three-bumps
- Dipole-Free Transverse Lattices
  - Review: FODO cell, without dipoles
  - Periodic triplet cell
  - $\pi/2$ and imaging insertions
  - Coupling (Mobius) insertion
  - Low-beta insertions (collision point, ion stripping, ...)
- Review: Dispersion
- Bending Transverse Lattices (FODO)
  - Review: FODO cell, with dipoles
  - FODO cell dispersion suppressors

# Monday (Feb 8): 1D+ and 2-3D+

- Localizing Dispersion: Achromats
  - Achromatic doglegs/chicanes
  - Bunch compressors
  - Double bend achromat
  - Triple bend achromat
  - Multi-bend achromat (HMBA)
  - Lead in to Fri Feb 12 lecture on 3G light source lattices
- (2D/3D manipulation):
  - Flat to round/round to flat transforms
  - Longitudinal/transverse emittance exchange
- Chromaticity correction blocks (insertions)??
  - Lead in to Tue Feb 9 lecture on sextupoles and chromaticity

# Review: General Linear Transport Matrix

Book section 3.3

- We can parameterize a general non-periodic transport matrix from $s_1$ to $s_2$ using lattice parameters and $\Delta\phi \equiv \phi(s_2) - \phi(s_1)$

$$M_{s_1 \to s_2} = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1}}[\cos\Delta\phi + \alpha(s_1)\sin\Delta\phi & \boxed{\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi} \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\phi + [1+\alpha(s_1)\alpha(s_2)]\sin\Delta\phi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi] \end{pmatrix}$$

- This does not have a pretty form like the periodic matrix

However both can be expressed as

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

where the C and S terms are cosine-like and sine-like.
(The second row is the s-derivative of the first row!)

A common use of this matrix is the $m_{12}$ term:

$$\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\phi)\,\Delta x'(s_1)$$

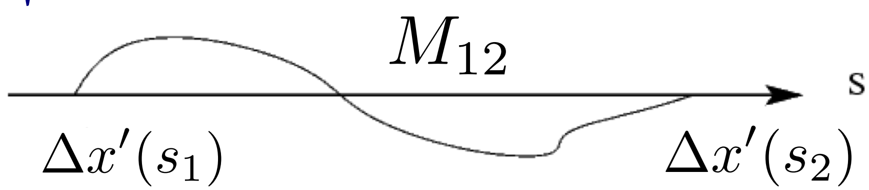
Effect of angle kick on downstream position

# Orbit Control: Two-Bump

$$\Delta x(s_2) = \Delta x'(s_1)\sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi$$

$$M_{12} = \begin{pmatrix} C_{12} & S_{12} \\ C'_{12} & S'_{12} \end{pmatrix}$$

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi]$$

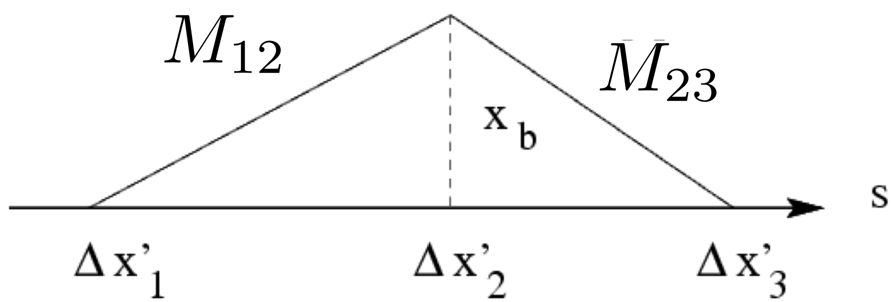



- A single orbit error changes all later positions and angles
  - Add another dipole corrector at a location where $\Delta\phi = k\pi$
    At this point the distortion from the original dipole corrector is all x' that we can cancel with the second dipole corrector.

$$\Delta x'(s_2) = \Delta x'(s_1)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}} + \text{angle from } \mathrm{s}_2 \text{ dipole}$$

  - Called a **two-bump**: localized orbit distortion from two correctors
  - But requires $\Delta\phi = k\pi$ between correctors

# Orbit Control: Three-Bump (another view of HW 9.2)




- A general local orbit distortion from three dipole correctors
  - Constraint is that net orbit change from sum of all three kicks must be zero

$$\begin{pmatrix} C_{23} & S_{23} \\ C'_{23} & S'_{23} \end{pmatrix} \left[ \begin{pmatrix} C_{12} & S_{12} \\ C'_{12} & S'_{12} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta x'_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x'_2 \end{pmatrix} \right] + \begin{pmatrix} 0 \\ \Delta x'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Delta x'_1 = \frac{x_b}{S_{12}} \qquad \Delta x'_2 = -\left( \frac{C_{23}S_{12} + S_{23}S'_{12}}{S_{12}S_{23}} \right) x_b \qquad \Delta x'_3 = \frac{S_{23}}{S_{12}^2} x_b$$

  - Bump amplitude $x_{\rm b} = S_{12}\Delta x'_1$
  - Only **three-bump** requirement is that $S_{12}$, $S_{23} \neq 0$

# Review: Matrices of Magnetic Elements

- For our purposes this morning:
  - All motion is linearized
$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad x' \equiv \frac{p_x}{p_0}$$
  - Linear transport matrices:
$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
Book A.1.1
$$M_{\mathrm{quad}} \approx \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$
Book A.1.6 for thin quads
  - (Sector) dipole includes constant fractional momentum offset
$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_2 = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ \frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_1 \qquad \delta \equiv \frac{\Delta p}{p_0}$$
Book A.1.2 for subset of phase space

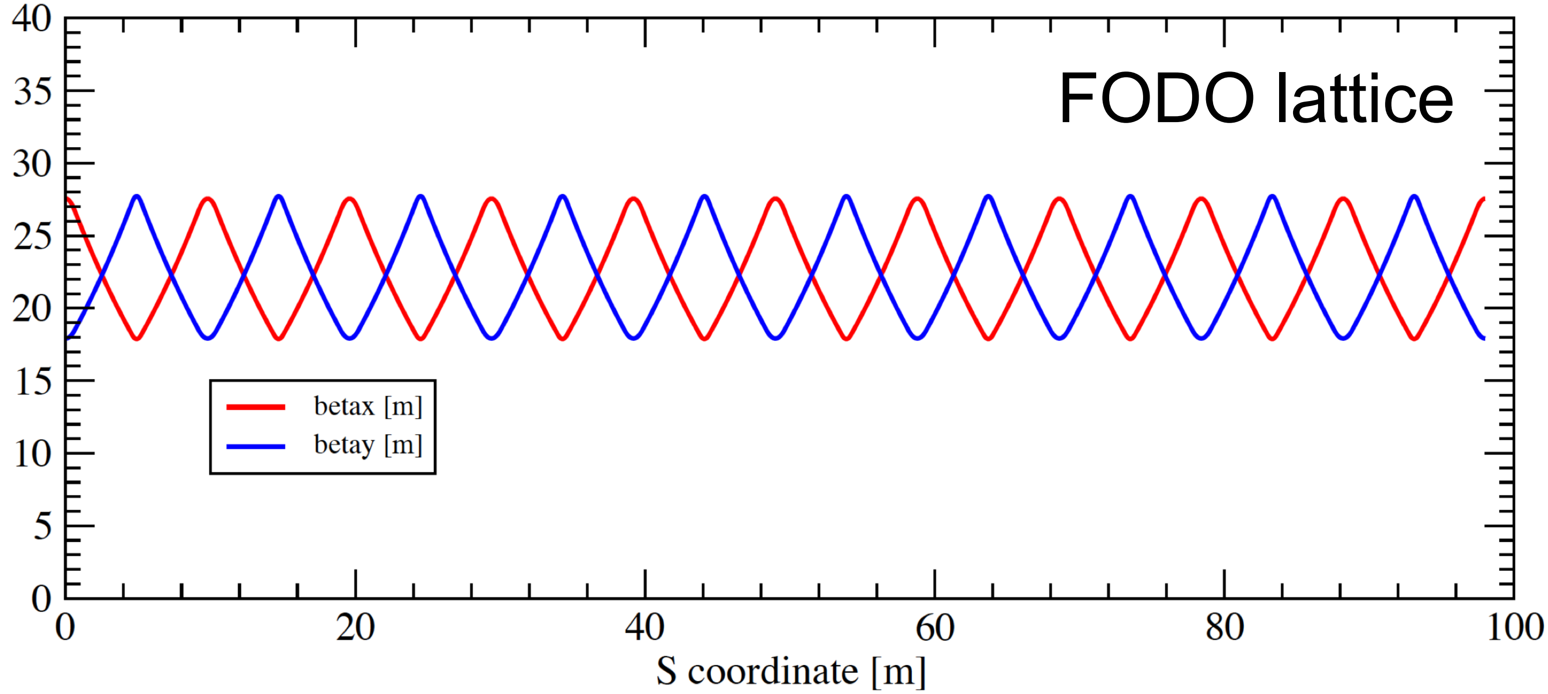
# Review: Periodic Transport Matrix Parameterization

- Periodic transport matrices can be parameterized as

$$M = I\cos\mu + J\sin\mu = e^{J\mu} \qquad J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad J^2 = -I$$

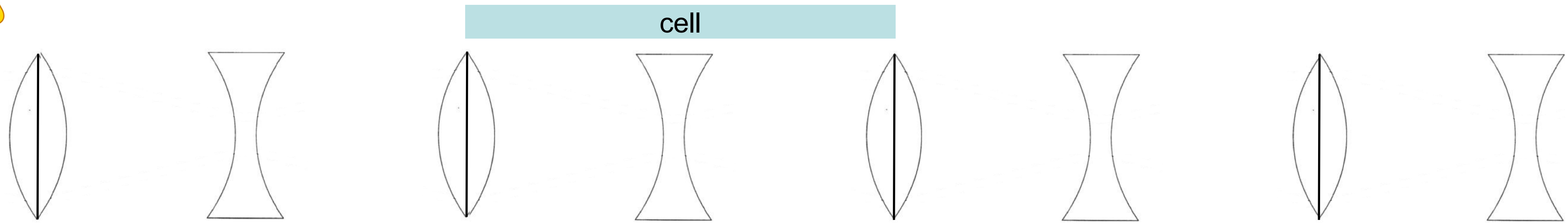
$(\beta, \alpha, \gamma \equiv (1+\alpha^2)/\beta)$ all depend on s location

all have the periodicity of the system



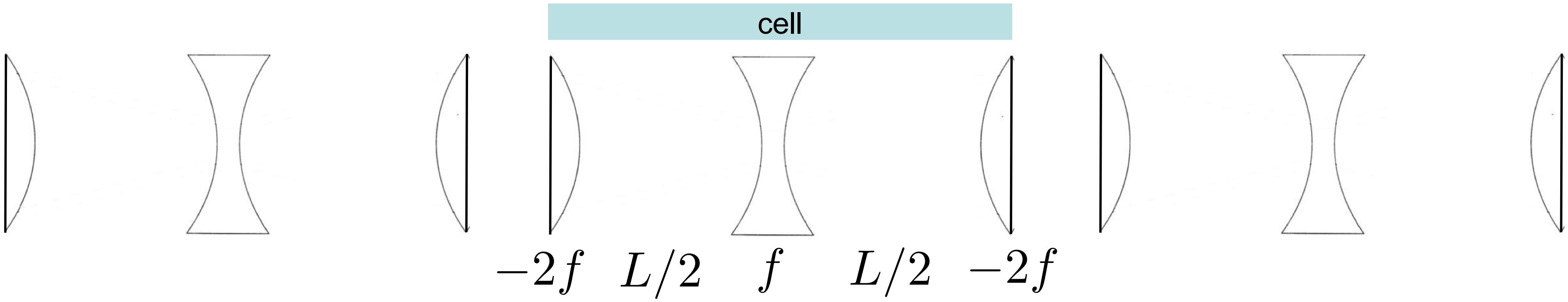
# Dipole-Free Transverse Lattices: FODO Review

**(Be very careful about comparisons to book: L is not the same L!)**




- Most accelerator lattices are designed in modular ways
  - Design and operational clarity, separation of functions
- One of the most common modules is a FODO module
  - Alternating focusing and defocusing "strong" quadrupoles
  - Spaces between are combinations of drifts and dipoles
  - Strong quadrupoles dominate the focusing
  - Periodicity is one FODO "cell" so we'll investigate that motion
  - Horizontal beam size largest at centers of focusing quads
  - Vertical beam size largest at centers of defocusing quads

# Dipole-Free Transverse Lattices: FODO Review




- Select periodicity between centers of **focusing** quads
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

Check: f large case

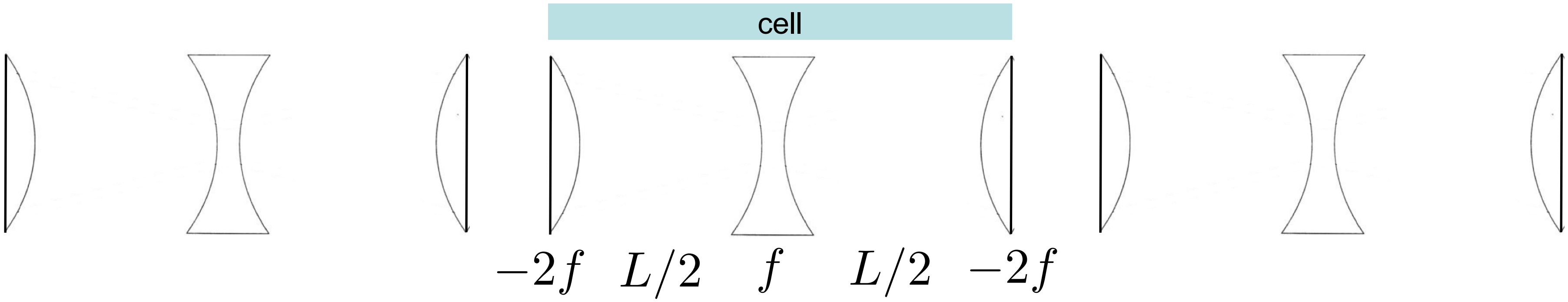
$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\mu = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\mu = 1 - 2\sin^2\frac{\mu}{2} \quad \Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f}$$

  - $\mu$ only has real solutions (stability) if $\frac{L}{4} < f$

(Be very careful about comparisons to book: L is not the same L! This is same as Accelerator Handbook)

# Dipole-Free Transverse Lattices: FODO Review




- What is the maximum beta function, $\hat{\beta}$ ?
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad m_{12} = \beta \sin\mu$$

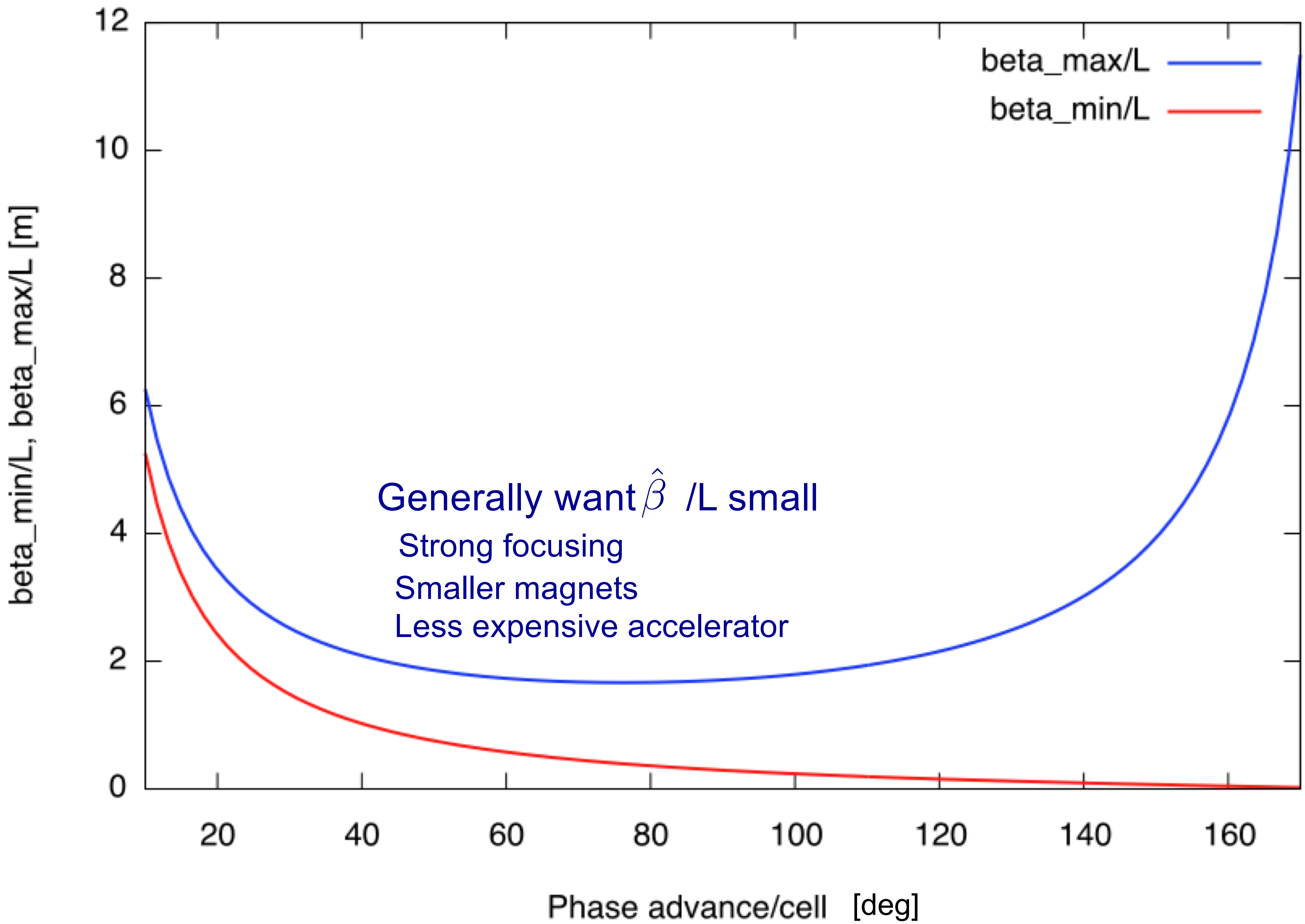
$$\hat{\beta} \sin\mu = \frac{L^2}{4f} + L = L\left(1 + \sin\frac{\mu}{2}\right)$$

$$\hat{\beta} = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right)$$

  - Follow a similar strategy reversing F/D quadrupoles to find the minimum β (s) within a FODO cell (center of D quad)

$$\check{\beta} = \frac{L}{\sin\mu}\left(1 - \sin\frac{\mu}{2}\right)$$

# FODO Betatron Functions vs Phase Advance

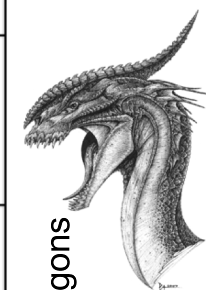



Generally want $\hat{\beta}$ /L small

Strong focusing

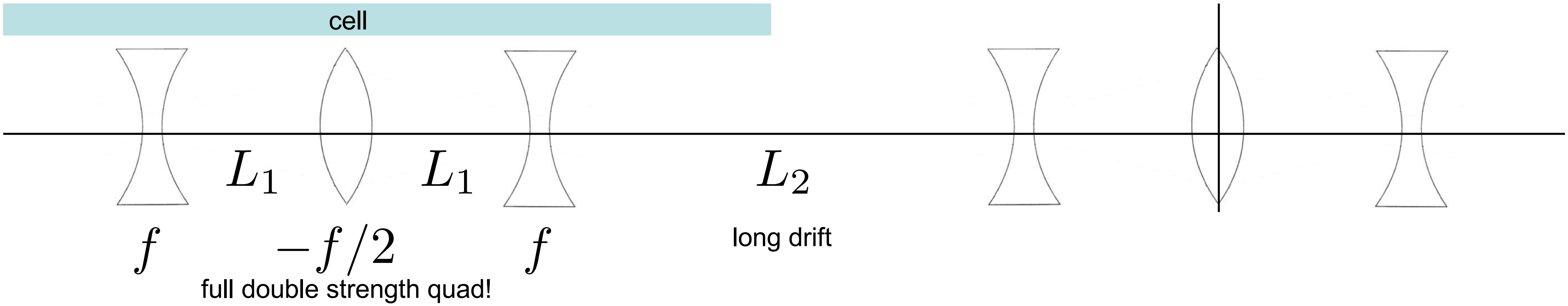
Smaller magnets

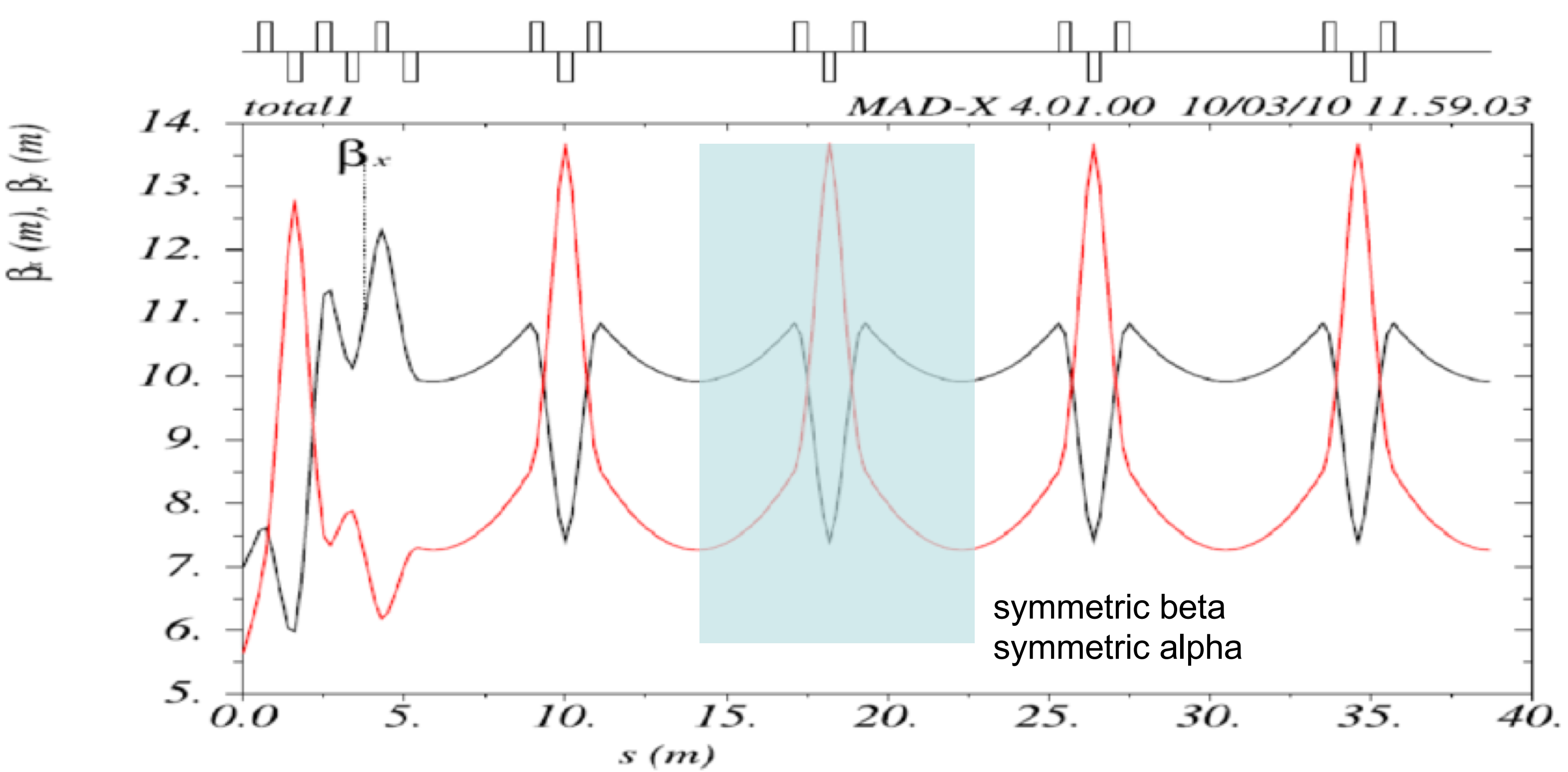
Less expensive accelerator



Here there be dragons

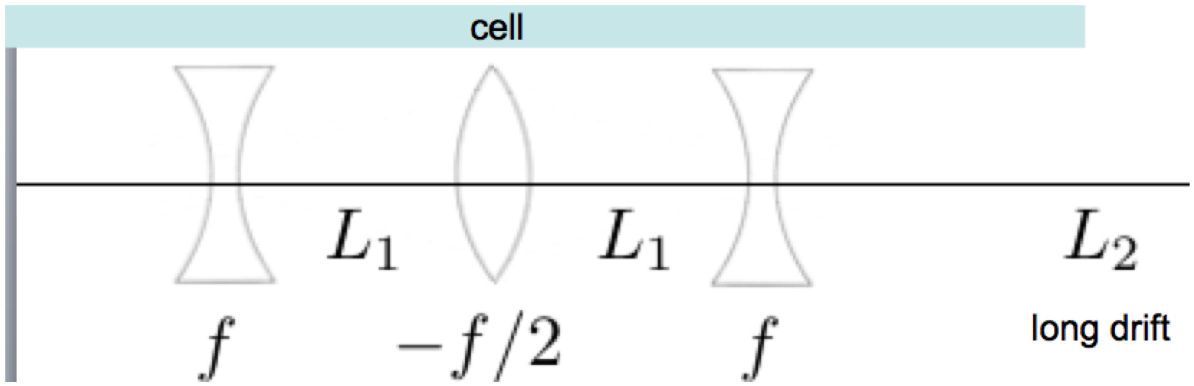
# Triplet Optics: Extra Straight Space




From R. Chehab et al., “The CLIC Positron Capture and Acceleration in the Injector Linac”, 2010.

# Triplet Cell Strategy: Not Exactly FODO

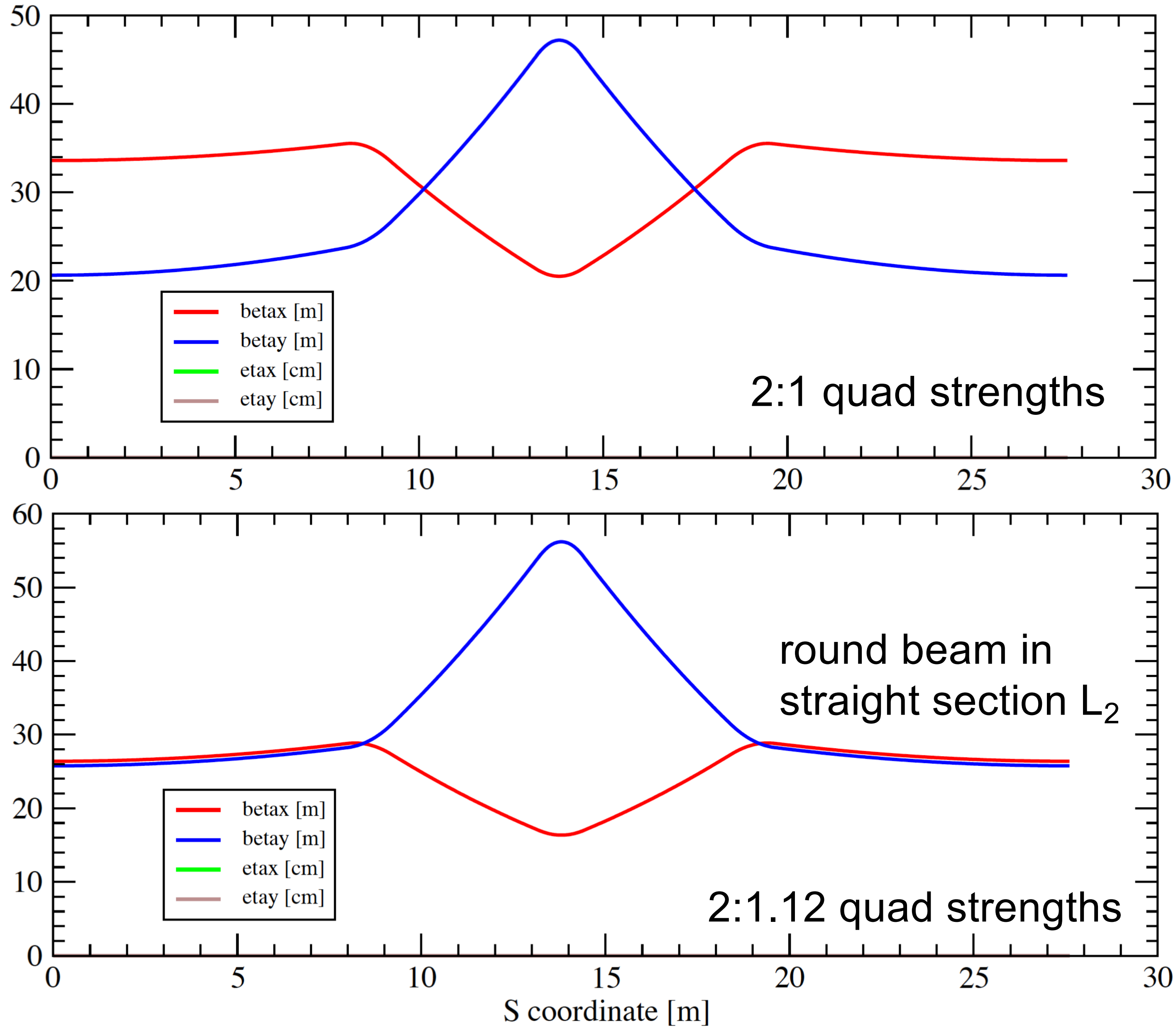



- Calculate transport matrix in terms of $L_1$, $L_2$, f
  - Three degrees of freedom
  - Can use our (now-familiar?) Twiss transport:

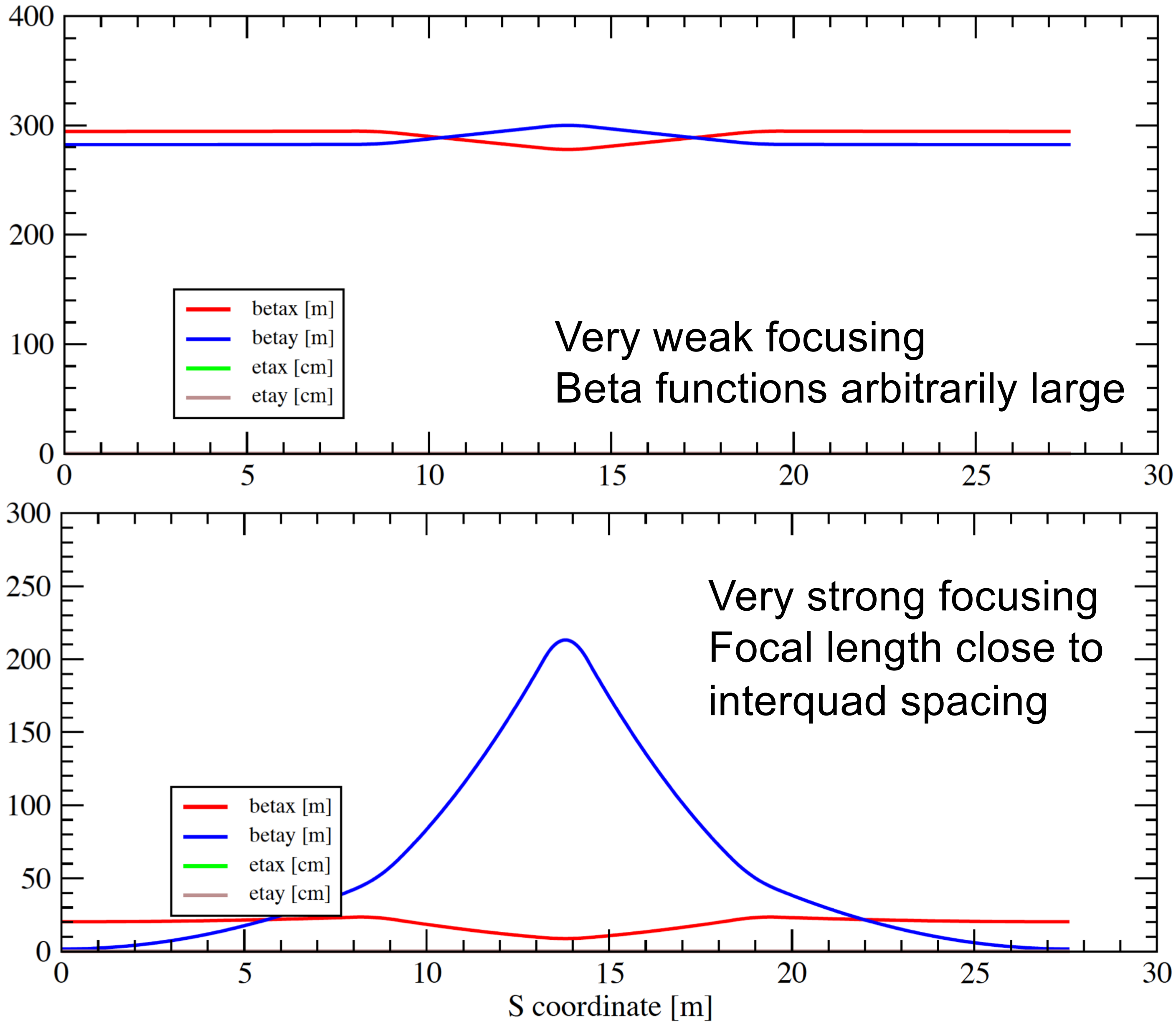
$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{21}m_{11} & 1+2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Emphasis on periodic solutions for repeating cells

# Triplet Focusing: Periodic Cell Examples

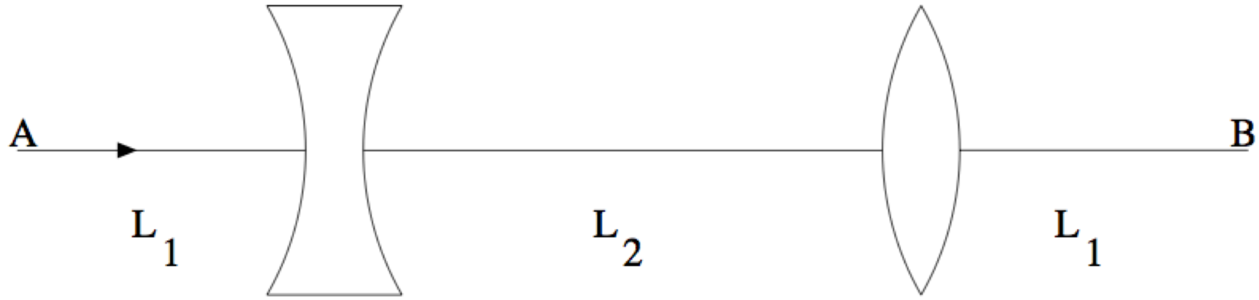


# Triplet Focusing: Absurd Extremes

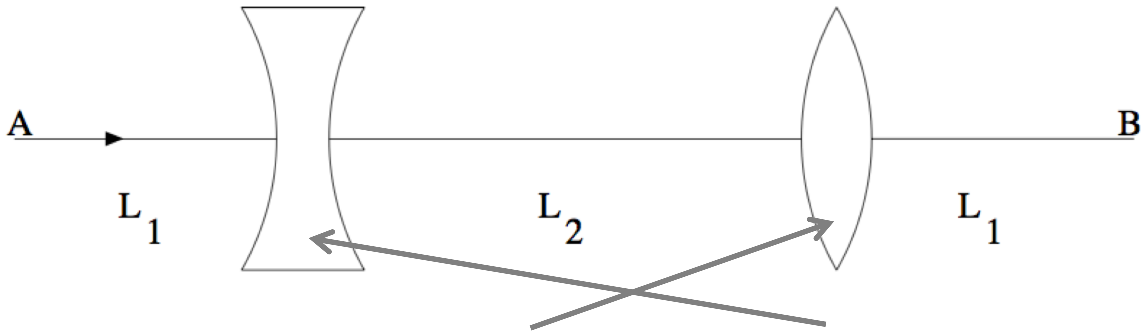


# π/2 Insertion

- Insertions and matching: **modular** accelerator design
- FODO sections have very regular spacings of quads
  - Periodicity of quadrupoles => periodicity of focusing
- But we may need some long quadrupole-free sections
  - RF, injections, extraction, experiments, long instruments
- Can we design a periodic "module" that fits in a FODO lattice with a long straight section, and matches to FODO optics?
  - Yes: the minimal periodic option is the π/2 insertion
  - Matching lattice functions $(\beta,\alpha)_{x,y}$ at locations A,B



# π/2 Insertion




$$M = \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + \frac{L_2}{f} - \frac{L_1 L_2}{f^2} & 2L_1 + L_2 - \frac{L_1^2 L_2}{f^2} \\ -\frac{L_2}{f^2} & 1 - \frac{L_1 L_2}{f^2} - \frac{L_2}{f} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

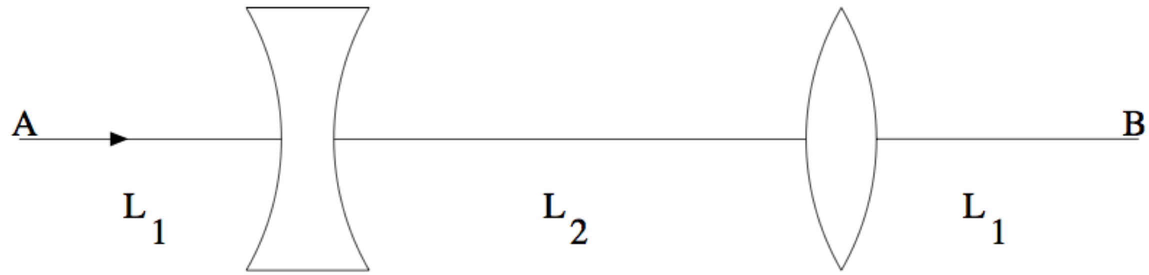
periodic boundary conditions

$$\cos\mu = 1 - \frac{L_1 L_2}{f} \qquad \beta \sin\mu = \left(2 - \frac{L_1 L_2}{f^2}\right) L_1 + L2 \qquad \gamma \sin\mu = \frac{L_2}{f^2}$$

$$m_{21} \text{ term}: \quad L_2 = f^2 \gamma \sin\mu \qquad (\text{recall } \gamma \equiv (1+\alpha^2)/\beta > 0)$$

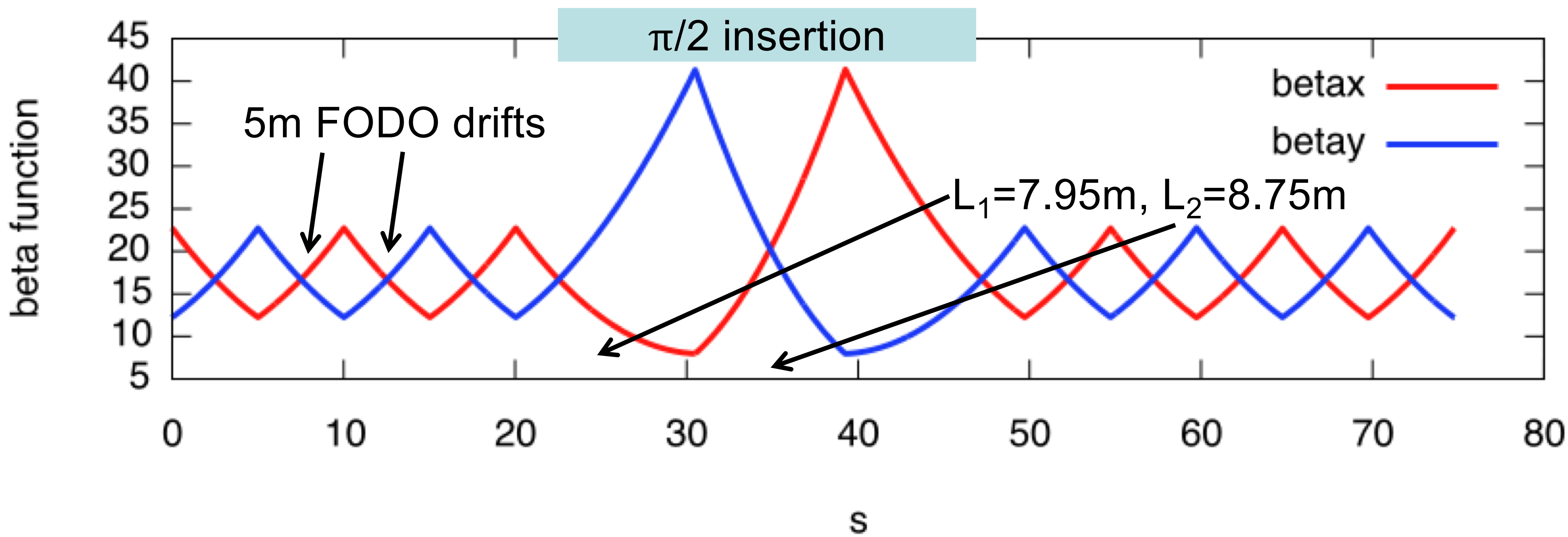
$$\text{Maximum } L_2 \text{ when} \qquad \sin\mu = 1 \qquad \mu = \frac{\pi}{2} \qquad \cos\mu = 0$$

# π/2 Insertion




$$\text{Design constraints} \ : \ f = \frac{\alpha}{\gamma} \quad L_2 = \frac{\alpha^2}{\gamma} \quad L_1 = \beta - L_2$$

$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J \qquad (\text{recall } J^2 = -I)$$



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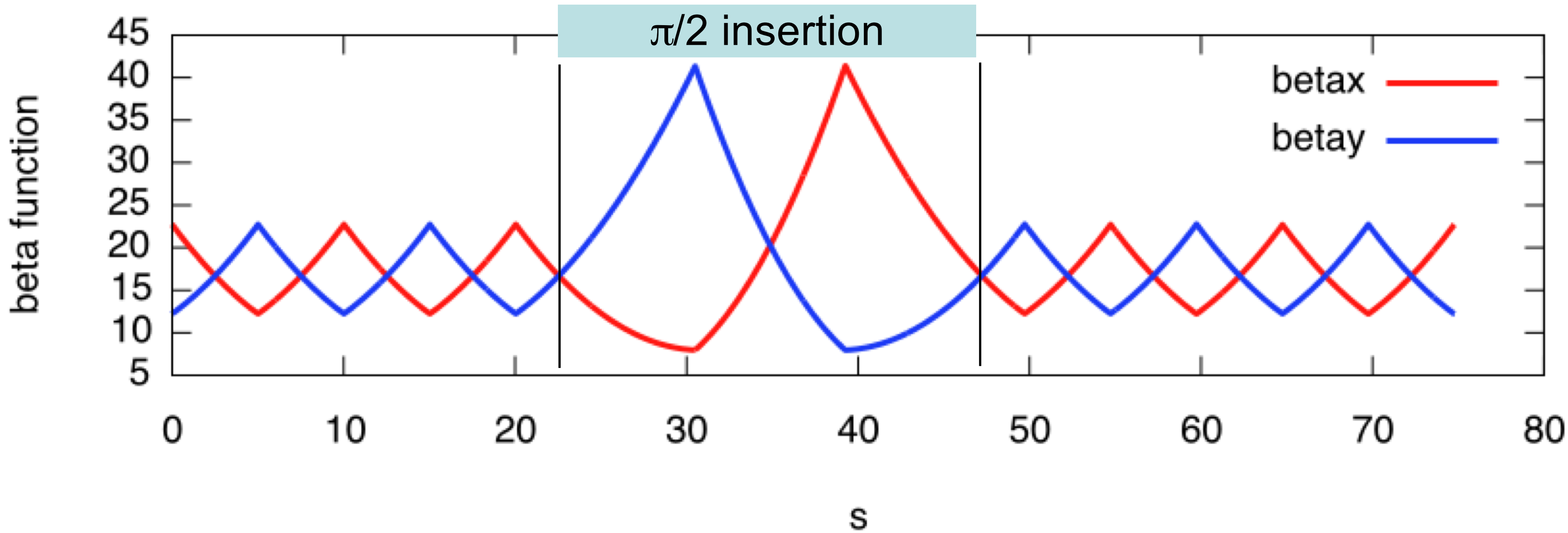
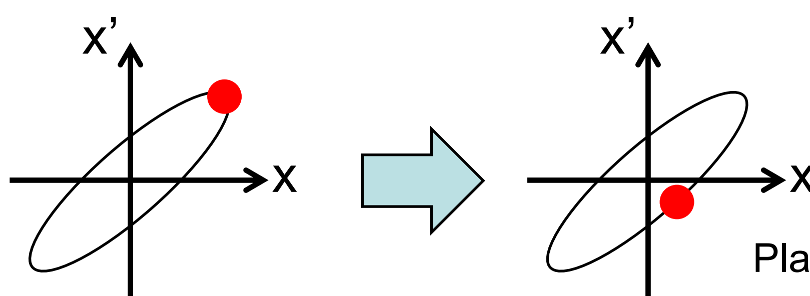


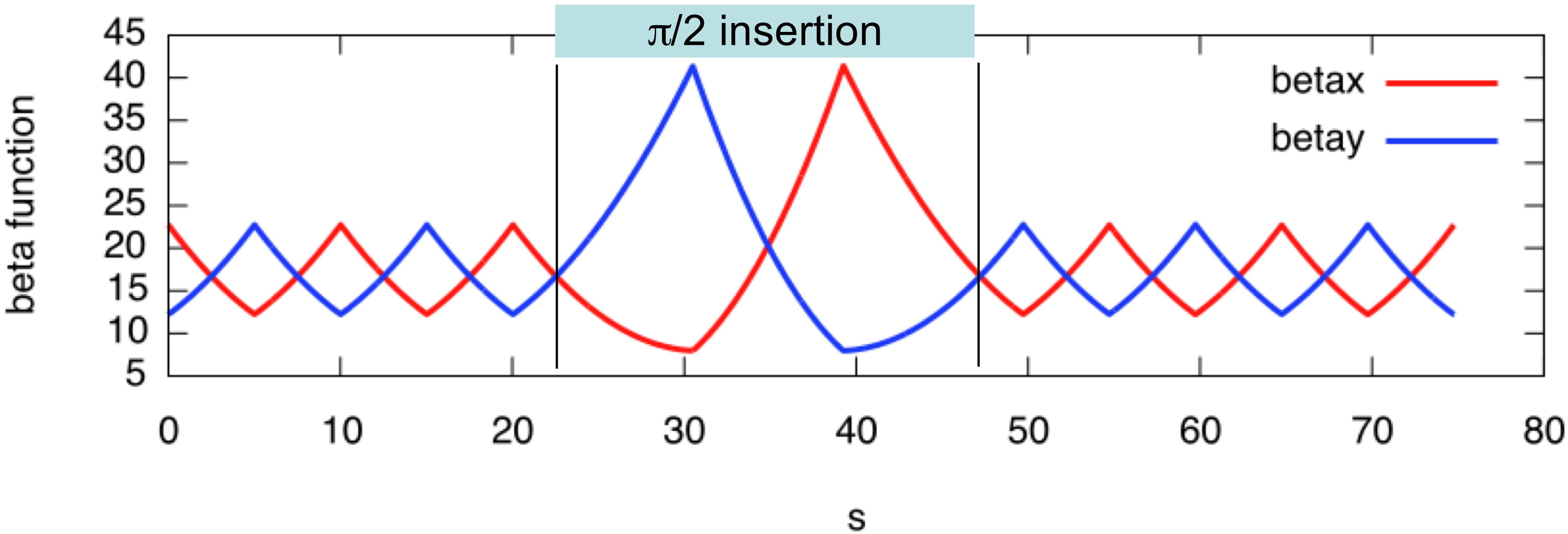

Particles advance 90 degrees (π/2) in phase in this insertion but the Twiss parameters are completely periodic




Play with Twiss propagation and components of $M_{\pi/2}$

# π/2 Insertion

$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J \qquad (\text{recall } J^2 = -I)$$

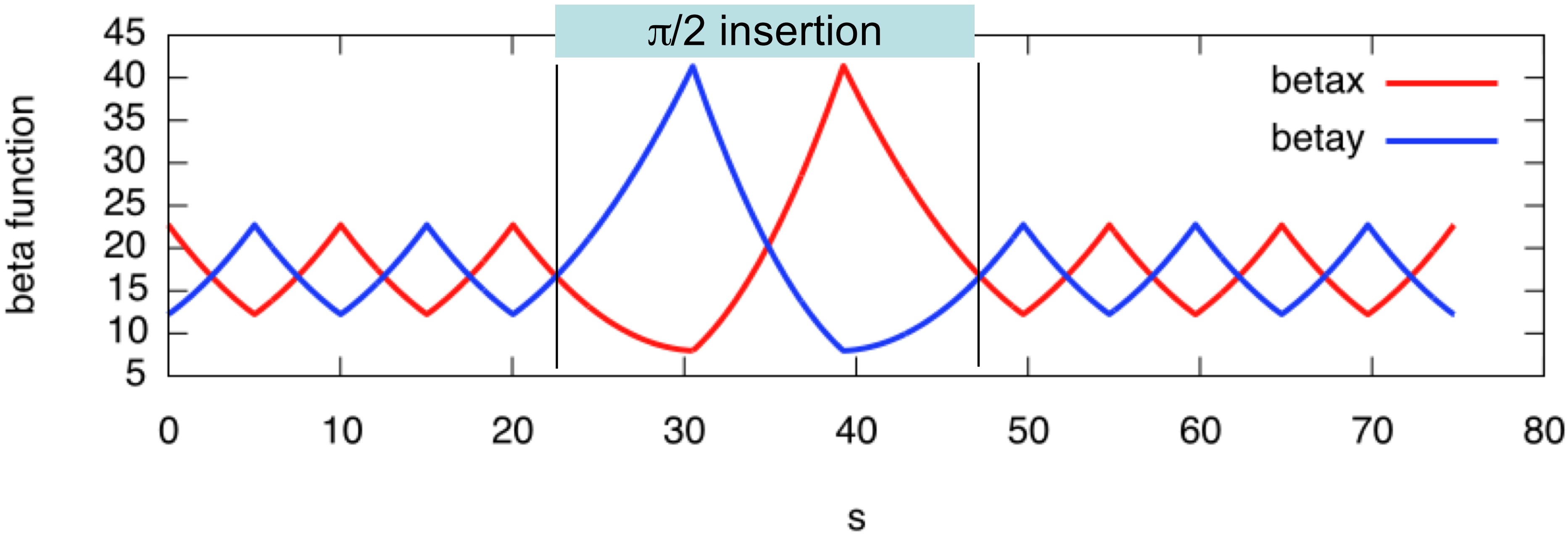



Q1: Why does this work for both planes even though we just designed for one plane?

Hint: $\text{Design constraints} \ : \ f = \frac{\alpha}{\gamma} \quad L_2 = \frac{\alpha^2}{\gamma} \quad L_1 = \beta - L_2$

# π/2 Insertion

$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J \qquad (\text{recall } J^2 = -I)$$

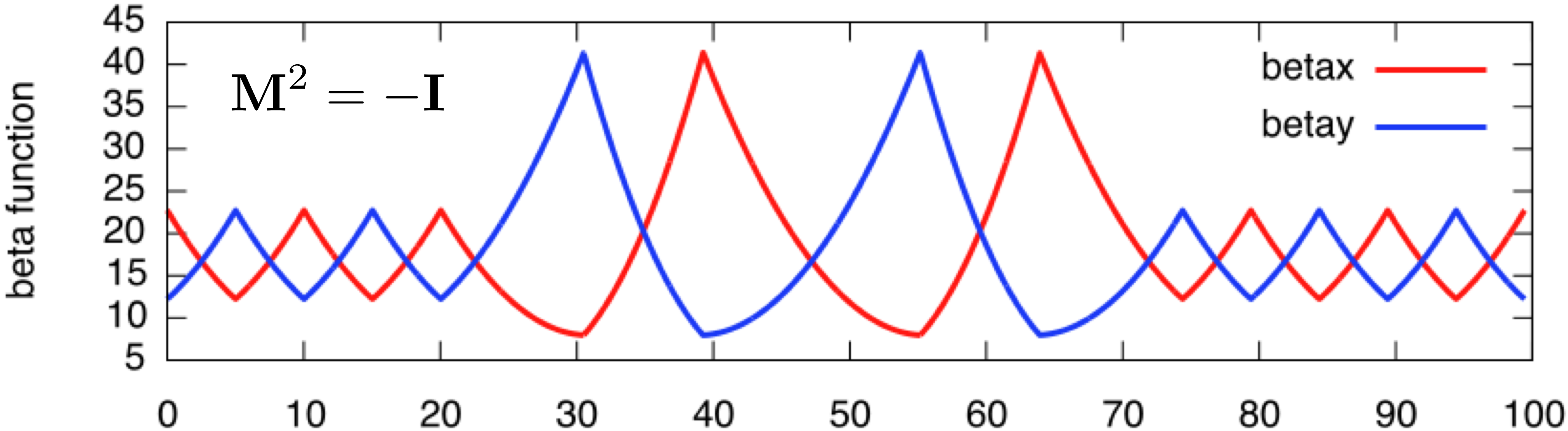


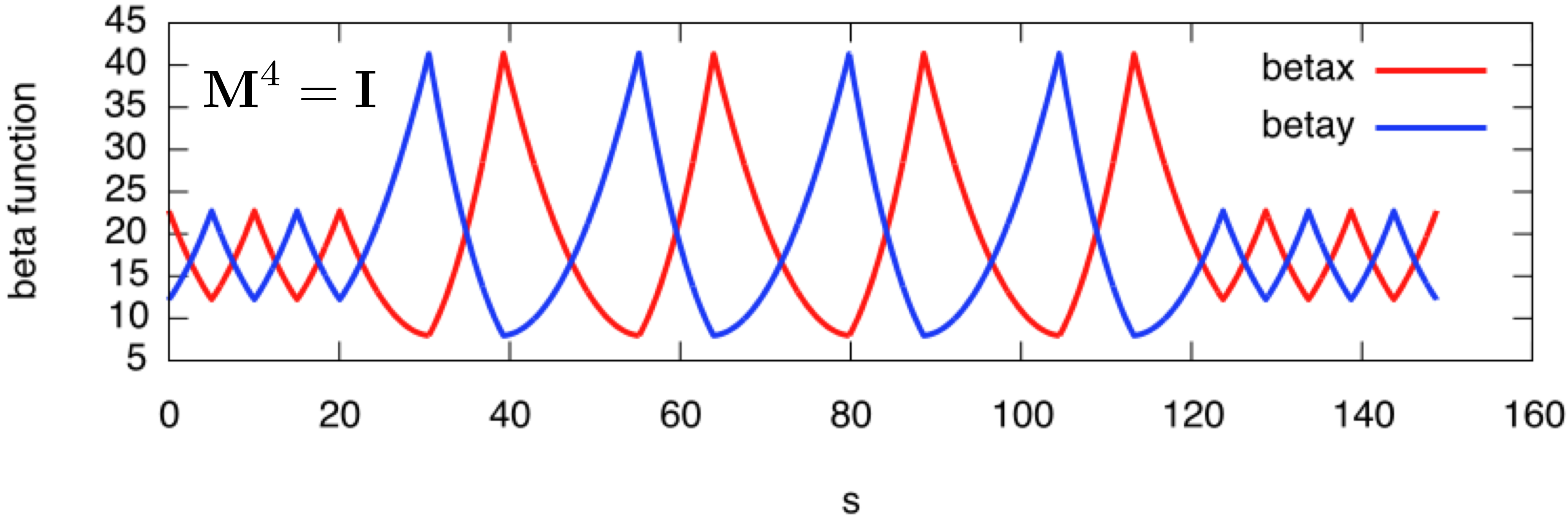

Q2: Can we set α=0 so this becomes an (x,x') exchanger?

$$M_{\mathrm{xx'\ exchange}} = \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix}$$

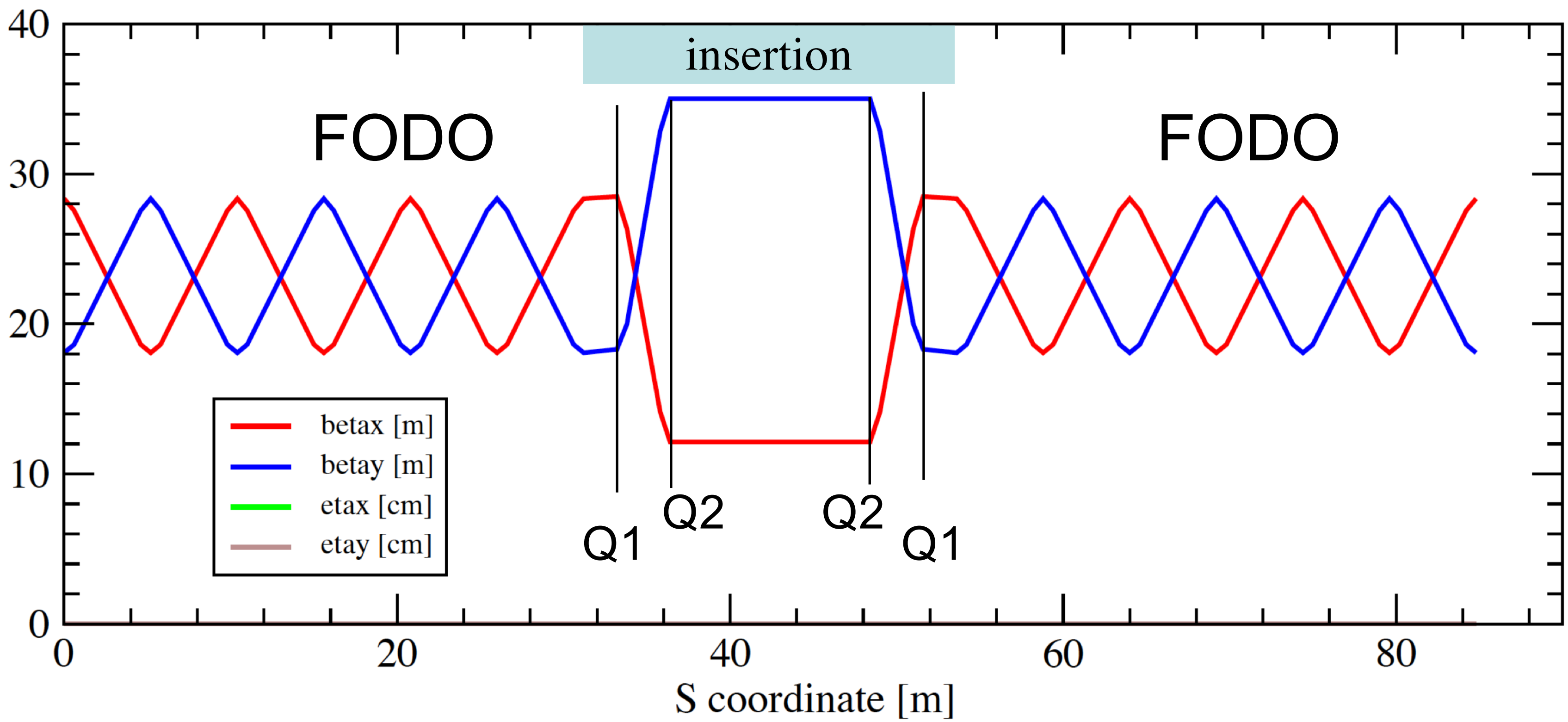
Hint: $\text{Design constraints} \ : \ f = \frac{\alpha}{\gamma} \quad L_2 = \frac{\alpha^2}{\gamma} \quad L_1 = \beta - L_2$

# Multiple π/2 Insertions



# Symmetric Two-Doublet Insertion




Q: What does the symmetry imply for optics behavior?
Q: What about an antisymmetric two doublet insertion?

# (From (x,x') Exchange to (x,y) Exchange)

- The π/2 solution prompted a question about (x,x') exchange
- Steve briefly discussed coupling from a theoretical and practical standpoint yesterday...
- Q: is it possible to construct a lattice insertion that exchanges horizontal and vertical phase spaces?
- A: Yes. This was developed in the 90s at Cornell and is called a Mobius insertion.
  - Could "trivially" be implemented with a very long solenoid

# (Mobius Insertion)

- Fully coupled equal-emittance optics for $e^+e^-$ CESR collisions (round beam $e^+e^-$ collisions)
  - Symmetrically exchange horizontal/vertical motion in insertion
  - Horizontal/vertical motion are coupled
    - Only one transverse tune degree of freedom!

$$Q_{x,y}: \text{ unrotated tunes} \qquad Q_{1,2} = \frac{Q_x + Q_y}{2} \pm \frac{1}{4} \qquad Q_1 - Q_2 = \frac{1}{2}$$

  - Match insertion to points where $\beta_x = \beta_y$ and $\alpha_x = \alpha_y$ with phase advances that differ by $\pi$ between planes
    - Normal insertion: $\mathbf{M}_{\text{erect}} = \begin{pmatrix} \mathbf{T} & 0 \\ 0 & -\mathbf{T} \end{pmatrix}$
    - Rotated by 45 degrees around s axis: $\mathbf{M}_{\text{mobius}} = \begin{pmatrix} 0 & \mathbf{T} \\ \mathbf{T} & 0 \end{pmatrix}$
  - A purely transverse example of an **emittance exchanger**

S. Henderson, R. Talman, et al., "Investigation of the Möbius Accelerator at CESR", Proc. of the 1999 Particle Accelerator Conference, New York, NY; R. Talman, "A Proposed Möbius Accelerator", Phys. Rev. Lett **74**, 1590-3 (1995).

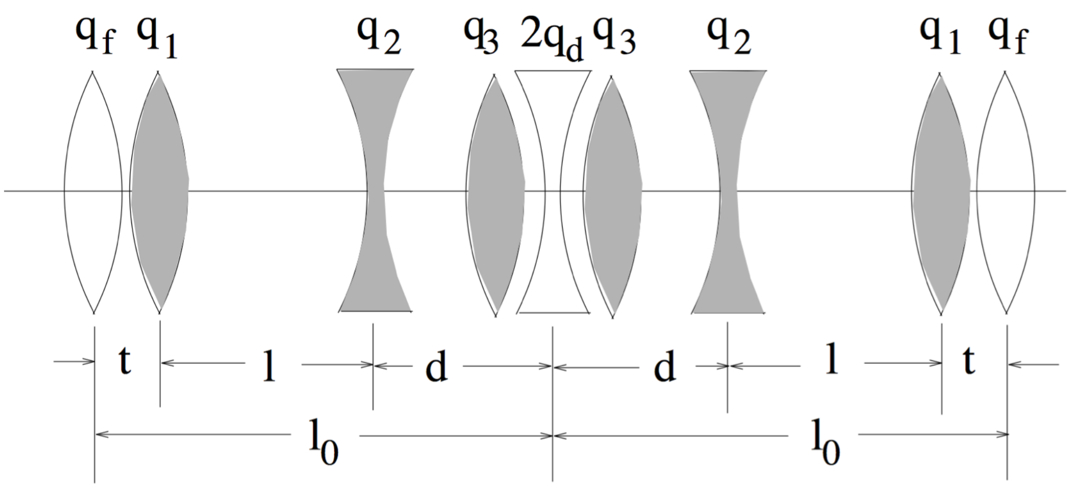
# (Talman 1993 Mobius Paper)

- https://www.classe.cornell.edu/public/CBN/1993/Mobius.ps

**The MÖBIUS ACCELERATOR**

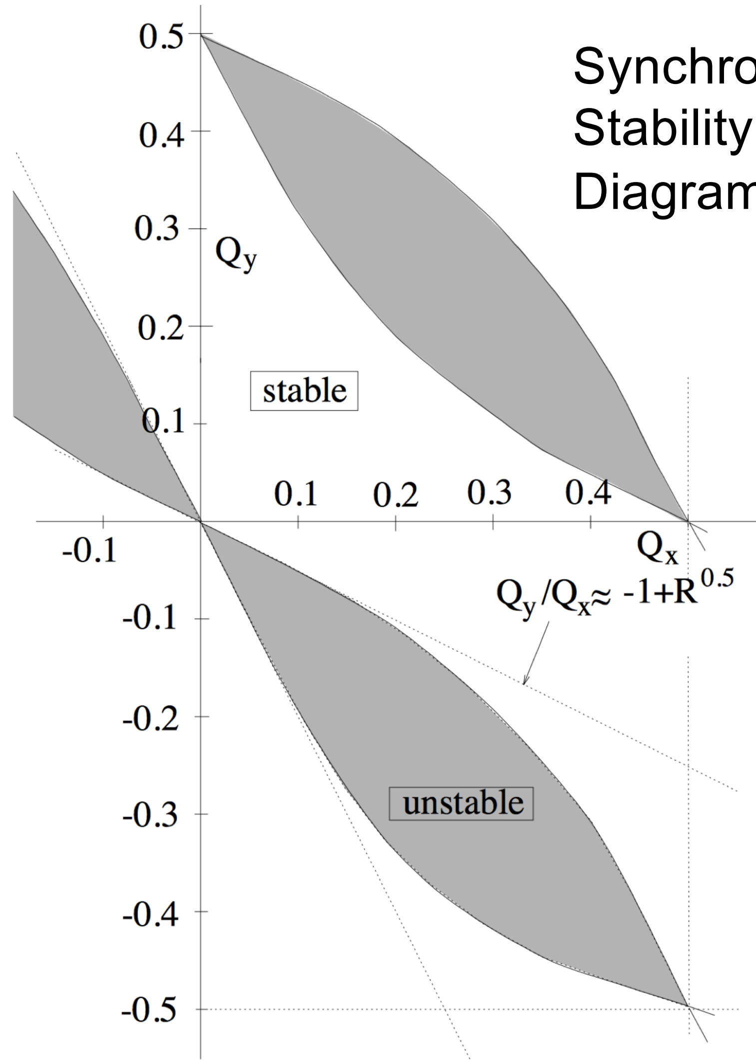

**Richard Talman**

Laboratory of Nuclear Studies
Cornell University
Ithaca, NY 14853


Shaded = Skew quadrupole

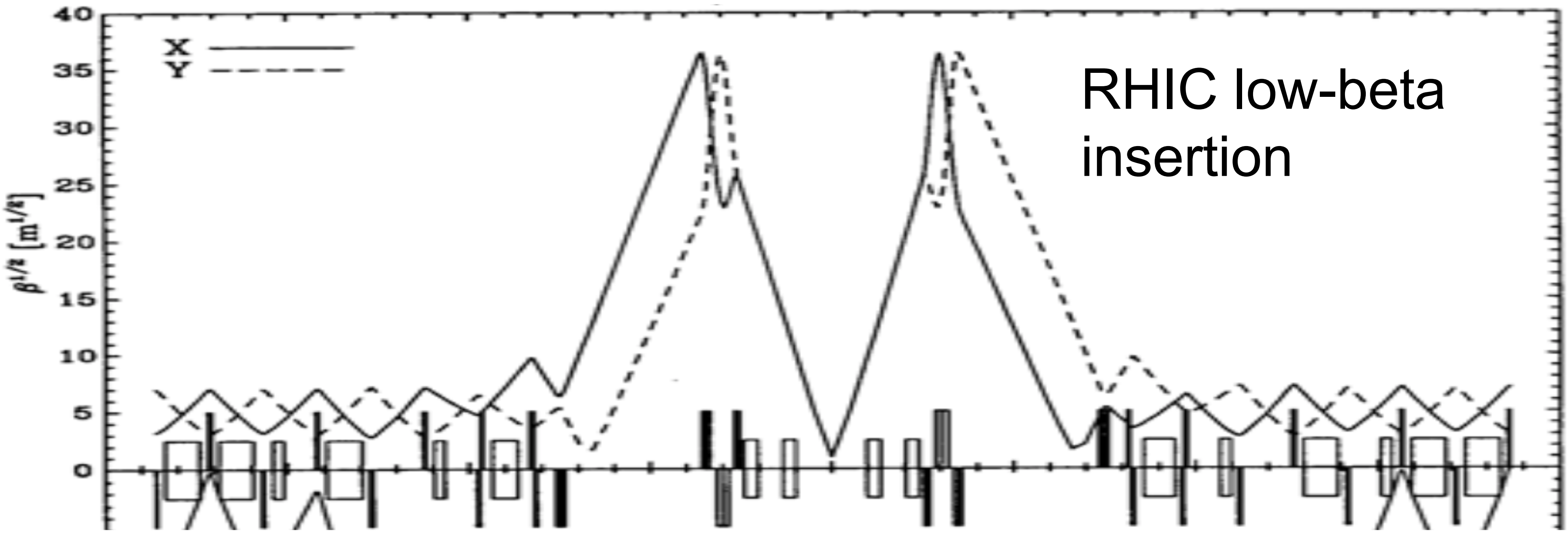
**Figure 2:** Lattice section needed to switch between ordinary and Möbius operation. For ordinary operation the unshaded elements are run as normal equal-tune FODO elements. For Möbius operation the central element $q_d$ is turned off and the shaded, skew quadrupole elements are powered.




Synchrotron Stability Diagram

# Low-beta Insertions: intro

- We have one final “dipole-free” insertion to discuss
  - (In practice it may well not be dipole free)
- Low beta insertions are fundamentally quads that focus into special long drift spaces
  - To this point we have avoided overfocusing
  - Low beta insertions are **intentionally** overfocused to create a **minimum beam size** (or waist) in a drift

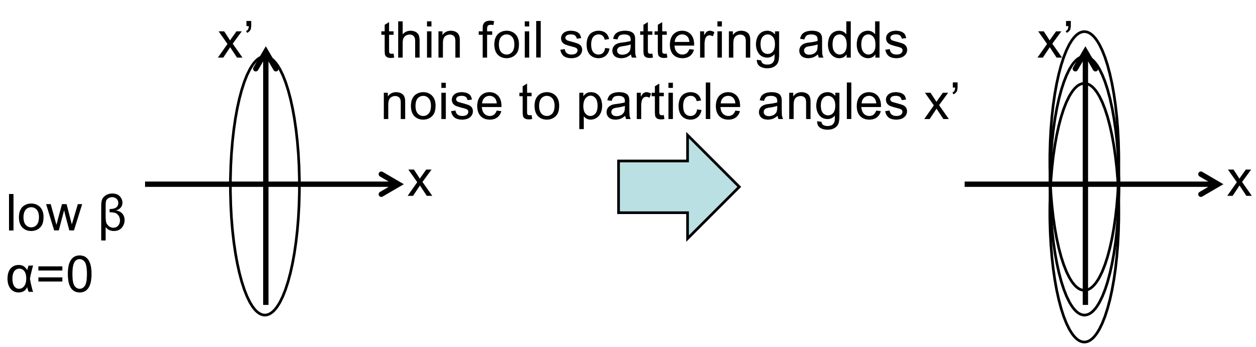


# Low-beta Insertions: Uses

- Low-beta is most famously used to maximize collider luminosity by minimizing beam size at interaction point

$$L = f_{rev}\, M \frac{N^2}{4\pi \sigma_H^* \sigma_V^*} \qquad (1.11)$$

- Also used to maximize beam divergence
  - Minimizes emittance growth from interactions with materials, e.g. ion stripping foils or diagnostic screens



# Low-beta Insertions

- Recall class comments from Steve about β evolution and phase advance in a drift

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star} \qquad \begin{aligned} \alpha^\star &= 0 \\ \gamma^\star &= 1/\beta^\star \end{aligned}$$

where β* is the minimum value of β and s is the s-coordinate distance from this minimum

Smaller $\beta^\star$ gives steeper parabolic increase!

β must be quite large at the quadrupoles surrounding the low-beta insertion to create a small β*

$$\text{Phase advance across straight section : } 2\arctan\left(\frac{L_{\text{insertion}}}{\beta^\star}\right)$$

$$\text{For } \mathrm{L}_{\text{insertion}} \gg \beta^\star, \;\; \text{, phase advance is } \pi$$

# (Low-beta Insertion guidelines)

1. Calculate the periodic solution in the arc
2. Start from the IP, introduce the drift space needed for the insertion device (detector ...)
3. Install a quadrupole triplet (or doublet?) fix the aperture requirements and the achievable field gradient
4. Set the desired beta*, drive the triplet at high field, so that the beam is focused back
5. Introduce additional quadrupoles to match the beam parameters to the values at the beginning of the arc

Parameters to be optimized & matched to the periodic solution:

$$\begin{matrix} \beta_x & \alpha_x & D_x & \mu_x \\ \beta_y & \alpha_y & D_y & \mu_y \end{matrix}$$

Use a code (e.g. madx) to optimize and match!

($D'$ is normally accepted at the IP)

8 (at least) individually powered quad magnets are needed to match the insertion

Slide from D. Pellegrini, 2018 USPAS course

# (Combining Beam Separation and Low Beta Quads)

Both dipoles and quadrupoles need to be close to the IP, not always integrable into the detector.

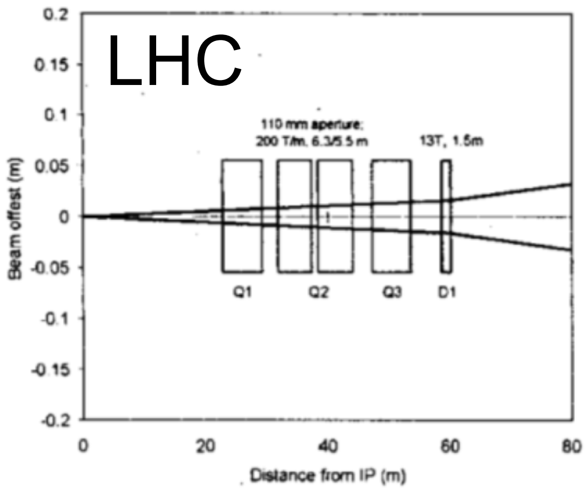



Figure 1: Quadrupole-first IR.

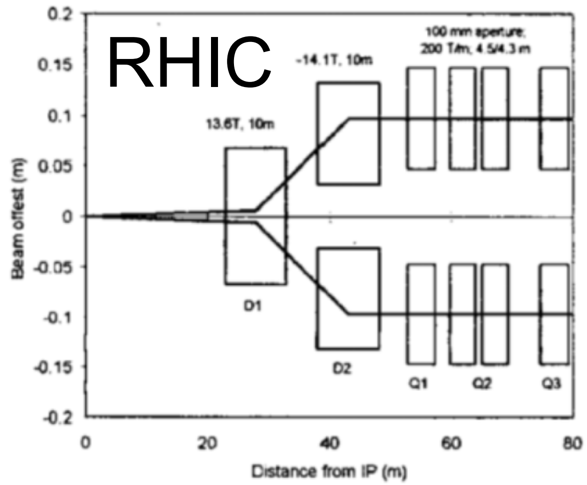



Figure 2: Dipoles-first IR.

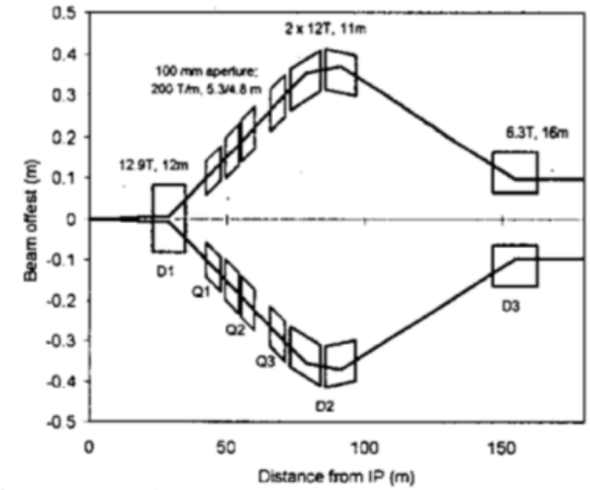



Figure 3: IR with quads between the separation dipoles.

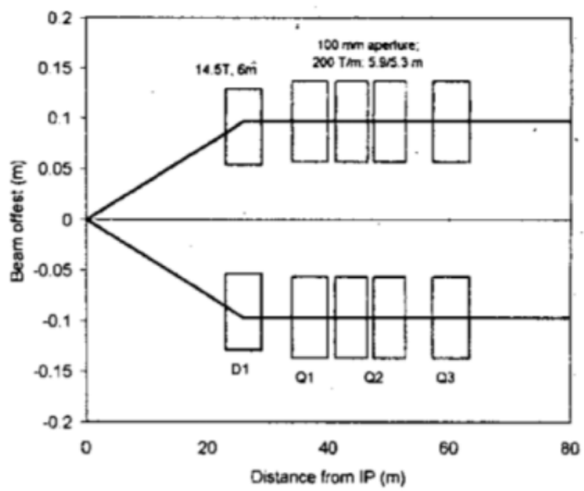



Figure 4: Dipole-first IR with large crossing angle.

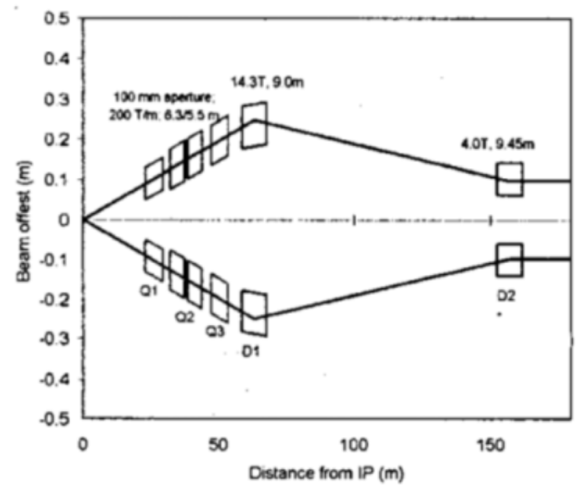



Figure 5: Quadrupole-first IR with large crossing angle.

LHC went for case 1

J. Strait et al., proceedings of PAC 2003

Slide from D. Pellegrini

# (Dispersion Review)

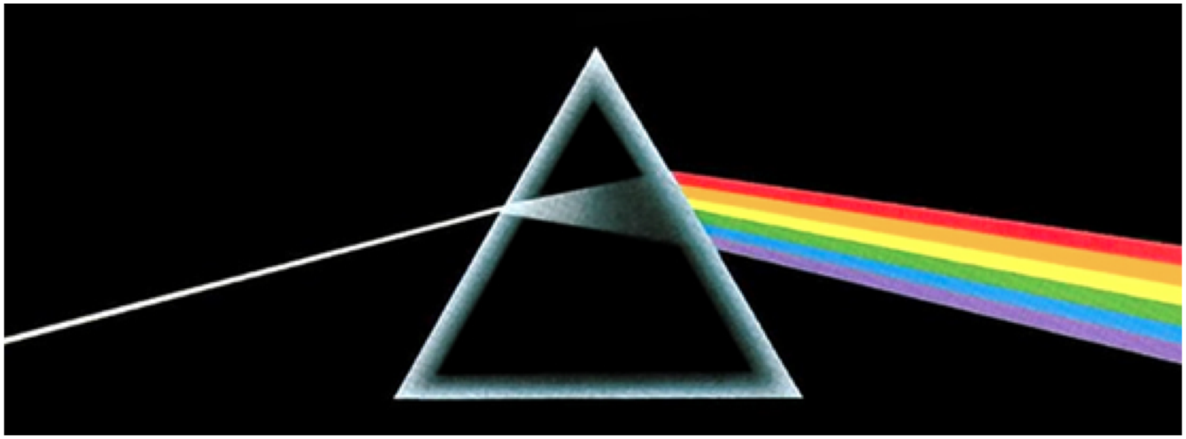
- Review and reformulation of Tuesday PM material (a long time ago)
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

$$x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$$

Dispersion originates from momentum dependence of dipole bends

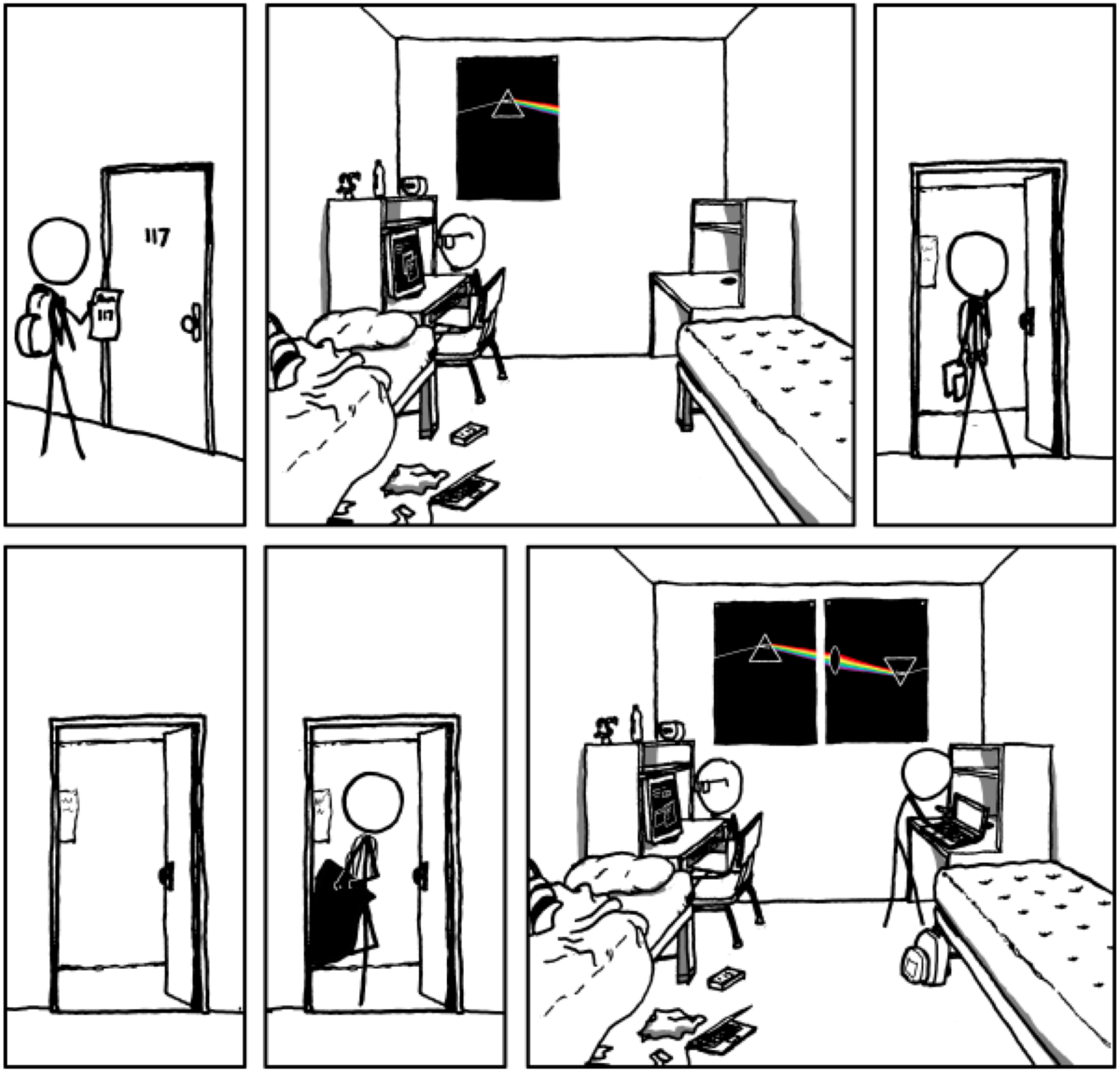
Equivalent to separation of optical wavelengths in prism

White light with many frequencies (momenta) enters, all with same initial trajectories (x,x')



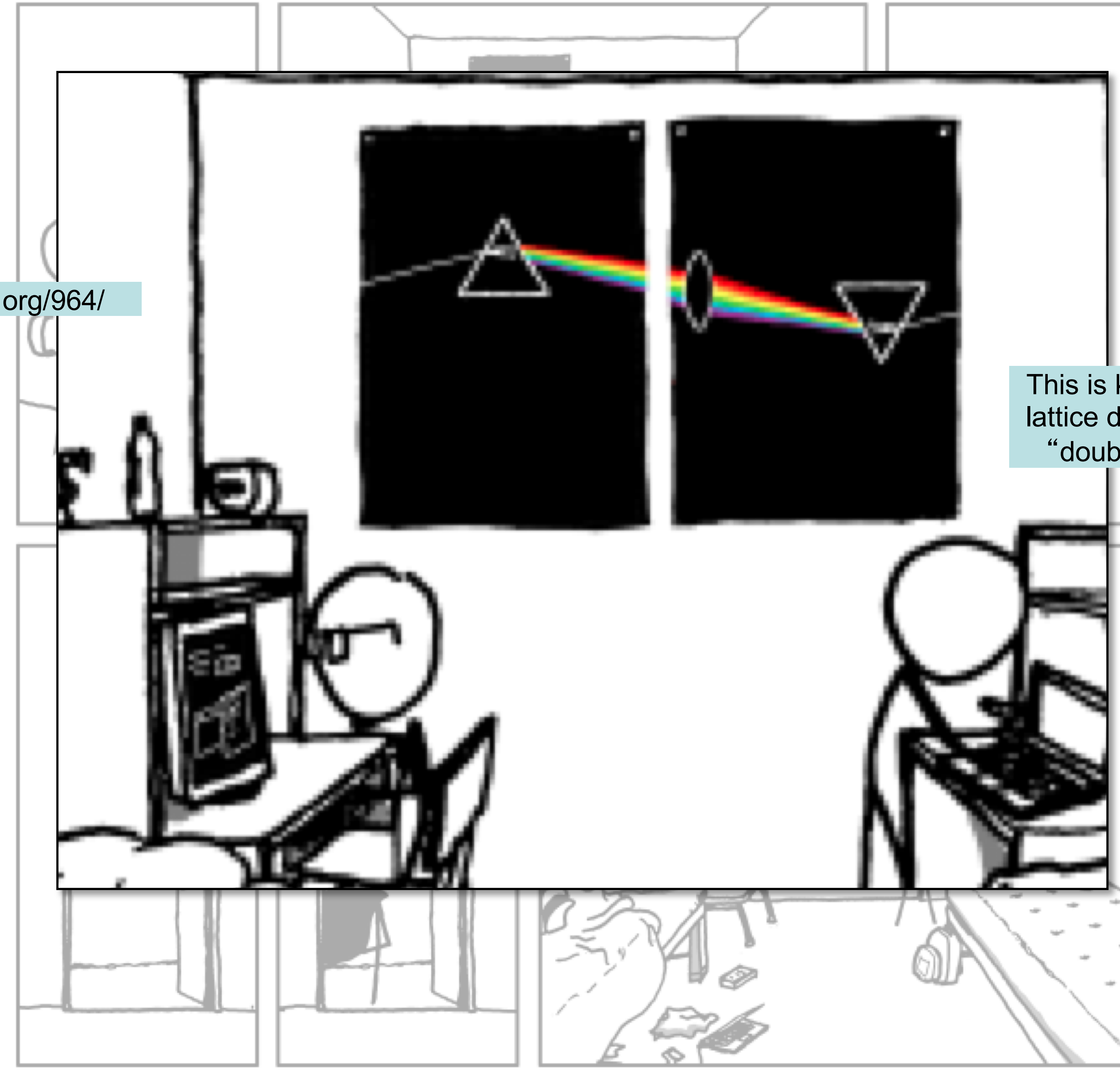
Different positions due to different bend angles of different wavelengths (frequencies, momenta) of incoming light

# (xkcd interlude)



# (xkcd interlude)

http://www.xkcd.org/964/



This is known in accelerator lattice design language as a “double bend achromat”

# (Dispersion Review)

- Add explicit momentum dependence to equation of motion

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

Perturb our zero-dispersion solution to find

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

The trajectory has two parts:

$$x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$$

# (Dispersion Review)

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat} : D = D' = 0$$

  - If we take $\delta_0 = 1$ we can solve this in a clever way

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

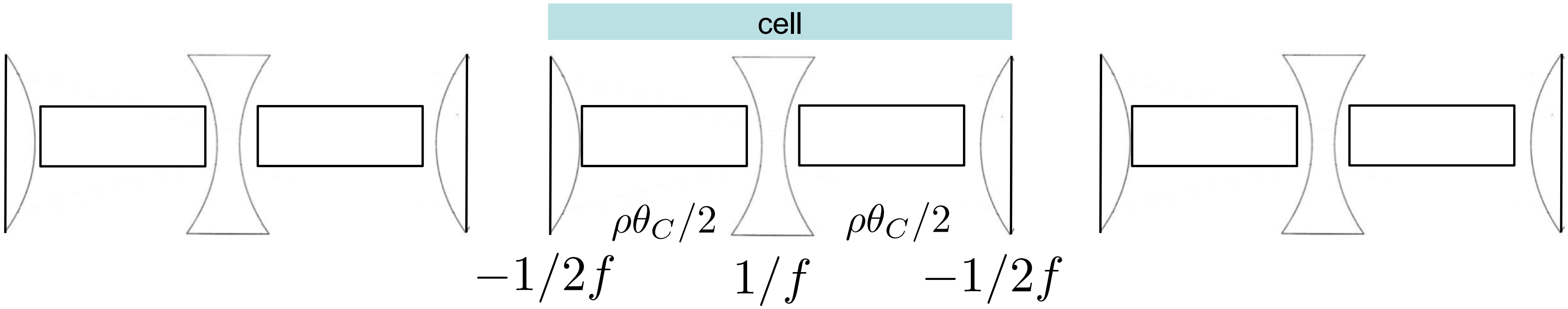
$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

  - Solving gives

$$\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\mu)}$$

$$\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\mu)}$$

# FODO with dipoles




- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

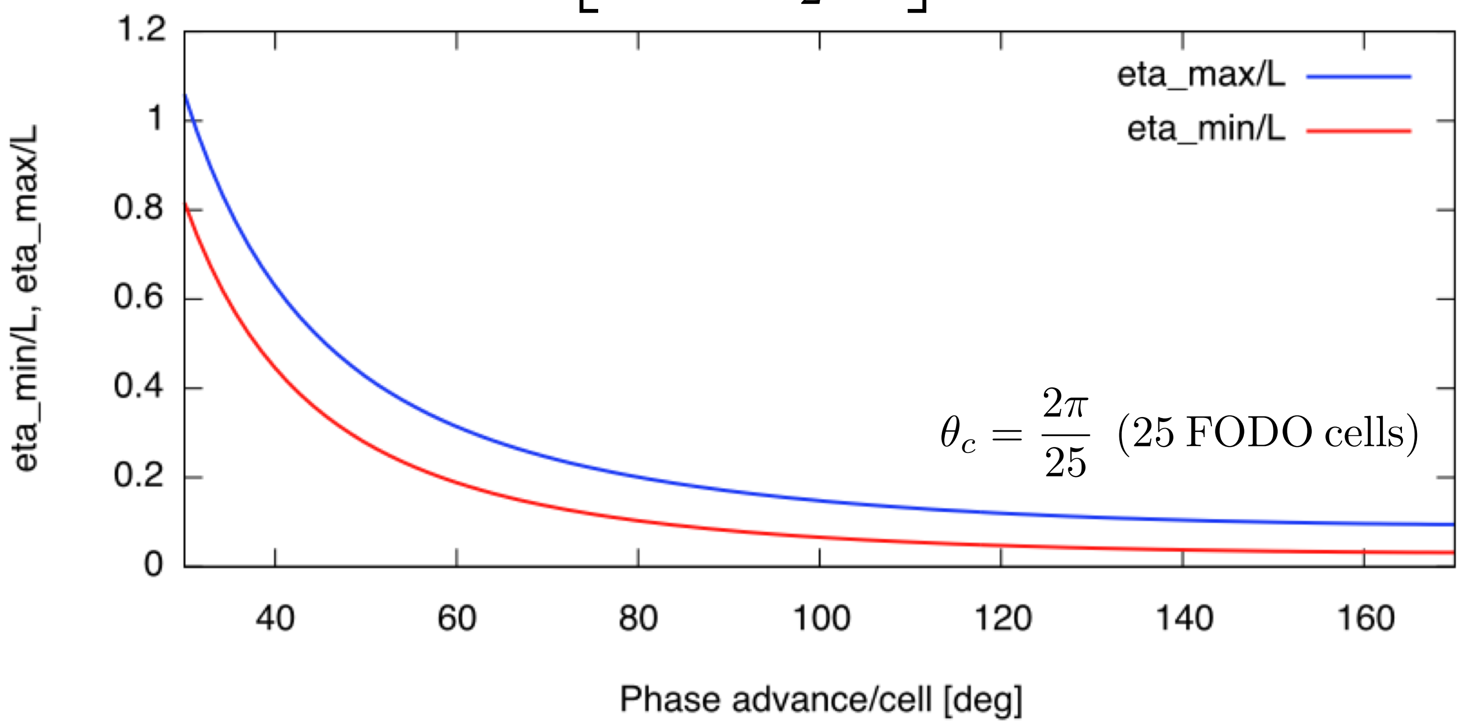
# FODO with dipoles

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

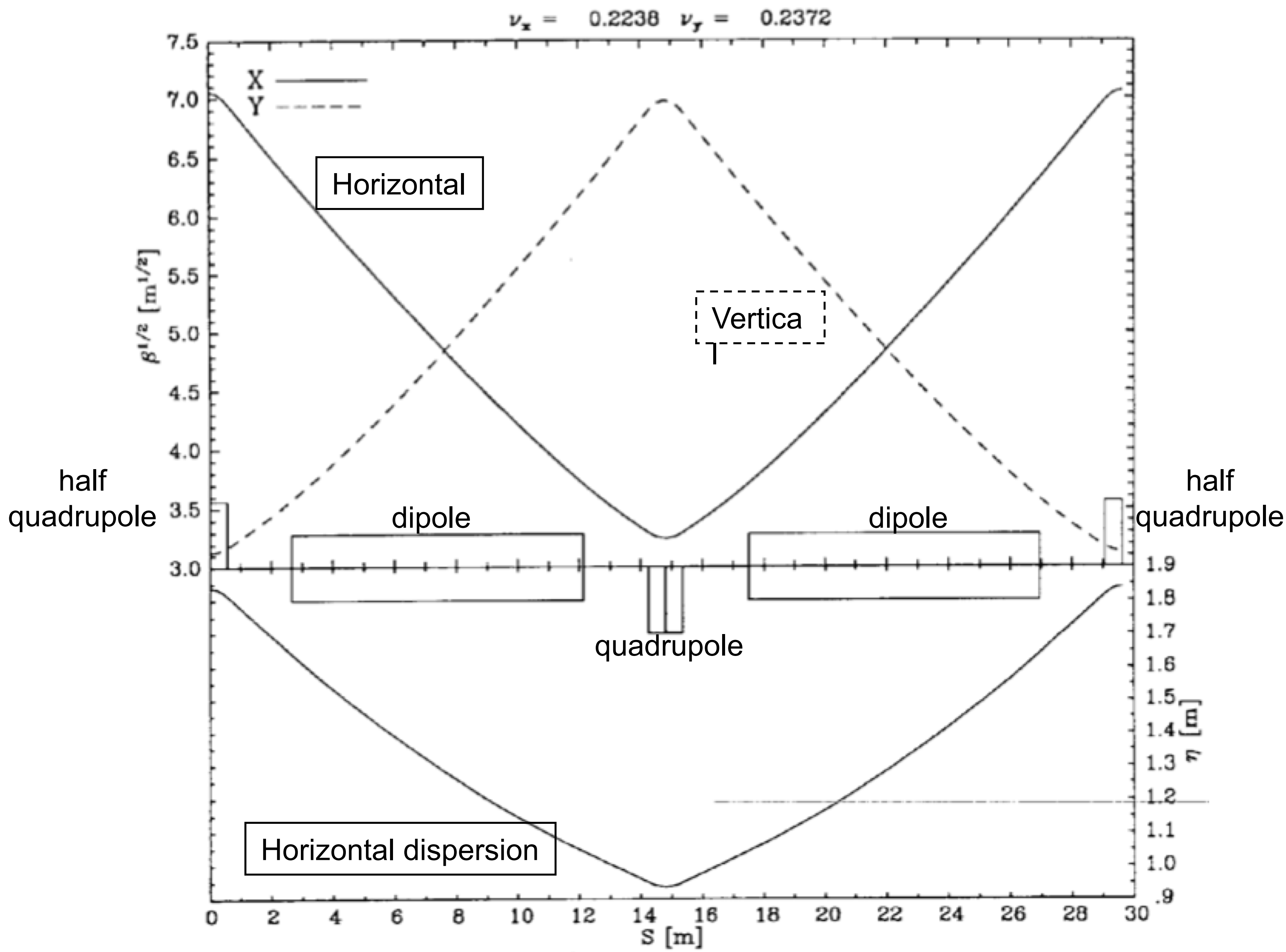
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at min}$$

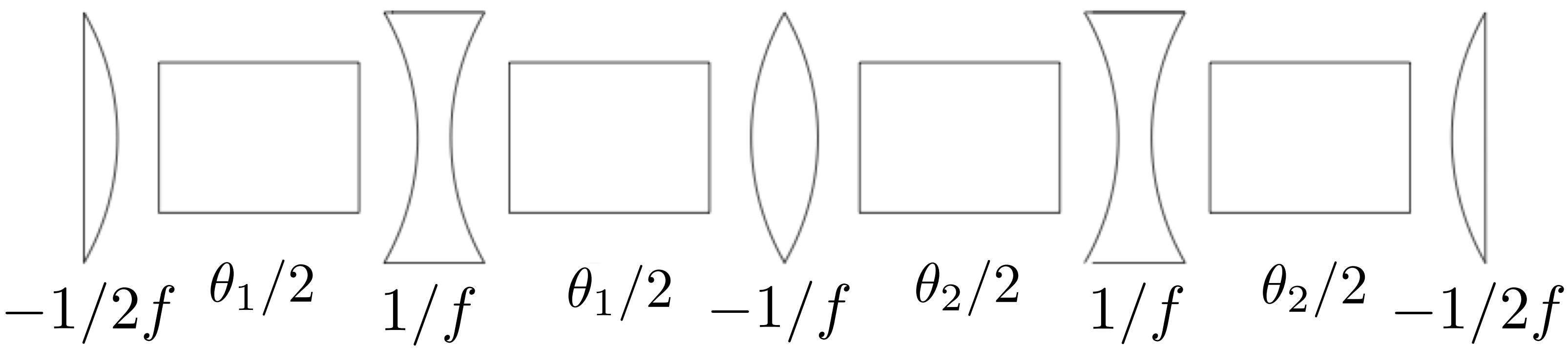


# RHIC FODO Cell



# Dispersion suppressors

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO $(\eta_x, \eta'_x)$ to zero.




  - Consider two FODO cells with different total bend angles $\theta_1, \theta_2$
    - Same quadrupole focusing to not disturb $\beta_x,\ \mu_x$ much
    - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
    - $\alpha_x = 0$ at ends to simplify periodic matrix

# Dispersion suppressors

Zero dispersion area

slope η' =0

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\mu_x & \beta_x \sin 2\mu_x & D(s) \\ -\frac{\sin 2\mu_x}{\beta_x} & \cos 2\mu_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

FODO peak dispersion,

slope η' =0

multiply matrices $\Rightarrow$

$$D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

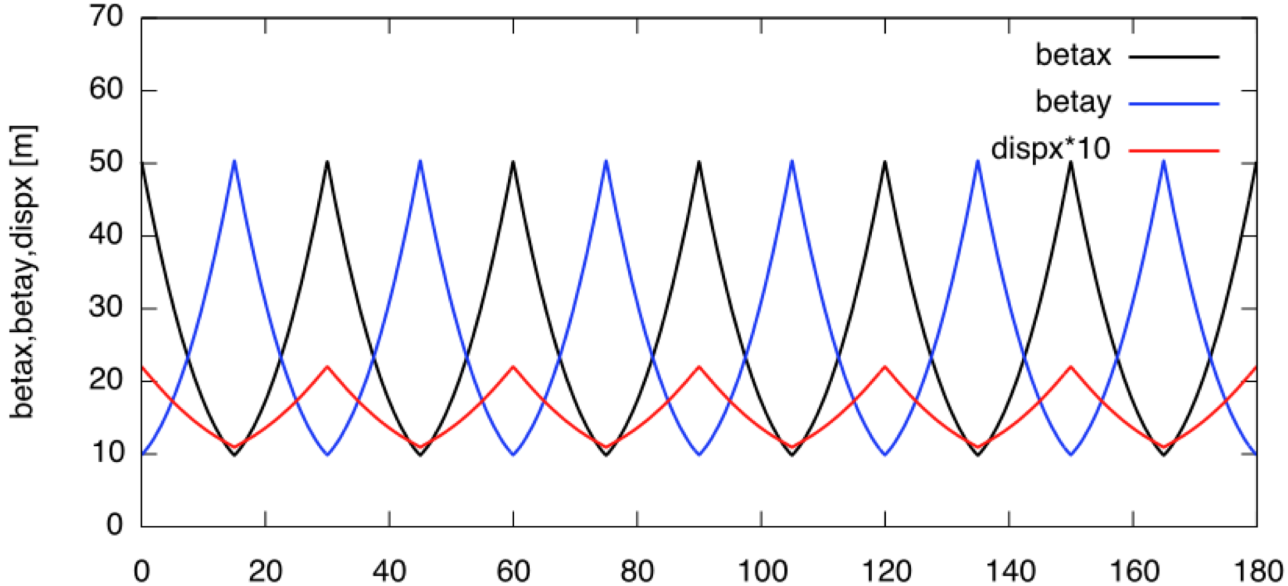
$$D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

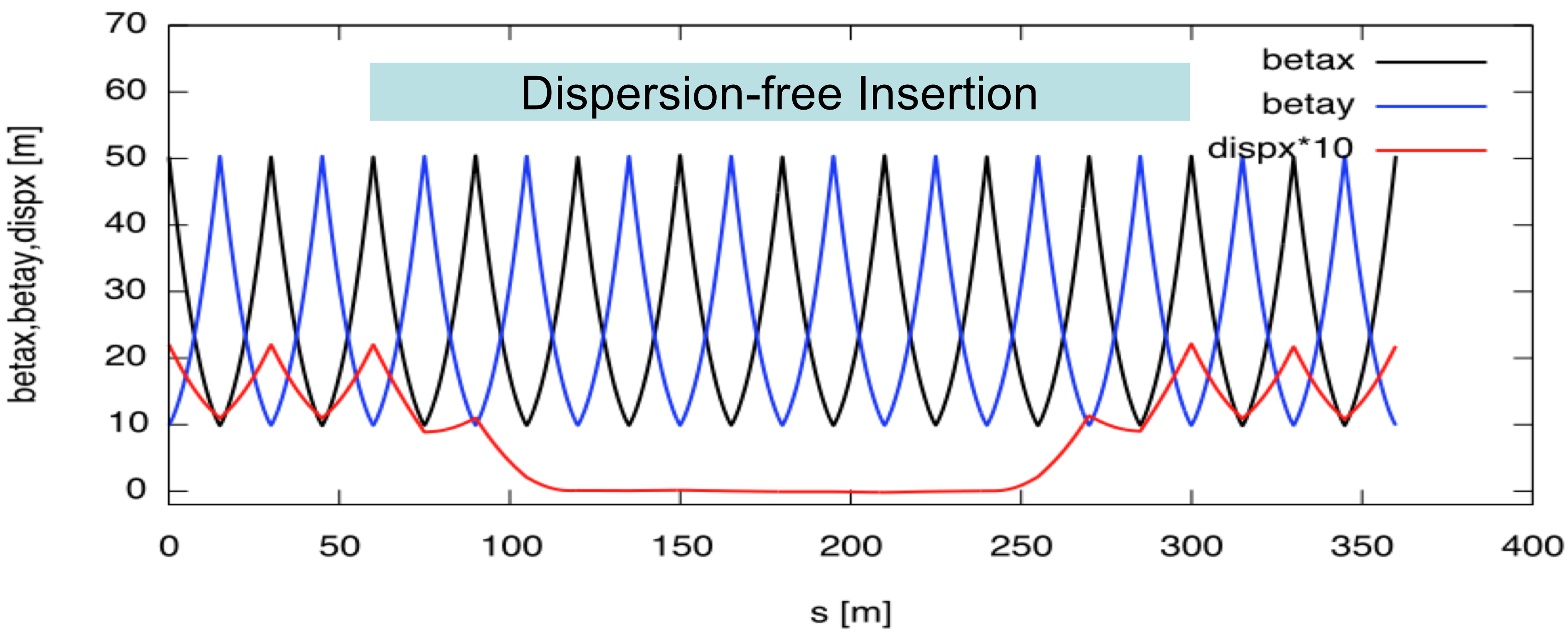
$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

$$\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta$$

$$\theta = \theta_1 + \theta_2$$ two cells, one FODO bend angle $\rightarrow$ reduced bending

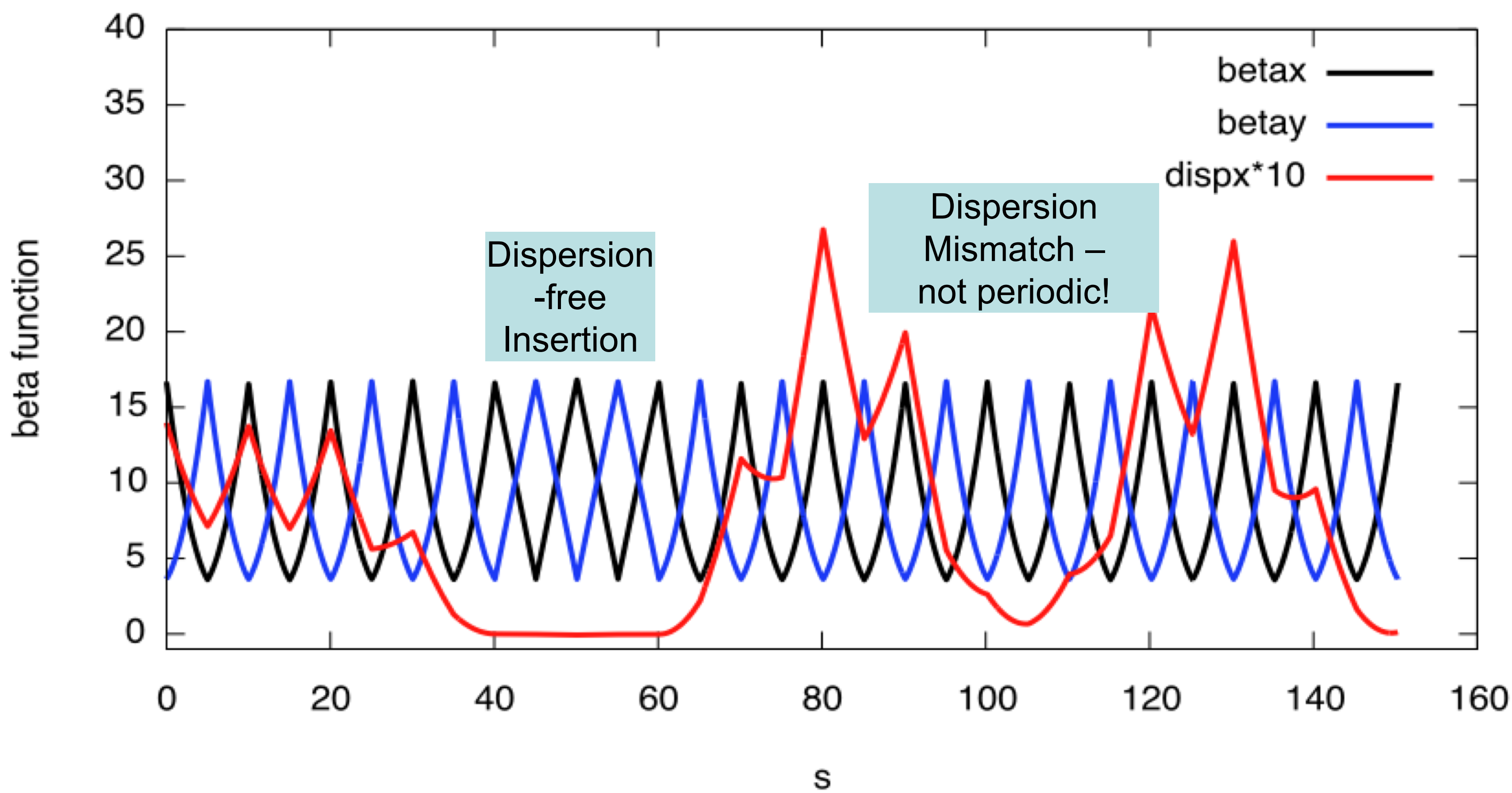
# FODO Cell Dispersion and Suppressor



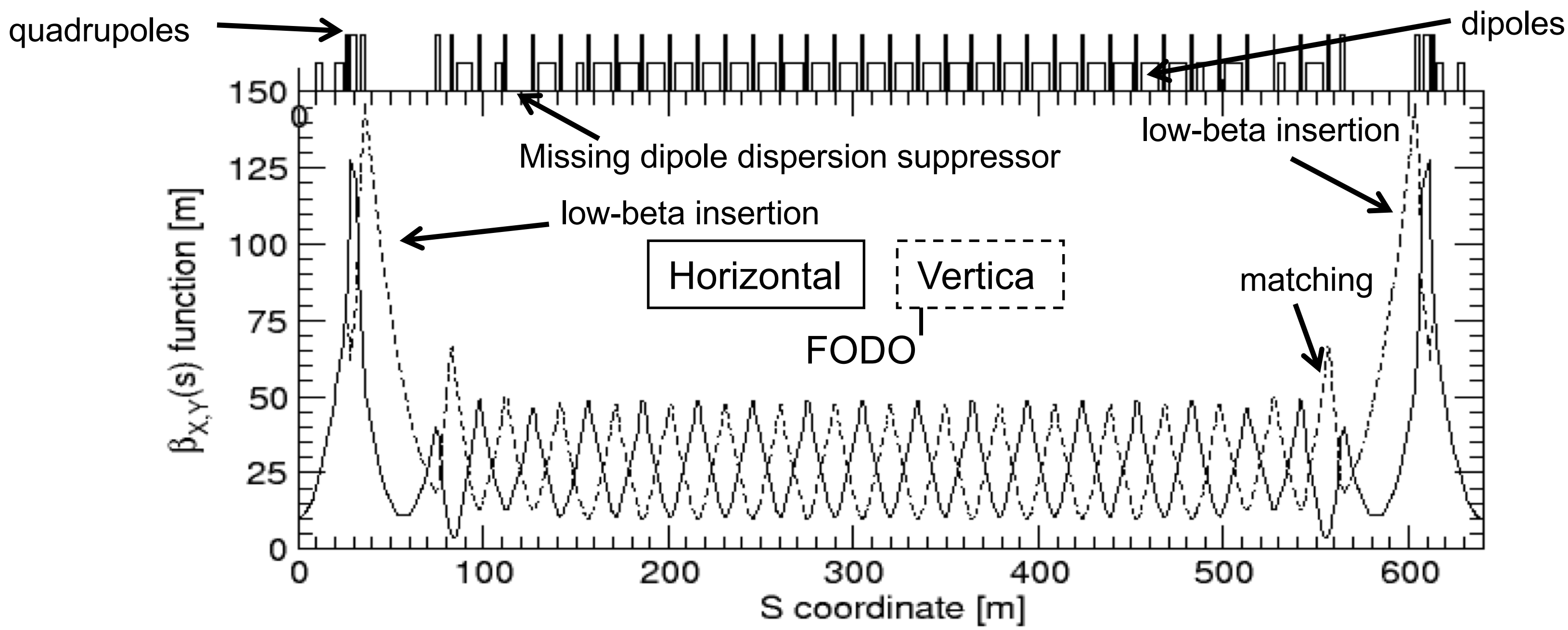



# Mismatched Dispersion

- What does mismatched dispersion look like?
  - For example, this is what happens when the second dispersion suppressor is eliminated and the dipole-free FODO cells run right up against the FODO cells with dipoles



# RHIC Lattice Revisited




- Note modular design, including low-beta insertions
  - Used for experimental collisions
  - Minimum beam size σ (with zero dispersion)
    - **maximize luminosity**
  - Large σ, beam size in "low beta quadrupoles"
  - Other facilities also have longitudinal bunch compressors (this afternoon)