CHAPTER 7

MODULATIONAL DIFFUSION

In this chapter a summary of the requirements for modulational (thick-layer) diffusion to exist in a particle synchrotron is presented and applied to a simple tune-modulated collider model of the Fermilab Tevatron where the only nonlinearities present are two beam-beam kicks. This is presented as an example of how nonlinearities combined with tune modulation can cause individual particle amplitude growth, leading to emittance growth and possible lifetime limitations in a storage ring.

Modulational diffusion has been the subject of many investigations in the past ten years, since it provides a particle loss mechanism in many-dimensional dynamical systems such as particle accelerators over timescales that are longer than those from pure resonant loss (typically hundreds of turns), but shorter than the timescales of Arnol'd (or thin-layer) diffusion (typically hundreds of millions of turns). Most of the salient features and quantitative analysis can be found in assorted publications (Chirikov et. al. 1985, Vivaldi 1984, Lichtenberg and Lieberman 1983); in § 7.1 the requirements for modulational diffusion to exist in a synchrotron are outlined, and the qualitative characteristics of amplitude growth created by this diffusion are described. The simulation lattice, a mockup of the Fermilab Tevatron collider lattice, and the beam-beam force are described in § 7.2. The results of Evol simulation of such a circumstance in an area of phase space where modulational diffusion is expected are described in § 7.3, which shows that amplitude growth in this circumstance is exponential instead of root-time as classically predicted by diffusion theory. These results and possible future directions are summarized in § 7.4.

7.1 CHARACTERISTICS OF MODULATIONAL DIFFUSION

Consider the one-dimensional tune modulation parameter plane of Figure (4.1). For appropriate tune modulation parameters falling within the "Chaos" region, sets of sidebands are created and overlap with the primary resonance and create a band of chaotic motion. Particles located at amplitudes within this chaotic band have phases that oscillate highly irregularly; in modulational diffusion models this chaos serves as a noise source for regular motion in the other transverse plane, coupled to this phase through a weak nonlinear coupling resonance.

Here we assume that there is horizontal tune modulation creating a localized chaotic region in the horizontal phase space and examine what would nominally be regular motion in the vertical dimension, influenced by one or more of these "weak" coupling resonances. (For these purposes such resonances are considered to be "weak" if their amplitudes are much smaller than that of the primary resonance in the horizontal plane driving the horizontal stochasticity.) The motion in the vertical plane is now that of a very weakly driven oscillator, where the driving force is now chaotic due to its weak coupling to the horizontal stochastic motion. Such motion is similar to that of a random walk problem; the stochastically driven vertical motion can "diffuse" out to large amplitudes in finite time.

Chirikov, Lieberman, Vivaldi and Shepelyanski (1985) write the Hamiltonian for the standard modulational diffusion model as

$$\begin{aligned} H_1(\theta_x, I_x, \theta_y, I_y) = &\frac{1}{2} I_x^2 - \epsilon \cos[(k+1)\theta_x + \lambda \sin \Omega t] + \\ &\frac{1}{2} I_y^2 - \mu \cos[k\theta_x + \theta_y] \ , \end{aligned} \tag{7.1}$$

where (θ, I) are action-angle variables in each plane. Ignoring the weak two-dimensional coupling, the motion in the horizontal dimension of this model is similar to that of the tune-modulated resonance islands in Chapter 4 as given by the N-turn Hamiltonian of Equation (4.27) — ϵ represents the square of the island

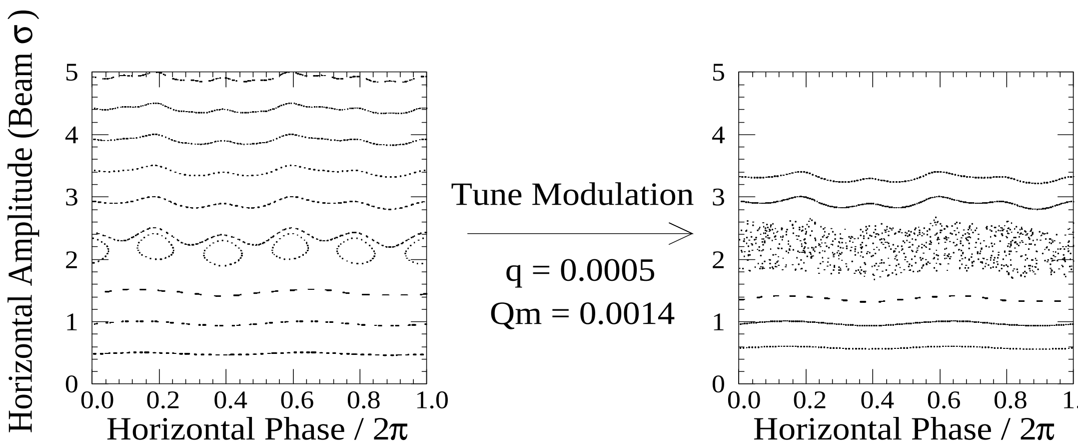

Figure 7.1: 1 dimensional phase space on the $Q_x = 3/5$ resonance, for moderately realistic parameters in the Fermilab Tevatron. Particle tracking is done in Evol with the B0D0 lattice, showing resonance island structures without tune modulation and a thick stochastic band with tune modulation induced by chromaticity.

tune, and λ and Ω represent the strength and frequency of the tune modulation, respectively. Modeling this system with a lattice to be described in the next section produces the one-dimensional phase-space plots shown in Figure (7.1). Motion in the vertical plane in Equation (7.1) is that of a pendulum, weakly coupled to the horizontal motion via the coupling strength $\mu \ll \epsilon$.

A significant difference between the modulational diffusion Hamiltonian (7.1) and the tune modulation nonlinear resonance Hamiltonian (4.27) is the presence of amplitude dependence in the resonance strengths of the latter. In particular, though the horizontal amplitude only varies through the stochastic band, the vertical amplitude growth predicted by modulational diffusion may affect the global motion. Such growth may either carry the horizontal tune off the primary resonance that drives the chaos through detuning, or alter the strength of the coupling resonance, thus changing the vertical amplitude growth rate as the vertical amplitude increases.

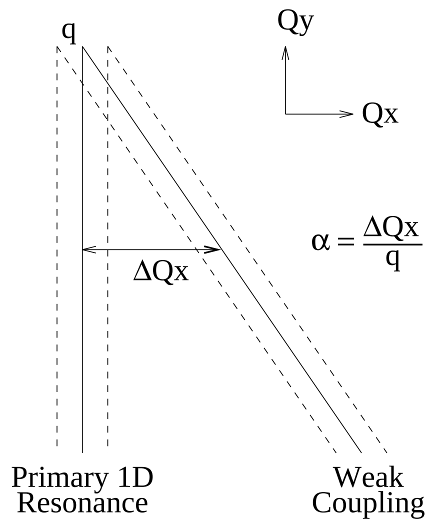

Figure 7.2: Resonance structure for modulational diffusion. α is the horizontal tune distance between the primary driving resonance and the secondary weak coupling resonance, scaled by the modulation depth q.

One prediction of modulational diffusion theory is that this diffusive growth in the vertical dimension will scale as the square root of turn number. A diffusion coefficient can be defined,

$$D \equiv \frac{\langle(\Delta I_y(t))^2\rangle}{2T} , \tag{7.2}$$

where ΔI_y is the vertical action excursion from the initial action and T is the time width of the averaging, in turns. The averaging should be performed over a time T short compared to the vertical diffusion time (so $\Delta I_y \ll I_y(t = 0)$ but long compared to the timescales of motion across the thick horizontal chaotic band (typically hundreds of turns). As the vertical tune is varied along the horizontal one-dimensional resonance, the proximity of the weak coupling resonance changes, as given by the dimensionless quantity

$$\alpha \equiv \frac{|Q_x(\text{weak coupling resonance}) - Q_x(\text{primary resonance})|}{q} . \tag{7.3}$$

A plot of the logarithm of the diffusion coefficient D versus this scaled proximity α shows a series of descending plateaus and sudden drops (Chirikov et. al. 1985).

Operational Parameter	**Symbol**	**Value**
Horizontal and vertical chromaticities	(ξ_x, ξ_y)	3.0
Typical momentum offset	$\Delta p/p$	0.0003
Synchrotron (modulation) frequency	Q_M	0.00078
Beam-beam linear tune shift per crossing	ξ	0.005
Revolution frequency	f_{rev}	47.7 kHz

Table 7.1. Typical Fermilab Tevatron 1992 operational parameters at 900 GeV.

This strange structure has sharp drops in D at even integer values of α for the case where both the driving resonance and the coupling resonance are modulated, and it is this sort of structure we attempt to qualitatively reproduce here within the operational framework of the Fermilab Tevatron.

7.2 THE TEVATRON SITUATION AND AN OPERATIONAL MODEL

In the Fermilab Tevatron during the 1992 collider run with separators, there were two strong beam-beam interactions every turn — one at the CDF experimental site at ring location B0 and one at the D0 experimental site. The operating estimate of the linear beam-beam tune shift ξ is approximately $\xi \approx 0.005$ per interaction, and with planned upgrades including the Fermilab Main Injector, this value may very well rise even further (Holmes 1991). With the exceptions of these beam-beam kicks and chromaticity-correction sextupoles (which are neglected for the sake of simplicity of the tracking model), the Tevatron is quite a linear machine, and so its transverse dynamics in this situation here can be modeled extremely simply in the tracking program Evol using only linear phase advances and beam-beam kicks. Typical operating parameters for the 1992 collider run are listed in Table 7.1.

The base tunes of the Tevatron in typical collider run circumstances are $Q_{x0} \approx 20.586$ and $Q_{y0} \approx 20.575$, running at a horizontal tune between the 12th order resonance, $Q_x = 20.583$ and the 5th, $Q_x = 20.600$. For the purposes of this study, however, a worst case scenario is investigated, where the driving resonance for the horizontal stochasticity necessary for modulational diffusion is the 5th order resonance and single particles are launched at a variety of vertical tunes along this resonance. If ξ ever exceeds .009 with two collisions in the Tevatron, the available space between the 12th and the 5th becomes too small for the entire beam, and a significant portion of the beam could be strongly affected by one of these resonances. The relevant portion of the tune plane diagram and the strange shape of the beam-beam footprint are shown in Figure (7.3).

The beam-beam force used here and within the tracking program Evol uses the weak-strong approximation and assumes both beams have round gaussian distributions of equal transverse size σ. For the horizontal beam-beam kick,

$$\frac{\Delta x'}{\sigma} = \frac{-4\pi\xi}{\beta_x^\star R^2}\left[1 - e^{-R^2/2}\right]\frac{x}{\sigma}\ , \tag{7.4}$$

where x is the transverse position relative to the opposing beam center, $x' \equiv dx/ds$, $\beta_x^\star$ is the beta function at the interaction point, and R is the distance from the center of the opposing beam scaled to the beam size σ:

$$R^2 \equiv \left(\frac{x}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^2\ ; \tag{7.5}$$

a kick similar to Equation (7.4) is seen in the vertical plane. Salient features relevant to this study can be noted:

The detuning is drastically different from the model of Equation (7.1), where there is explicitly no coupling other than the weak resonance. The explicit variation of resonance strength with particle amplitude is also a difference between these two models. As vertical amplitude grows in the beam-beam situation, one of two mechanisms will halt modulational diffusion: the vertical amplitude growth will either pull the horizontal tune off the primary driving resonance or it will

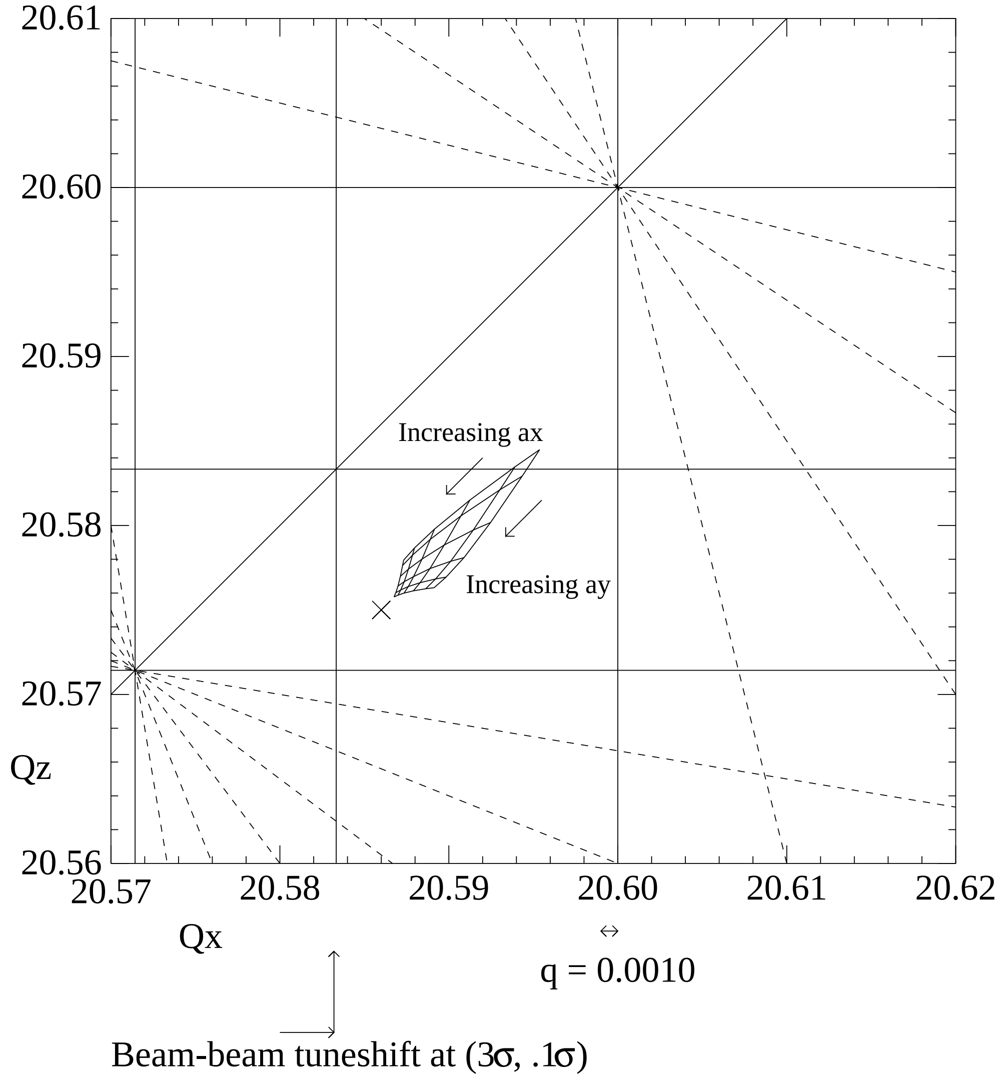

Figure 7.3: The tune plane for typical Fermilab Tevatron 1992 collider operations, showing the 5th, 7th and 12th order resonances. The nominal operating tunes are indicated by the cross, and the beam-beam footprint is shown for $\xi = 0.005$ with two collisions at B0 and D0. Footprint contours of constant amplitude range from $.1\sigma$ to 5.1σ in 1σ increments.

- The variation of tune with amplitude (detuning) given by the beam-beam force is nonlinear and strongly coupled, quite unlike the octupole detuning observed in the previous chapter.
- From the form of R, even-order resonances are driven to first order in ξ. Odd-order resonances of order N are driven as even order resonances of order $2N$.
- Resonance strengths vary with particle amplitude, or action.
- There is no beam-beam tuneshift or resonance driving at infinite amplitudes, so global motion of the unperturbed beam-beam system is stable.

suppress the coupling resonance strength. It remains to be conclusively shown whether such vertical amplitude growth will cause significant particle loss; however, even collective vertical amplitude growth without loss will raise the vertical beam emittance and result in luminosity degradation.

7.3 SIMULATION RESULTS

The tracking program Evol was used for all simulations, using the B0D0 lattice described in the previous section. In order to drive the 5th order resonance strongly for the worst-case scenario, a small $.1\sigma$ beam-beam offset was included; closed orbit alignment errors of this magnitude at the collision points are quite possible. Tracking this lattice with no tune modulation with the beam-beam tuneshift given in Table 1 on the $Q_x = 20.6$ resonance finds an island tune of $Q_I = 1.51 \cdot 10^{-3}$. Since the synchrotron frequency of the Tevatron at this energy is approximately $Q_M = 7.8 \cdot 10^{-4}$ (with a period $T_M = 1/Q_M = 1280$ turns), the chaotic region of the tune modulation parameter space is quite accessible for moderate tune modulation depths q.

Tracking was performed with tune modulation depth $q = 0.0010$, present only in the horizontal plane for comparison to the results originating in the similarly modulated Hamiltonian of Equation (7.1). This tune modulation amplitude corresponds to a horizontal chromaticity of about 3 units with a momentum offset

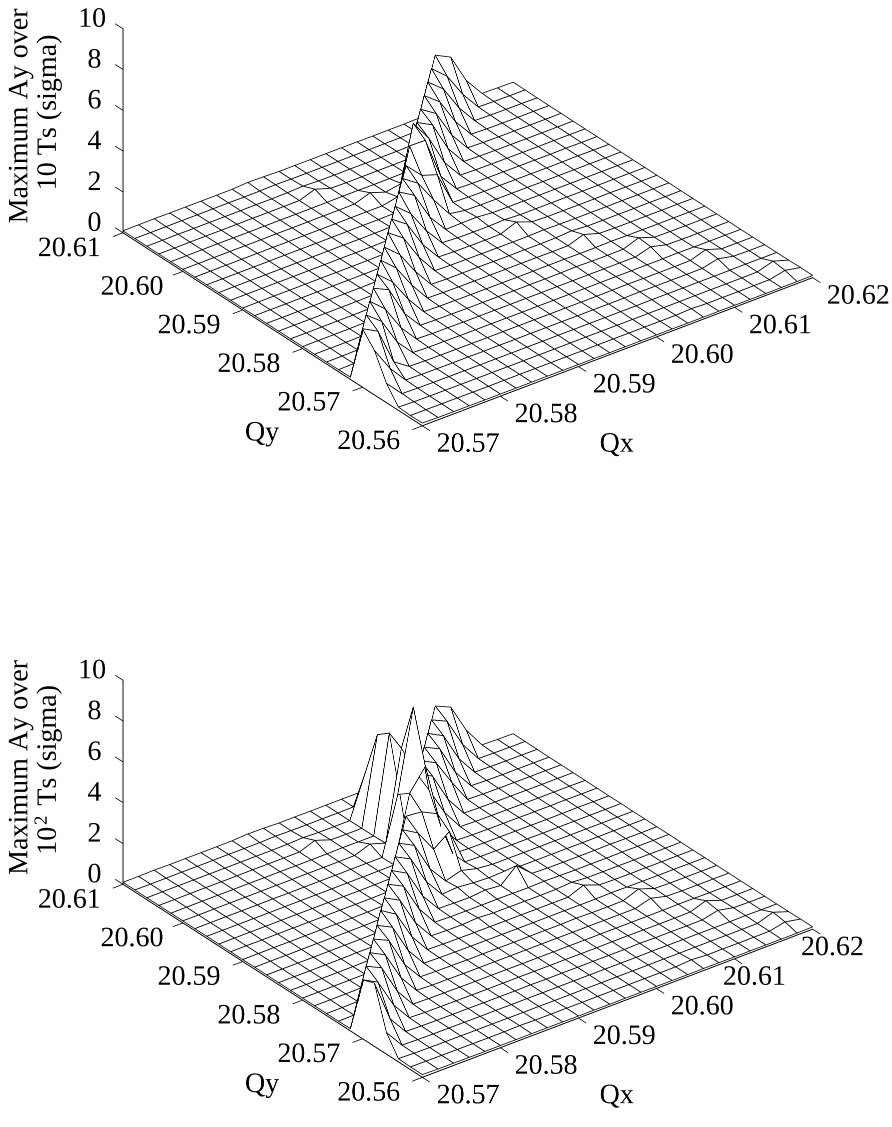

Figure 7.4: Maximum vertical amplitudes of single particles with initial vertical amplitudes $.1\sigma$, tracked over 10 and 100 synchrotron periods. The particle's horizontal amplitude is 3σ, inside a chaotic band.

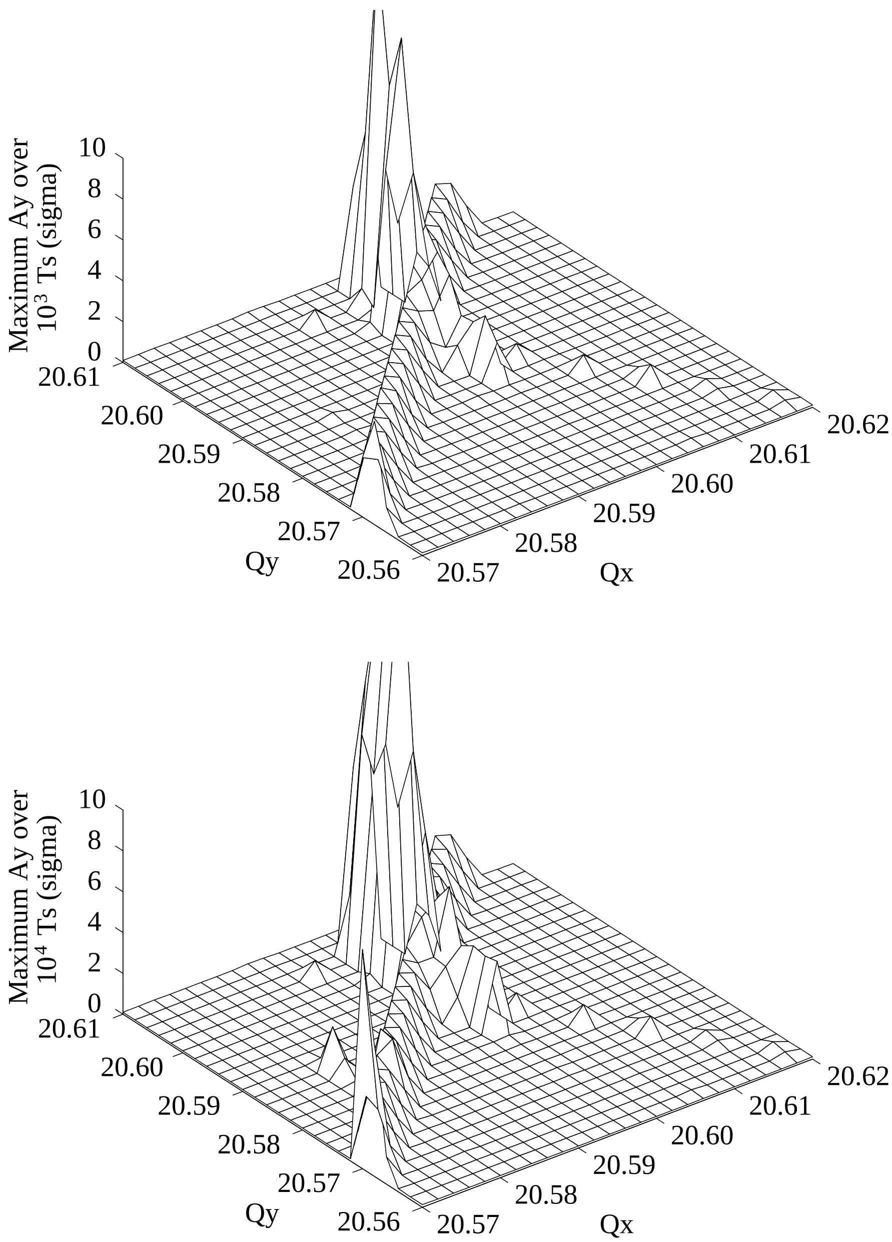

Figure 7.5: Maximum vertical amplitudes of single particles with initial vertical amplitudes $.1\sigma$, tracked over 10^3 and 10^4 synchrotron periods. Conditions for tracking are otherwise the same as those in Figure (7.4).

$\Delta p/p$ of $3 \cdot 10^{-4}$, realistic values for the Tevatron. Particles were launched with horizontal amplitude of 3 σ, with base tune $Q_{x0} = 20.597$, and vertical amplitude of .1 σ with various base tunes. Tracking was stopped when a finite number of synchrotron periods had been tracked (10^4, corresponding to nearly 5 minutes of real particle evolution), or the vertical amplitude had reached 1.0 σ. The one sigma vertical cutoff was introduced because the influence of the vertical motion on the horizontal stochastic band was expected to become non-negligible at moderate vertical amplitudes.

To establish the timescales of the relevant amplitude growth mechanisms, the maximum vertical amplitude was recorded for single particles launched at the above initial conditions over a mesh on the tune plane, for tracking times ranging from 10 to 10^4 synchrotron periods. The tune mesh limits used were the same as those shown in the tune plane diagram, Figure (7.3), and the results of this tracking are shown in Figures (7.4) and (7.5). In these figures there is a quite definite amplitude growth near the intersections of the $Q_x - Q_y$ and $5Q_x$ resonances that evolves over timescales of thousands of synchrotron periods, consistent with the naive timescales of modulational diffusion. Such growth is completely absent with modulation turned off ($q = 0$), where only amplitude growth on the $Q_x - Q_y$ resonance is seen due to energy exchange between the unbalanced horizontal and vertical amplitudes; this is a conclusive indication that the modulation drives this vertical amplitude growth. The growth also displays structure along the horizontal resonance, consistent with the modulational diffusion expectations of the dependence of the amplitude growth rate on distance from the nearest coupling resonance. There is also some growth that appears on the $3Q_x + 2Q_y$ resonance; however, the structure along this resonance is quite minimal in comparison to the growth near the previously mentioned intersection of $Q_x - Q_y$ and $5Q_x$ resonances.

Once the timescales of amplitude growth have been established, there remains the question of how the vertical amplitude evolves with time. It has already been mentioned that classical diffusion predicts that the vertical amplitude will grow proportionally to $t^{1/2}$ — if this is the case a plot of $\log a_y$ versus $\log t$ should be a straight line with a slope of one half. However, if the vertical amplitude grows exponentially with time,

$$a_y(t) = a_{y0}\ e^{\gamma t}\ , \tag{7.6}$$

the plot of $\log a_y$ versus t, not $\log t$, should grow linearly, and the slope of this line is the exponential growth rate γ. γ has units of inverse synchrotron periods, because the natural time unit for problems involving direct modulation is the modulation period, not turns.

Figure (7.6) shows three examples of vertical amplitude evolution over relatively long timescales, each plotted on log-linear and log-log scales. It is clear from examining these evolutions (as well as those of many other particles at different distances α from the nearby weak coupling resonance $4Q_x + Q_y$) that the vertical amplitude is growing as an exponential of time, not a power law as one would expect from standard diffusion phenomenology. It has been suggested that this growth may be explained by the dependence of resonance strengths on particle amplitude — the change in amplitude creates a changing resonance strength which feeds back upon the amplitude growth, creating exponential growth.

The exponential growth coefficient γ can now be plotted versus the scaled distance to the weak coupling resonance as one varies the vertical tune along the $5Q_x$ resonance to investigate whether there is any structure present. Since γ is expected to vary over many orders of magnitude, we instead plot $\log\gamma$ versus α; α can be directly determined from the vertical base tune Q_{y0} via

$$\alpha = \frac{20.60 - Q_{y0}}{5q} \tag{7.7}$$

when considering the $4Q_x + Q_y$ resonance to be the source of weak coupling.

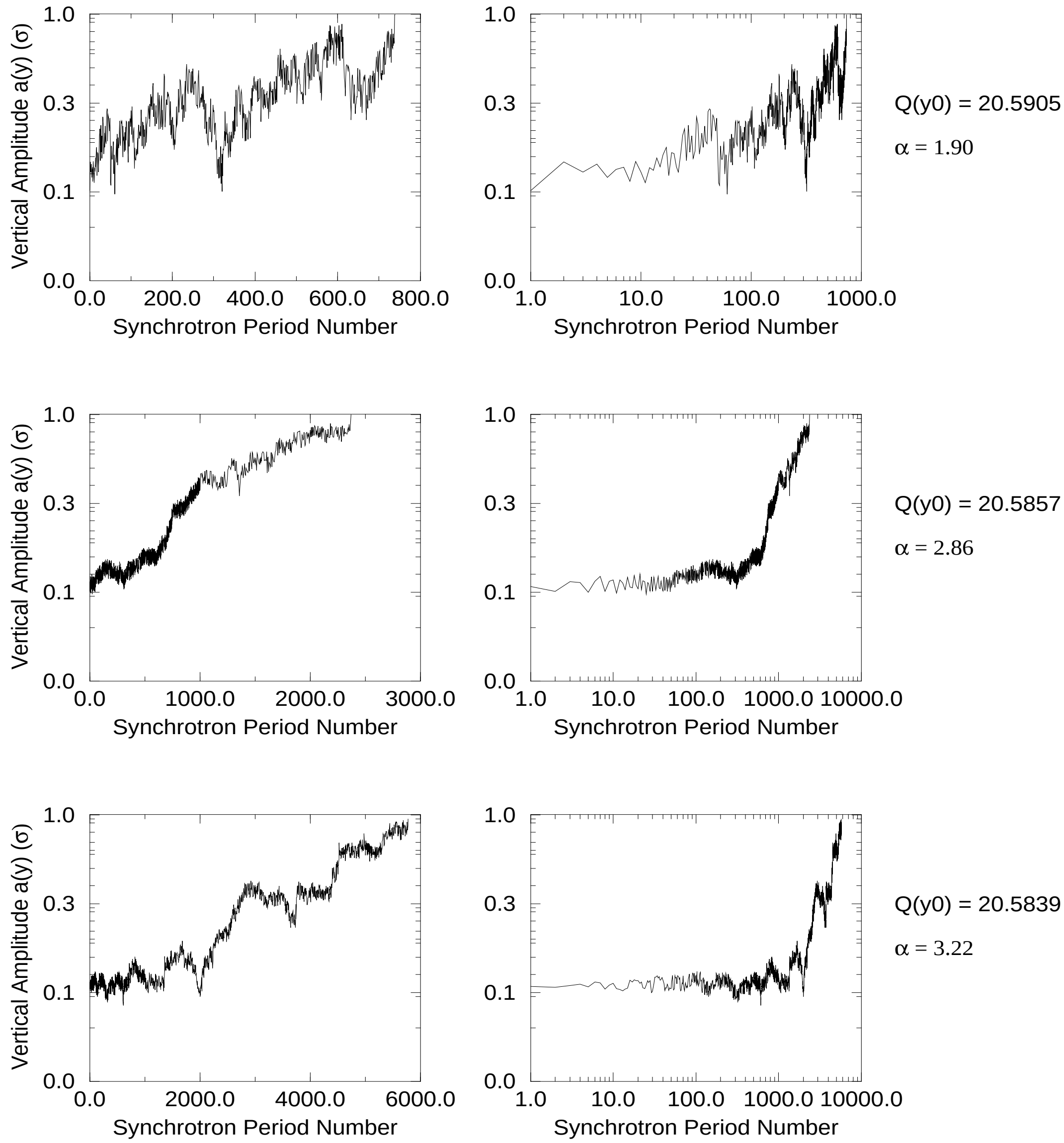

Figure 7.6: Character of vertical amplitude growth for particles launched within a horizontal stochastic band, and initial vertical amplitudes $.1\sigma$ and three different base tunes, or three values of the scaled coupling resonance proximity α. Tracking was stopped when the vertical amplitude reached 1σ or after 10^4 synchrotron periods. The vertical amplitude growth rate changes by an order of magnitude here.

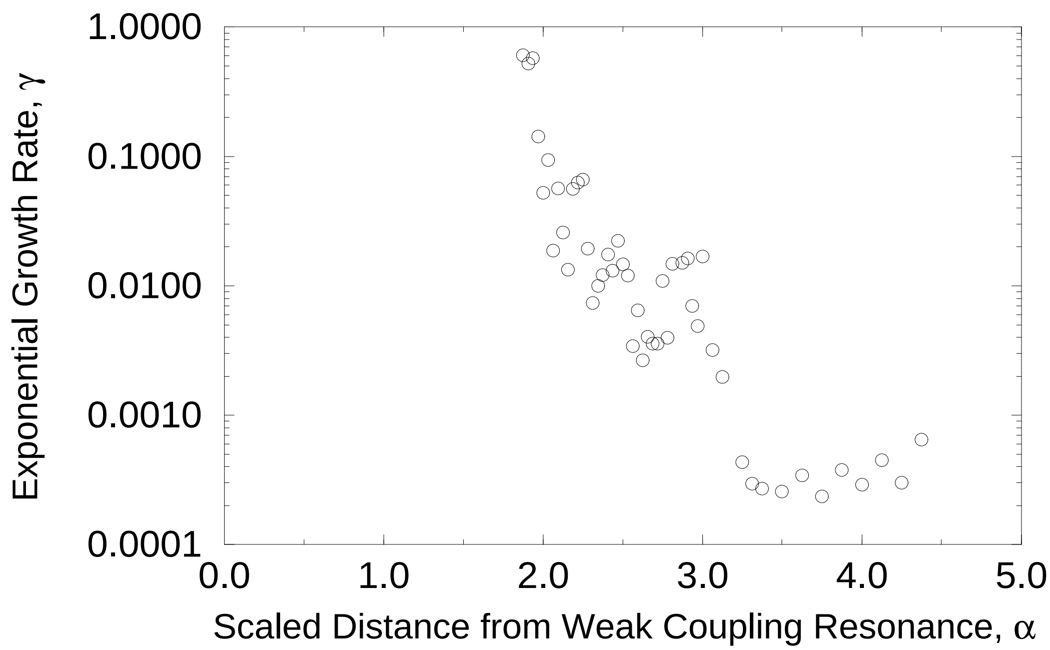

Figure 7.7: Exponential vertical amplitude growth rate γ plotted versus the scaled distance α from the $4Q_x + Q_y$ resonance.

Other resonances, such as $Q_x - Q_y$ and $3Q_x + 2Q_y$ are also nearby, but are farther away in horizontal tune distance α than this resonance as can be seen in Figure (7.3). γ is measured from a standard linear fit of tracked $\log a_y$ versus time data. Figure (7.7) shows this data; note the two distinct "plateaus" and the sudden drops in the growth rate at $\alpha = 2$ and $\alpha = 3$.

7.4 CONCLUSIONS AND FUTURE DIRECTIONS

Modulational diffusion has been investigated within a simple model of the beam-beam interaction in the Fermilab Tevatron collider. Realistic operational parameters indicate that particles subject to horizontal stochasticity, or naively those that are within the tune modulation depth distance in horizontal tune of the $5Q_x$ resonance, experience modulational diffusion that causes their vertical amplitudes to grow exponentially over timescales of thousands of synchrotron periods, or millions of turns, leading to possible long-term particle loss. The rate of this amplitude growth is also dependent on proximity of nearby coupling resonances, and shows a structural dependence similar to those of previous modulational diffusion studies, even though the vertical amplitude growth is not root-time as naively predicted in these models where resonance strengths are not action-dependent.

Under current operating conditions, no particles are expected to be affected by the $5Q_x$ resonance this severely unless the horizontal tune drifts upwards, dragging particles into the fifth, or the linear beam-beam tune shift ξ increases. However, with future luminosity upgrades, this tune shift per crossing will almost certainly rise and the operational space used in past runs may not be large enough to accommodate the entire tune spread of the beam. A significant portion of the beam would be influenced by the horizontal 7th and 5th integer resonances and vertical beam blowup could possibly occur. This circumstance would lead to luminosity degradation and intensity loss over a collider store as the beam size grows. These effects, were they present in an actual collider, would be difficult to diagnose due to their slow growth timescales.

Future studies should be twofold. First, a concrete theoretical structure of modulational diffusion should be investigated to conclusively show that in the case of amplitude-dependent coupling resonance strengths, vertical amplitude growth is exponential instead of root-time as in classical models. Second, the collective nature of the particle growth should be investigated to see what observable effects such a mechanism could have on the beam size (and thus luminosity) evolution over time. These collective effects and the emittance growth timescale dependence on proximity to driving resonances would be experimentally observable.

The octupole-decapole tracking model might also be used to investigate the amplitude growth mechanism once a theoretical framework is in place, to avoid the rather complex detuning and coupling of the beam-beam force. Tracking with this lattice has several distinct advantages — motion at large particle amplitudes is no longer stable, so no ad hoc aperture needs to be introduced, and parameters for a Hamiltonian description as in Equation (7.1) can easily be found to first order in the individual magnet strengths as described in Chapter 3.